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RESEARCH PAPER

Darboux integrability of a hyperjerk memristive system

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A B S T R A C T:

In this work, we investigate the following novel four-dimensional dynamical memristive system (see, (Prousalis et al., 2017))

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = w, \qquad \dot{w} = -z - (1 + x)y - a_1 w - a_2 z^2 w.$$

This system, in a certain area of the parameter space, exhibits hyper-jerk dynamics and a line of singularities passing through the origin, It may be regarded as a dynamical system including hidden attractors. The behavior of the proposed system is studied through polynomial first integrals and a Darboux first integral. We first prove that the system has a minimal polynomial first integral. In addition, all the invariant algebraic surfaces and exponential factors of this suggested system are computed. Then, we prove that this type of system doesn't have a first integral of the Darboux type.

KEY WORDS: Chaos system, Darboux first integrals, polynomial first integrals
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1. Introduction and the main results:

Memristors have attracted the curiosity of engineers due to their prospective uses, (Chua, 1971, Corinto, 2011, Zhao, 2019). They have been applied to produce chaotic signals as a dynamic element. Typically, a memristor results in the four-dimensional system (Bao, 2011, Li, 2018). Several the three dimensional memristive chaotic systems have been investigated (Itoh, 2008, Muthuswamy, 2010, Sun, 2016), but in those circumstances, the memristor was replaced with a complex operational amplifier based equivalent device.

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According to the link between the four fundamental circuit elements (voltage, current, magnetic flux, and charge), the fourth missing circuit element is hypothesized in theory as a memristor by Chua in 1971 [2]. Chua and Kang later developed a general idea of memristive systems in 1976 (Chua, 1976). Based on its characteristics, memristors offer a wide range of possible applications, including charge pump locked loops, chaotic oscillators, neural networks, and more(Itoh, 2008, Muthuswamy, 2010).

Electrical engineering, physical engineering, and other fields all involve numerous non-linear differential systems. The study on differential systems has been an interesting area for finding new ways to illustrate and understand the complexity of systems. One of the open problem in the qualitative theory of differential systems is the investigation on the existence and nonexistence of first integrals, or more broadly, the integrability problem of a differential system (Christopher, 2019, Hussein, 2022). In the four-dimensional nonlinear differential systems, it can be challenging to recognize the existence of first integrals. Here, we consider the hyperjerk memristive system which is described in (Prousalis et al., 2017)

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -z-(1+x)y-a_1w-a_2z^2w, \end{aligned} \tag{1}$$

where a_1 and a_2 are nonzero parameters, the variables x, y, z and w are state variables in R, and $\cdot = d/dt$ is the derivative with respect to time t. The vector field corresponding to system (1) is defined as

$$\chi = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} - ((1+x)y$$
$$+a_1w + a_2z^2w + z)\frac{\partial}{\partial w}. \tag{2}$$

The degree of the vector field χ is $deg(\chi) = 3$.

System (1) is one of the most intriguing dynamical electronic systems. For every a_1 and a_2 , the system has a line of singularities $E_x(x, 0,0,0)$ as in(Jalal, 2020). Authors, in (Prousalis et al., 2017), investigated the behavior of system (1) using standard methods of dynamical system theory, such as Lyapunov exponents, phase pictures, and bifurcation diagrams. Here, we first investigate the minimal polynomial first integrals of system (1). Recall that a (non-constant) continuously-differentiable function $H(x, y, z, w): D \subset \mathbb{R}^4 \to \mathbb{R}$ (D an open set) is a first integral of the vector field χ on D if $H(x(t), y(t), z(t), w(t))$ is a constant for all $t \in \mathbb{R}$, for which $x(t), y(t), z(t), w(t)$ is a solution of the vector field χ defined on D. That is, $H(x(t), y(t), z(t), w(t))$ on D is a first integral of system (1) if and only if the following holds

$$\chi(H) = y\frac{\partial H}{\partial x} + z\frac{\partial H}{\partial y} + w\frac{\partial H}{\partial z} - ((1+x)y$$
$$+a_1w + a_2z^2w + z)\frac{\partial H}{\partial w} = 0, \text{ on } D.$$

Recall that $H(x, y, z, w)$ is a polynomial first integral if it is a polynomial function. If there is a polynomial first integral $F(x, y, z, w)$ of the system such that $\deg(H) \le \deg(F)$, then $H(x, y, z, w)$ is said to be minimal polynomial first integral, (Llibre, 2001). In following theorem which is the first result, we study a polynomial first integral of system (1).

Theorem 1. *System (1) has a minimal polynomial first integral of degree 3 which is*

$$H(x, y, z, w) = x + y + a_1z + w + \frac{1}{2}x^2 + \frac{1}{3}a_2z^3.$$

We prove Theorem 1 in Section 2.

The second main result is concerned to the non-existence of a Darboux first integral of system (1) and characterizes all of its invariant algebraic surfaces and exponential factors. In order to prove Theorem 2, we declare and demonstrate two more results that characterize all Darboux invariant polynomials and exponential factors of system (1). we first need the following.

Let $\mathbb{C}[x, y, z, w]$ be the ring of complex polynomials, then a non-constant element $f(x, y, z, w) \in \mathbb{C}[x, y, z, w]$ is said to be a Darboux invariant polynomial of system (1) if there is a non-zero cofactor $K_f(x, y, z, w) \in \mathbb{C}[x, y, z, w]$ such that

$$y\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} - ((1+x)y$$
$$+a_1w + a_2z^2w + z)\frac{\partial f}{\partial w} = K_f\, f. \tag{3}$$

In this work, the degree $deg(K_f) \le 3$. If $K_f = 0$, then recall that a Darboux invariant polynomial $f = f(x, y, z, w)$ becomes a polynomial first integral, see (Lima, 2014). An exponential factor has the same role as the Darboux invariant polynomial. So, to determine the Darboux first integral, an exponential factor will be required. Therefore, $E = exp(\frac{f}{g})$, where $f, g \in \mathbb{C}[x, y, z, w]$, is an exponential factor if it satisfies

$$y\frac{\partial E}{\partial x} + z\frac{\partial E}{\partial y} + w\frac{\partial E}{\partial z} - ((1+x)y$$

$$+a_1 w + a_2 z^2 w + z)\frac{\partial E}{\partial w} = K_E\, E\,, \qquad (4)$$

for some cofactor $K_E \in \mathbb{C}[x,y,z,w]$ with $deg(K_E) \le 2$, for more detail, see (Christopher et al., 2007). If the differential system has no non-constant Darboux invariant polynomial, then $E = exp(g)$ can be an exponential factor which derives from the multiplicity of the infinite invariant surface, see (Christopher et al., 2007, Christopher and Llibre, 2000). To find that the differential system has first integrals according to Darboux's idea, one must look for a function of the type (6), and K_{f_i}, K_{E_j} that satisfy the following equation

$$\sum_{i=1}^{p} \lambda_i \mathrm{K}_{\mathrm{f_i}} + \sum_{j=1}^{q} \mu_j K_{E_j} = 0, \qquad (5)$$

where λ_i, and μ_j not all zero are in $\mathbb{C}$, for more details see (Llibre and Zhang, 2009). In order to Prove Theorem 2, we also need two further results that explain all Darboux invariant polynomials and exponential factors of system (1). By a Darboux first integral Darboux first integral, we mean that the next expression is a first integral in Darboux type

$$\widetilde{H}(x,y,z,w) = h_1^{\lambda_1} \cdots h_p^{\lambda_{\mathrm{p}}}\, E_1^{\mu_1} \cdots E_q^{\mu_q}, \qquad (6)$$

where h_i for $i = 1,\cdots,p$ are Darboux invariant polynomials and E_j for $j = 1,\cdots,q$ are exponential factors, with λ_j for $j = 1,\cdots,p$ and μ_k for $k = 1,\cdots,q$ are constants in $\mathbb{C}$, for more details, see Section 2. The Darboux theory of integrability, which is basically dependent on the number of Darboux invariant polynomials and exponential factors of a differential system, contains a principle that may be found in the form (6), see (Christopher et al., 2007, Llibre and Zhang, 2012). Namely, the existence of a Darboux first integral is relied on the sufficient number of h_i and E_i. However, in Section 2, we show that system (1) has no Darboux invariant polynomials, but some exponential factors are found.

In Theorem 2, we have the minimal polynomial first integral, $H(x,y,z,w) = x + y + a_1 z + w + \frac{1}{2}x^2 + \frac{1}{3}a_2 z^3$ which is a particular case of the Darboux first integral. In the following theorem we want to show that the system (1) has no other Darboux first integral except $H(x,y,z,w)$.

We summarize, in the following, the second result of system (1)

Theorem 2. *System (1) has no first integrals of the Darboux type except*
$H(x,y,z,w) = x + y + a_1 z + w + \frac{1}{2}x^2 + \frac{1}{3}a_2 z^3.$
We prove Theorem 2 in Section 2.

2. Auxiliary results

In this section, we provide some definitions and theorems as a basis to this study in order to prove our main results.

Lemma 1. *The polynomial function $H(x,y,z,w) = x + y + a_1 z + w$ is the minimal polynomial first integral of the following linear vector field*

$$\tilde{\chi} = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} - (y + z + a_1 w)\frac{\partial}{\partial w}. \qquad (7)$$

Proof. The proof is obvious that it becomes a minimal polynomial first integral.
By L_1 we denote the following linear partial differential operator

$$L_1 = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} - (y + z + a_1 w)\frac{\partial}{\partial w}. \qquad (8)$$

Next result concerns with the polynomial first integral of the following quadratic vector field

$$\chi_q = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} - ((1+x)y$$
$$+z + a_1 w)\frac{\partial}{\partial w}. \qquad (9)$$

Lemma 2 *The polynomial function*
$H(x,y,z,w) = x + y + a_1 z + w + \frac{1}{2}x^2$ *is the minimal polynomial first integral of the quadratic vector field* (9).

Proof. Let $H=\sum_{i=1}^{n} h_i(x,y,z,w)$ be a non-zero polynomial first integral of (9) where h_i is a homogeneous polynomial of degree i. Using operator L_1 then, H satisfies

$$\begin{aligned}
&L_1(h_1)=0,\\
&L_1(h_2)=xy\frac{\partial h_1}{\partial w},\\
&\quad\vdots \qquad\qquad \vdots\\
&L_1(h_{n-1})=xy\frac{\partial h_{n-2}}{\partial w}, \qquad\qquad (10)\\
&L_1(h_n)=xy\frac{\partial h_{n-1}}{\partial w}
\end{aligned}$$

If $h_1(x,y,z,w)=x+y+a_1z+w$, then $L_1(h_1(x,y,z,w))=0$ holds. Using the fact that $h_2(x,y,z,w)$ is the homogeneous polynomial of degree two, from the second equation in (10), we obtain

$$h_2(x,y,z,w)=C(a_1z+w+x+y)^2-\frac{1}{2}(a_1z+w+y)(a_1z+w+2\,x+y),$$

where C is a constant if we put $C=\frac{1}{2}$. Then, we obtain $h_2(x,y,z,w)=\frac{1}{2}x^2$. By the same way, we proceed to find a solution to the other equation in (10), we find that the following are solution

$$\begin{aligned}
h_3&=c_0(za_1+w+x+y)^3,\\
h_4&=c_0\frac{3}{2}x^2(za_1+w+x+y)^2,\\
h_5&=c_0\frac{3}{4}x^4(za_1+w+x+y),\\
h_6&=c_1\frac{1}{8}x^6,\\
h_7&=c_1(za_1+w+x+y)^7,\\
h_8&=c_1\frac{7}{2}x^2(za_1+w+x+y)^6,\\
h_9&=c_1\frac{21}{4}x^4(za_1+w+x+y)^5,\\
h_{10}&=c_1\frac{35}{8}x^6(za_1+w+x+y)^4,\\
h_{11}&=c_1\frac{35}{16}x^8(za_1+w+x+y)^3,\\
h_{12}&=c_1\frac{21}{32}x^{10}(za_1+w+x+y)^2,\\
h_{13}&=c_1\frac{7}{64}x^{12}(za_1+w+x+y),\\
h_{14}&=c_1\frac{1}{128}x^{14},\\
h_{15}&=c_2(za_1+w+x+y)^{15},\\
&\vdots \qquad\qquad \vdots
\end{aligned}$$

From (10), if $n=5$ then we have only one polynomial first integral which is

$$H_0=za_1+w+x+y+\frac{1}{2}x^2,$$

If $n=6$ then we have only one polynomial first integral which is

$$H_1=c_0\,H_0+c_3(H_0)^3,$$

where c_0 and c_3 are constants. If $n=13$, then H_1 is the polynomial first integral. If $n=14$, then we have only one polynomial first integral which is

$$H_3=c_0H_0+c_3(H_0)^3+c_7(H_0)^7$$

where c_0, c_3 and c_7 are constants. If $n=29$, then H_3 is the minimal polynomial first integral. If $n=30$, then we have only one polynomial first integral which is

$$H_4=c_0\,H_0+c_3\,(H_0)^3+c_7\,(H_0)^7+c_{15}\,(H_0)^{15},$$

where $c_{\mathrm{i}}, i=0,3,7,15$ are constants. We can proceed by the same way to find a first integral of form $H=\sum_{i=1,3,7,30,\cdots}^{n} c_{\mathrm{i}}(H_0)^i$. This means that

$$H=x+y+a_1z+w+\frac{1}{2}x^2$$

is the minimal polynomial first integral of the quadratic system (9). ■

Let $H(x,y,z,w)=\sum_{i=1}^{n} h_i(x,y,z,w)$ be a polynomial first integral of (9) where h_i is the homogeneous polynomial of degree i. Using operator L_1 then, $H(x,y,z,w)$ satisfies the following equations

$$\begin{aligned}
&L_1(h_1)=0,\\
&L_1(h_2)=xy\frac{\partial h_1}{\partial w},\\
&L_1(h_3)=xy\frac{\partial h_2}{\partial w}+a_2z^2w\frac{\partial h_1}{\partial w},\\
&L_1(h_4)=xy\frac{\partial h_3}{\partial w}+a_2z^2w\frac{\partial h_2}{\partial w},\\
&\quad\vdots \qquad\qquad \vdots\\
&L_1(h_n)=xy\frac{\partial h_{n-1}}{\partial w}+a_2z^2w\frac{\partial h_{n-2}}{\partial w}. \qquad (11)
\end{aligned}$$

If $h_1=x+y+a_1z+w$ and $h_2=\frac{1}{2}x^2$. Then, from Lemmas 1 and 2, the first two equations in (11) hold. The third equation in (11) becomes

$$L_1(h_3(x,y,z,w))=a_2z^2w. \qquad (12)$$

Also, using the fact that $h_3(x,y,z,w)$ is the homogeneous polynomial of degree three, from (12) we have a solution which is

$$h_3 = c(za_1 + w + x + y)^3 + \frac{1}{3}a_2 z^3.$$

We can take $c = 0$. This gives that

$$H(x,y,z,w) = x + y + a_1 z + w + \frac{1}{2}x^2 + \frac{1}{3}a_2 z^3$$

is the first integral of system (2).

We now use Lemmas 1 and 2 to prove that system (1) has a minimal polynomial first integral.

Now we turn to prove Theorem 1.

Proof. (Proof of Theorem 1:) By direct computation $H(x,y,z,w)$ is the first integral of system (1). we mean that $\chi(H) = 0$. The prove of Theorem 1 is completed. ■

Proposition 1. *The following statements hold.*

a) *If $e^{g/h}$ is an exponential factor for the polynomial differential system (1) and h is not a constant polynomial, then $h = 0$ is an invariant algebraic surface.*
b) *Eventually e^g can be an exponential factor, coming from the multiplicity of the infinite invariant plane.*

The proof of the next result can be found in (Christopher et al., 2007, Christopher and Llibre, 2000).

Theorem 3. *Suppose that a polynomial vector field χ defined in $\mathbb{R}^n$ of degree m admits p Darboux polynomials f_i with cofactors K_i for $i = 1, \dots, p$, and q exponential factors $F_j = exp(g_j/h_j)$ with cofactors L_j for $j = 1, \dots, q$. If there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i \mathrm{K_i} + \sum_{j=1}^{q} \mu_j L_j = 0 \tag{13}$$

then the following real (multivalued) function of Darboux type

$$\widetilde{H}(x,y,z,w) = h_1^{\lambda_1} \cdots h_p^{\lambda_p} E_1^{\mu_1} \cdots E_q^{\mu_q},$$

substituting $f_i^{\lambda_i}$ by $|f_i|^{\lambda_i}$ if $\lambda_i \in \mathbb{R}$, is a first integral of the vector field χ. For a proof of the next result see (Llibre and Zhang, 2009).

Proposition 2. *System (1) has no Darboux invariant polynomials with a non-zero cofactor.*

Proof. Let $H = \sum_{j=0}^{n} f_j(x,y,z,w)$ be a Darboux invariant polynomial of system (1) (where f_i is a homogeneous polynomial in its variables of degree i for $i = 1,2,\cdots,n$ such that not all $f_i(x,y,z,w)$ is zero). Here we can consider that K_H is as follows

$$\begin{aligned}\mathrm{K_H} = {} & \mathrm{k}_0 + \mathrm{k}_1 x + \mathrm{k}_2 y + \mathrm{k}_3 z + \mathrm{k}_4 w + \mathrm{k}_5 x^2 \\ & + \mathrm{k}_6 yx + \mathrm{k}_7 zx + \mathrm{k}_8 wx + \mathrm{k}_9 y^2 + \mathrm{k}_{10} zy \\ & + \mathrm{k}_{11} wy + \mathrm{k}_{12} z^2 + \mathrm{k}_{13} wz + \mathrm{k}_{14} w^2,\end{aligned} \tag{14}$$

where $k_i \in \mathbb{C}$ for $i = 0, \dots, 14$. Then, from (3), we have

$$\begin{aligned} & y\sum_{j=0}^{n}\frac{\partial f_j}{\partial x} + z\sum_{j=0}^{n}\frac{\partial f_j}{\partial y} \\ & + w\sum_{j=0}^{n}\frac{\partial f_j}{\partial z} - ((1+x)y + a_1 w \\ & + a_2 z^2 w + z)\sum_{j=0}^{n}\frac{\partial f_j}{\partial w} = \mathrm{K_H}(\sum_{j=0}^{n} f_j).\end{aligned} \tag{15}$$

In (15), we compute the terms of degree $n+2$ to obtain

$$\begin{aligned}-a_2 z^2 w\frac{\partial f_n}{\partial w} = {} & (\mathrm{k}_{14} w^2 + \mathrm{k}_8 wx + \mathrm{k}_{11} wy \\ & + \mathrm{k}_{13} wz + \mathrm{k}_5 x^2 + \mathrm{k}_6 yx + \mathrm{k}_7 zx \\ & + \mathrm{k}_9 y^2 + \mathrm{k}_{10} zy + \mathrm{k}_{12} z^2) f_n.\end{aligned}$$

That is

$$f_n(x,y,z,w) = G_1(x,y,z)\, w^{\frac{-z^2\mathrm{k}_{12} + (-x\mathrm{k}_7 - y\mathrm{k}_{10})z - x^2\mathrm{k}_5 - xy\mathrm{k}_6 - y^2\mathrm{k}_9}{a_2 z^2}}\, e^{-\frac{w(w\mathrm{k}_{14} + 2x\mathrm{k}_8 + 2y\mathrm{k}_{11} + 2z\mathrm{k}_{13})}{2a_2 z^2}},$$

where G_1 is a polynomial function in the variables x, y and z of degree $n - m$. Since, f_n must be a homogeneous polynomial of degree n, we obtain $\mathrm{k}_5 = \mathrm{k}_6 = \mathrm{k}_7 = \mathrm{k}_8 = \mathrm{k}_9 = \mathrm{k}_{10} = \mathrm{k}_{11} = \mathrm{k}_{13} = \mathrm{k}_{14} = 0$ and we set $\mathrm{k}_{12} = -a_2 m$ with $m \in \mathbb{N} \cup \{0\}$. It follows that

$$f_n(x,y,z,w) = G_1(x,y,z)w^m.$$

Furthermore, we compute the terms of degree $n+1$ in (15), to obtain

$$\begin{aligned} a_2 z^2 w\frac{\partial f_{n-1}(x,y,z,w)}{\partial w} = {} & a_2 m z^2 f_{n-1}(x,y,z,w) \\ & -(mxyw^{m-1} + (\mathrm{k}_1 x + \mathrm{k}_2 y + \mathrm{k}_3 z \\ & + \mathrm{k}_4 w)w^m)G_1(x,y,z),\end{aligned}$$

By solving we obtain

$$f_{n-1}=\frac{(-w(\mathrm{k}_1x+\mathrm{k}_2y+\mathrm{k}_3z)\ln(w)+mxy)}{a_2z^2}-\frac{\mathrm{k}_4w^2+mxy-\mathrm{k}_4w^2)}{a_2z^2}G_1w^{m-1}+G_2w^m,$$

where G_2 is an arbitrary function in the variables $x, y\ and\ z$ of degree $n-m-1$.

We determine that $\mathrm{k}_1=\mathrm{k}_2=\mathrm{k}_3=0$ because f_{n-1} must be a polynomial. Therefore

$$f_{n-1}=\frac{(mxy-\mathrm{k}_4w^2)}{a_2z^2}G_1w^{m-1}+G_2w^m.$$

This indicates that in order to prove the proposition, we just need to show that $k_0=k_4=k_{12}=0$. The demonstration of this claim will make use of the following techniques and concepts, [12].

A polynomial $f(x)$ with $x\in\mathbb{R}^4$ is said to be the weight-homogeneous polynomial if there is $s=(s_1,s_2,s_3,s_4)\in\mathbb{N}^4$ and $d\in\mathbb{N}$ such that for all non-zero $\mu\in\mathbb{R}$

$$f(\mu^s x)=f(\mu^{s_1}x,\mu^{s_2}y,\mu^{s_3}z,\mu^{s_4}w)=\mu^d f(x,y,z,w).$$

We should use the terms s for the weight of $f(x)$, d for the degree of weight, and $x\to\mu^d x$ for the change in weight of the variables.

We outline the basic procedure of the proof for a general system in order to make the computation easier to understand and make the proof easier to follow. The following four-dimensional polynomial differential system is then taken into consideration:

$$\dot{x}=f(x),\ x\in\mathbb{R}^4. \tag{16}$$

1. We first select a suitable weight change of the variables

$$X=\mu^s\,x=(\mu^{s_1}x_1,\mu^{s_2}x_2,\mu^{s_3}x_3,\mu^{s_4}x_4)$$

and rescale t by $T=\mu^{-s_0}t$, where $s_i\in\mathbb{N}$ for $i=0,\cdots,4$ such that system (16) can be transformed into

$$\mathbf{X}'=\sum_{j=0}^{r}\mu_{d-j}^{j}f(\mathbf{X}),\quad r\le d, \tag{17}$$

where $'$ is the derivative with respect to the variable T, and $f_{d-j}(\mathbf{X})$ is a polynomial vector, so that each component is a weighted-homogeneous polynomial.

2. Suppose that $H(\mathbf{x})$ is a invariant algebraic surface of system (16) with a cofactor $K_H(\mathbf{x})$. Set

$$\widetilde{H}(\mathbf{X})=\mu^l G(\mu^{-\mathbf{s}}\mathbf{X})=\sum_{i=0}^{p}\mu^i\widetilde{H}_i(\mathbf{X}),\quad \widetilde{K}(\mathbf{X})=\mu^h K_H(\mu^{-\mathbf{s}}\mathbf{X})=\sum_{j=0}^{q}\mu^j\widetilde{K}_j(\mathbf{X}),$$

Where $\mu^{-\mathbf{s}}\mathbf{X}=(\mu^{-s_1}X_1,\mu^{-s_2}X_2,\mu^{-s_3}X_3), p,q\in\mathbb{N}, l$ and h are the highest weight degrees of the weight-homogeneous components for H and K_H, respectively, and $\widetilde{H}_i$ and K_j are weight-homogeneous polynomials. Then $\widetilde{H}(\mathbf{X})$ is a Darboux invariant polynomial of system (16) with the cofactor as $\mu^{s_0-h}\widetilde{K}(\mathbf{X})$. From (3), we have

$$\frac{\partial\sum_{i=0}^{p}\mu^i\widetilde{H}_i}{\partial\mathbf{X}}\sum_{j=0}^{r}\mu^j f_{d-j}(\mathbf{X})=\mu^{s_0-h}\sum_{j=0}^{q}\mu^j\widetilde{K}_{H_j}\sum_{i=0}^{p}\mu^i\widetilde{H}_i, \tag{18}$$

and by equating the term's coefficients with the same power of μ^i in (18) (same weight degree), we obtain $i=0,1,\dots,\max\{r+p,s_0-h+q+p\}$ linear partial differential equations of the function $\widetilde{H}_i(\mathbf{X})$. The weighted-homogeneous polynomial solution $\widetilde{H}_i$ is then obtained by solving each linear partial differential equation.

3. Here, we solve a single PDE using Maple's pdsolve program.

First provide (18) with $i=0$. A general solution H_0^* of (18) is obtained with $i=0$ by solving the linear partial differential equations. Generally, H_0^* is not a weight-homogeneous polynomial with the given degree. As a result, we will set the components whose non-weight-homogeneous polynomials are of the equal degree to be zero, thus the resulting conditions and $\widetilde{H}_0$ are obtained. Second, we introduce $\widetilde{H}_0$ into equation (18) with $i=1$ and we work in a similar way to solve $\widetilde{H}_0$. Then $\widetilde{H}_1$ can be acquired with some requirements that must be satisfied. We then continue by introducing $\widetilde{H}_0$ and $\widetilde{H}_1$ into (18) with $i=2$, we obtain $\widetilde{H}_2$. This technique yields all of the weight-homogeneous polynomial solutions $\widetilde{H}_i$ for (18) as well as the requirements on the system coefficients.

4. From $H(\mathbf{x})=\widetilde{H}(\mathbf{X})|_{\mu=1}=\sum_{i=1}^{p}\widetilde{H}_i$, all the Darboux invariant polynomials for system (16) can be found.

We will now back to the proof of this proposition. We use the variable change of the coordinates

$$x=\mu^{-1}X,\quad y=Y,\quad z=Z,\quad w=W,\quad t=\mu\tau,$$

then, system (1) becomes

$$\begin{aligned}\dot{X}&=\mu^2 Y,\\ \dot{Y}&=\mu Z,\\ \dot{Z}&=\mu W,\\ \dot{W}&=-(Y+Z+(a_2Z^2+a_1)W)\mu-YX,\end{aligned}$$

where the dot denotes derivative with respect to the variable τ.
Set $F(X,Y,Z,W)=\mu^n f(\mu^{-1}X,Y,Z,W)$
And
$\mathrm{K}(X,Y,Z,W)=\mathrm{k}(\mu^{-1}X,Y,Z,W)=\mathrm{k}_{12}Z^2+\mathrm{k}_4W+\mathrm{k}_0$,
where as we considered above n is the highest weight degree in the weight homogeneous components of f in the variables x,y,z and w with weight degrees (1,0,0,0). Assume that

$$F=\sum_{i=0}^{n}\mu^i F_i(X,Y,Z,W),$$

where F_i is a weight homogeneous polynomial in X,Y,Z and W with weight degree $n-i$ for $i=0,1,\dots,n$. Note that in particular

$$F_j(X,Y,Z,W)=\mu^n f_{n-j}(x,y,z,w),$$

for $j=0,\dots,n$. From the definition of Darboux polynomial we have that

$$\begin{aligned}&\mu^2Y\sum_{i=0}^{n}\mu^i\frac{\partial F_i}{\partial X}+\mu Z\sum_{i=0}^{n}\mu^i\frac{\partial F_i}{\partial Y}\\ &+\mu W\textstyle\sum_{i=0}^{n}\mu^i\frac{\partial F_i}{\partial Z}-\big((Y+Z+(a_2Z+a_1)W\big)\mu \quad (19)\\ &+XY)\sum_{i=0}^{n}\mu^i\frac{\partial F_i}{\partial W}=(\mathrm{k}_{12}Z^2+\mathrm{k}_4W+\mathrm{k}_0)\sum_{i=0}^{n}\mu^iF_i,\end{aligned}$$

we compute the coefficient of μ^0 in equation (19), to obtain

$$YX\frac{\partial F_0(X,Y,Z,W)}{\partial W}=-(\mathrm{k}_{12}Z^2+\mathrm{k}_4W+\mathrm{k}_0)F_0(X,Y,Z,W).$$

This gives

$$F_0(X,Y,Z,W)=G_3(X,Y,Z)\mathrm{e}^{\frac{(2\mathrm{k}_{12}Z^2+\mathrm{k}_4W+2\mathrm{k}_0)W}{2YX}},$$

where $G_3(X,Y,Z)$ is an arbitrary polynomial function in the variables X,Y and Z. Since $F_0(X,Y,Z,W)$ is a weight homogeneity, it follows that $\mathrm{k}_{12}=\mathrm{k}_4=\mathrm{k}_0=0$. This gives that system (1) has no non-trivial Darboux invariant polynomials. ■

Proposition 3. *The exponential factors of system (1) is* $e^{\frac{a_2z^3}{3}+x^2+yx+zx+y^2+zy+z^2+2x+2y+za_1+z+w}$ *and* e^{H^n}*, with the cofactor* $L=wx+wy+2wz+xy+xz+y^2+3yz+z^2+w+y+z$ *and zero cofactor, respectively, where*

$$H(x,y,z,w)=x+y+a_1z+w+\tfrac{1}{2}x^2+\tfrac{1}{3}a_2z^3$$

is the minimal polynomial first integral.

Proof. Let $E=\exp(g/h)$ be an exponential factor of the hyper jerk system (1) with cofactor L of the degree two, where $g,h,L\in\mathbb{C}[x,y,z,w]$ with $\gcd(g,h)=1$. Then, we consider two cases.

Case 1. If h is a constant, we can take $h=1$. Thus $E=\exp(g)$, where $g\in\mathbb{C}[x,y,z,w]$. Then, g satisfies

$$\begin{aligned}&y\frac{\partial e^g}{\partial x}+z\frac{\partial e^g}{\partial y}+w\frac{\partial e^g}{\partial z}-((1+x)y+a_1w\\ &+a_2z^2w+z)\frac{\partial e^g}{\partial w}=L(x,y,z,w)e^g. \qquad (20)\end{aligned}$$

Simplify equation (20), we obtain

$$\begin{aligned}&y\frac{\partial g}{\partial x}+z\frac{\partial g}{\partial y}+w\frac{\partial g}{\partial z}-((1+x)y+a_1w\\ &+a_2z^2w+z)\frac{\partial g}{\partial w}=L(x,y,z,w), \qquad (21)\end{aligned}$$

where

$$\begin{aligned}L=\;&c_{2,0,0,0}x^2+c_{1,1,0,0}xy+c_{1,0,1,0}xz+c_{1,0,0,1}xw\\ &+c_{0,2,0,0}y^2+c_{0,1,1,0}yz+c_{0,1,0,1}yw+c_{0,0,2,0}z^2\\ &+c_{0,0,1,1}zw+c_{0,0,0,2}w^2+c_{1,0,0,0}x+c_{0,1,0,0}y\\ &+c_{0,0,1,0}z+c_{0,0,0,1}w+c_{0,0,0,0}.\end{aligned}$$

We can write g as

$$g(x,y,z,w)=\sum_{j=1}^{n}g_j(x,y,z,w),$$

where each $g_j(x,y,z,w)$ is a polynomial of its variables. We first assume $n>3$. To proceed, we initiate the computation of the terms with degree $n+2$ in equation (21). By doing so, we acquire

$$-a_2z^2w\frac{\partial}{\partial w}g_n(x,y,z,w)=0.$$

That is

$$g_n(x,y,z,w) = g_n(x,y,z). \tag{22}$$

Additionally, we perform computations for the terms with degree $n+1$ in equation (21). This computation yields the following results

$$-xy\frac{\partial}{\partial w}g_n(x,y,z) - a_2z^2w\frac{\partial}{\partial w}$$
$$g_{n-1}(x,y,z,w) = 0.$$

That is

$$g_{n-1}(x,y,z,w) = g_{n-1}(x,y,z).$$

Furthermore, by calculating the terms with degree n in equation (21), we obtain the following results

$$(y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})g_n(x,y,z)$$
$$-(y + a_1w + z)\frac{\partial}{\partial w}g_n(x,y,z)$$
$$-xy\frac{\partial}{\partial w}g_{n-1}(x,y,z)$$
$$-a_2z^2w\frac{\partial}{\partial w}g_{n-2}(x,y,z,w) = 0.$$

Simplify, we have

$$\left(y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)g_n(x,y,z)$$
$$-a_2z^2w\frac{\partial}{\partial w}g_{n-2}(x,y,z,w) = 0,$$

By solving this differential equation, we arrive at the following solution

$$g_{n-2}(x,y,z,w) = \frac{\ln(w)(\frac{\partial}{\partial x}g_n(x,y,z))y}{a_2\,z^2}$$
$$+\frac{\ln(w)(\frac{\partial}{\partial y}g_n(x,y,z))}{a_2\,z}$$
$$+\frac{w(\frac{\partial}{\partial z}g_n(x,y,z))}{a_2\,z^2} + K_1(x,y,z), \tag{23}$$

that is

$$g_n(x,y,z) = g_n(z).$$

where K_1 is a polynomial function. We now introduce a change of variables known as the weight transformation

$$x = \mu^{-1}X,$$
$$y = Y,$$
$$z = Z,$$
$$w = \mu^{-1}W, \qquad t = \mu\tau, \text{ with } \mu \in \mathbb{C}\backslash\{0\}.$$

Then the four dimensions Hyper-Jerk system becomes

$$\dot{X} = \mu^2Y,$$
$$\dot{Y} = \mu Z,$$
$$\dot{Z} = W,$$
$$\dot{W} = -\mu((Y+Z)\mu + (Z^2a_2 + a_1)W + YX).$$

Set

$$G(X,Y,Z,W) = \mu^n g(\mu^{-1}X,Y,Z,\mu^{-1}W)$$
$$= \sum_{i=0}^{n}\mu^iG_i(X,Y,Z,W),$$

where G_i is the weight homogeneous part with weight degree $n-i$ of G and n is the weight degree of G with weight exponent $s = (1,0,0,1)$, then from definition of exponential factor we have From the coefficient of the term μ^0 in equation (24), we have

$$\mu^2Y\sum_{i=0}^{n}\mu^i\frac{\partial G_i}{\partial X} + \mu Z\sum_{i=0}^{n}\mu^i\frac{\partial G_i}{\partial Y} + W\sum_{i=0}^{n}\mu^i\frac{\partial G_i}{\partial Z}$$
$$-\mu((Y+Z)\mu + (Z^2a_2 + a_1)W + YX)$$
$$\sum_{i=0}^{n}\mu^i\frac{\partial G_i}{\partial W} = L(X,Y,Z,W). \tag{24}$$

From the coefficient of the term μ^0 in equation (24), we have

$$((\frac{\partial}{\partial Z}G_0(X,Y,Z,W))W = c_{0,2,0,0}Y^2 + (c_{0,1,1,0}Z$$
$$+c_{0,1,0,0})Y + c_{0,0,2,0}Z^2$$
$$+c_{0,0,1,0}Z + c_{0,0,0,0}.$$

Solving differential equation we get

$$G_0(X,Y,Z,W) = \frac{c_{0,2,0,0}Y^2Z + \frac{1}{2}c_{0,1,1,0}YZ^2}{W}$$
$$+\frac{\frac{1}{3}c_{0,0,2,0}Z^3 + c_{0,1,0,0}YZ + \frac{1}{2}c_{0,0,1,0}Z^2 + c_{0,0,0,0}Z}{W} + K_2(X,Y,W),$$

where $K_2(X,Y,Z)$ is an arbitrary polynomial function in the variables X,Y and Z. Since, $G_0(X,Y,Z,W)$ is a weight homogeneity, it follows that

$$c_{0,2,0,0} = c_{0,1,1,0} = c_{0,0,2,0} = c_{0,1,0,0} = c_{0,0,1,0} = c_{0,0,0,0} = 0,$$

then we have

$$G_0(X,Y,Z,W) = K_2(X,Y,W). \tag{25}$$

From equations (22) and (25), we must have $g_n(x,y,z,w) = g_n(x,y)$. Then, equation (23) becomes

$$g_{n-2}(x,y,z,w) = \frac{\ln(w)(\frac{\partial}{\partial x}g_n(x,y))y}{a_2z^2}$$

$$+\frac{\ln(w)(\frac{\partial}{\partial y}g_n(x,y))}{a_2 z}+K_1(x,y,z).$$

Since $g_{n-2}(x,y,z,w)$ is a homogeneous polynomial of degree $n-2$, then must be

$$\frac{\partial}{\partial x}g_n(x,y)=0, \quad \text{and} \quad \frac{\partial}{\partial y}g_n(x,y)=0,$$

which implies that $g_n(x,y,z,w)$ is a constant for $n>3$, and this is contradiction to our assumption. Thus $g_n(x,y,z,w)$ must be a polynomial of degree at most three satisfying equation (21). Put

$$\begin{aligned}
g(x,y,z,w) &= b_{3,0,0,0}x^3+b_{2,1,0,0}x^2y\\
&+b_{2,0,1,0}x^2z+b_{2,0,0,1}x^2w+b_{1,2,0,0}xy^2\\
&+b_{1,1,1,0}xyz+b_{1,1,0,1}xyw+b_{1,0,2,0}xz^2\\
&+b_{1,0,1,1}xzw+b_{1,0,0,2}xw^2+b_{0,3,0,0}y^3\\
&+b_{0,2,1,0}y^2z+b_{0,2,0,1}y^2w+b_{0,1,2,0}yz^2\\
&+b_{0,1,1,1}yzw+b_{0,1,0,2}yw^2+b_{0,0,3,0}z^3\\
&+b_{0,0,2,1}z^2w+b_{0,0,1,2}zw^2+b_{0,0,0,3}w^3\\
&+b_{2,0,0,0}x^2+b_{1,1,0,0}xy+b_{1,0,1,0}xz\\
&+b_{1,0,0,1}xw+b_{0,2,0,0}y^2+b_{0,1,1,0}yz\\
&+b_{0,1,0,1}yw+b_{0,0,2,0}z^2+b_{0,0,1,1}zw\\
&+b_{0,0,0,2}w^2+b_{1,0,0,0}x+b_{0,1,0,0}y\\
&+b_{0,0,1,0}z+b_{0,0,0,1}w,
\end{aligned}$$

where $b_{i,j,k,l}$, $i,j,k,l=0,\ldots,3$ are constants. From (21), we have

$$\begin{aligned}
&-a_2b_{2,0,0,1}x^2z^2w-a_2b_{1,1,0,1}xyz^2w\\
&-a_2b_{1,0,1,1}xz^3w-2a_2b_{1,0,0,2}xz^2w^2\\
&-a_2b_{0,2,0,1}y^2z^2w-a_2b_{0,1,1,1}yz^3w\\
&-2a_2b_{0,1,0,2}yz^2w^2-a_2b_{0,0,2,1}z^4w\\
&-2a_2b_{0,0,1,2}z^3w^2-3a_2b_{0,0,0,3}z^2w^3\\
&-b_{2,0,0,1}x^3y-b_{1,1,0,1}x^2y^2-b_{1,0,1,1}x^2yz\\
&-2b_{1,0,0,2}x^2yw-b_{0,2,0,1}xy^3-b_{0,1,1,1}xy^2z\\
&-2b_{0,1,0,2}xy^2w-b_{0,0,2,1}xyz^2-2b_{0,0,1,2}xyzw\\
&-3b_{0,0,0,3}xyw^2-a_2b_{1,0,0,1}xz^2w\\
&-a_2b_{0,1,0,1}yz^2w-a_2b_{0,0,1,1}z^3w\\
&-2a_2b_{0,0,0,2}z^2w^2-b_{1,0,0,1}x^2y-b_{2,0,0,1}x^2y\\
&+3b_{3,0,0,0}x^2y-b_{2,0,0,1}x^2z+b_{2,1,0,0}x^2z\\
&-a_1b_{2,0,0,1}x^2w+b_{2,0,1,0}x^2w-b_{0,1,0,1}xy^2\\
&-b_{1,1,0,1}xy^2+2b_{2,1,0,0}xy^2-b_{0,0,1,1}xyz\\
&-b_{1,0,1,1}xyz-b_{1,1,0,1}xyz+2b_{2,0,1,0}xyz\\
&+2b_{1,2,0,0}xyz-a_1b_{1,1,0,1}xyw-2b_{0,0,0,2}xyw\\
&-2b_{1,0,0,2}xyw+2b_{2,0,0,1}xyw+b_{1,1,1,0}xyw\\
&-b_{1,0,1,1}xz^2+b_{1,1,1,0}xz^2-a_1b_{1,0,1,1}xzw\\
&-2b_{1,0,0,2}xzw+b_{1,1,0,1}xzw+2b_{1,0,2,0}xzw\\
&-2a_1b_{1,0,0,2}xw^2+b_{1,0,1,1}xw^2+b_{1,2,0,0}y^3\\
&-b_{0,2,0,1}y^3-b_{0,1,1,1}y^2z-b_{0,2,0,1}y^2z\\
&+b_{1,1,1,0}y^2z+3b_{0,3,0,0}y^2z-a_1b_{0,2,0,1}y^2w\\
&-2b_{0,1,0,2}y^2w+b_{1,1,0,1}y^2w+b_{0,2,1,0}y^2w\\
&-b_{0,0,2,1}yz^2-b_{0,1,1,1}yz^2+b_{1,0,2,0}yz^2\\
&+2b_{0,2,1,0}yz^2-a_1b_{0,1,1,1}yzw-2b_{0,0,1,2}yzw\\
&-2b_{0,1,0,2}yzw+b_{1,0,1,1}yzw+2b_{0,2,0,1}yzw\\
&+2b_{0,1,2,0}yzw-2a_1b_{0,1,0,2}yw^2-3b_{0,0,0,3}yw^2\\
&+b_{1,0,0,2}yw^2+b_{0,1,1,1}yw^2-b_{0,0,2,1}z^3\\
&+b_{0,1,2,0}z^3-a_1b_{0,0,2,1}z^2w-a_2b_{0,0,0,1}z^2w\\
&-2b_{0,0,1,2}z^2w+b_{0,1,1,1}z^2w+3b_{0,0,3,0}z^2w\\
&-2a_1b_{0,0,1,2}zw^2-3b_{0,0,0,3}zw^2+b_{0,1,0,2}zw^2\\
&+2b_{0,0,2,1}zw^2+b_{0,0,1,2}w^3-3a_1b_{0,0,0,3}w^3\\
&-b_{0,0,0,1}xy-b_{1,0,0,1}xy+2b_{2,0,0,0}xy-b_{1,0,0,1}xz\\
&+b_{1,1,0,0}xz-a_1b_{1,0,0,1}xw+b_{1,0,1,0}xw\\
&-b_{0,1,0,1}y^2+b_{1,1,0,0}y^2-b_{0,0,1,1}yz-b_{0,1,0,1}yz\\
&+b_{1,0,1,0}yz+2b_{0,2,0,0}yz-a_1b_{0,1,0,1}yw\\
&-2b_{0,0,0,2}yw+b_{1,0,0,1}yw+b_{0,1,1,0}yw-b_{0,0,1,1}z^2\\
&+b_{0,1,1,0}z^2-a_1b_{0,0,1,1}zw-2b_{0,0,0,2}zw\\
&+b_{0,1,0,1}zw+2b_{0,0,2,0}zw+b_{0,0,1,1}w^2\\
&-2a_1b_{0,0,0,2}w^2-b_{0,0,0,1}y+b_{1,0,0,0}y\\
&-a_1b_{0,0,0,1}w-(c_{2,0,0,0}x^2+c_{1,1,0,0}xy\\
&+c_{1,0,1,0}xz+c_{1,0,0,1}xw+c_{0,2,0,0}y^2\\
&+c_{0,1,1,0}yz+c_{0,1,0,1}yw+c_{0,0,2,0}z^2\\
&+c_{0,0,1,1}zw+c_{0,0,0,2}w^2+c_{1,0,0,0}x+c_{0,1,0,0}y\\
&+c_{0,0,1,0}z+c_{0,0,0,1}w+c_{0,0,0,0})=0.
\end{aligned}$$

We simplify with collect the equal power of variables, we obtain linear equations of coefficients. We find that

$$\begin{aligned}
g(x,y,z,w) &= \frac{2a_2b_{2,0,0,0}z^3}{3}-\frac{a_2c_{1,1,0,0}z^3}{3}\\
&+b_{2,0,0,0}x^2+c_{1,0,1,0}xy+c_{1,0,0,1}xz+b_{0,2,0,0}y^2\\
&+c_{0,1,0,1}yz+b_{0,0,2,0}z^2+2b_{2,0,0,0}x\\
&+c_{0,1,0,0}x-c_{1,1,0,0}x+2b_{2,0,0,0}y\\
&+c_{0,0,1,0}y-c_{1,1,0,0}y+2a_1b_{2,0,0,0}z\\
&-a_1c_{1,1,0,0}z+c_{0,0,0,1}z+2b_{2,0,0,0}w\\
&-c_{1,1,0,0}w,
\end{aligned}$$

with cofactor

$$\begin{aligned}
L &= c_{1,1,0,0}xy+c_{1,0,1,0}xz+c_{1,0,0,1}xw\\
&+c_{1,0,1,0}y^2+2b_{0,2,0,0}yz+c_{1,0,0,1}yz\\
&+c_{0,1,0,1}yw+c_{0,1,0,1}z^2+2b_{0,0,2,0}zw\\
&+c_{0,1,0,0}y+c_{0,0,1,0}z+c_{0,0,0,1}w,
\end{aligned}$$

we can take

$$\begin{aligned}
&c_{1,1,0,0}=c_{1,0,1,0}=c_{1,0,0,1}=c_{1,0,1,0}=b_{0,2,0,0}\\
&=c_{1,0,0,1}=c_{0,1,0,1}=c_{0,1,0,1}=b_{0,0,2,0}\\
&=c_{0,1,0,0}=c_{0,0,1,0}=c_{0,0,0,1}=1.
\end{aligned}$$

Thus

$$E = e^{g(x,y,z,w)}, \quad g = \frac{a_2 z^3}{3} + (x+y+z)^2$$
$$-(x+y-a_1-1)z - (y-2)x + w + 2y + 1, \quad (26)$$

is the exponential factor with the cofactor

$$L = wx + wy + 2wz + xy + xz + y^2$$
$$+3yz + z^2 + w + y + z.$$

Since

$$h = H(x,y,z,w)$$
$$= x + y + a_1 z + w + \frac{1}{2}x^2 + \frac{1}{3}a_2 z^3,$$

is the minimal polynomial first integral of system (1). And the polynomial first integral is a particular case of a Darboux first integral. Then e^{H^n} is also the exponential factor with zero cofactor, this gives that e^{H^n+g} is the exponential factor of system.

Case 2. If h is not constant. Let $E = e^{g/h}$ be an exponential factor of the system with $\gcd(g,h) = 1$, where $g = \sum_{i=0}^{n} g_i$, each g_i is a non-constant homogeneous polynomial of degree $i = 1,\dots,n$. Then, from the definition of the exponential factor equation (4), there exists a polynomial K_E such that equation (27) below holds, see in (Zhang, 2022)

$$h\left(y\frac{\partial g}{\partial x} + z\frac{\partial g}{\partial y} + w\frac{\partial g}{\partial z} - \left((1+x)y + a_1 w + a_2 z^2 w + z\right)\frac{\partial g}{\partial w}\right) - \mathrm{g}\left(y\frac{\partial h}{\partial x} + z\frac{\partial h}{\partial y} + w\frac{\partial h}{\partial z} - \left((1+x)y + a_1 w + a_2 z^2 w + z\right)\frac{\partial h}{\partial w}\right) = \mathrm{h}^2 K_E(x,y,z,w).$$

Since, from Proposition 2 and Theorem 1,

$$h = H = x + y + a_1 z + w + \tfrac{1}{2}x^2 + \tfrac{1}{3}a_2 z^3$$

is the invariant algebraic surface of system (1) with zero cofactor. This implies that

$$y\frac{\partial h}{\partial x} + z\frac{\partial h}{\partial y} + w\frac{\partial h}{\partial z}$$
$$-\left((1+x)y + a_1 w + a_2 z^2 w + z\right)\frac{\partial h}{\partial w} = 0.$$

Since, $gcd(g,h) = 1$, there exists a polynomial $K_E \in \mathbb{C}[x,y,z,w]$, such that

$$y\frac{\partial g}{\partial x} + z\frac{\partial g}{\partial y} + w\frac{\partial g}{\partial z} - ((1+x)y + a_1 w$$
$$+a_2 z^2 w + z)\frac{\partial g}{\partial w} = K_E\, h, \quad (27)$$

we mean h is a Darboux polynomial with zero cofactor, where h is the minimal polynomial first integral defining Theorem 1 with degree three and the degree K_E is two. By the same way as proved Case 1 that g_n is constant if $\deg(G_n) > 6$. Thus $g_n(x,y,z,w)$ must be a polynomial of degree at most six satisfying equation (27). Put

$$g(x,y,z,w) = \sum_{j=1}^{6} g_j(x,y,z,w),$$

where each $g_J = \sum_{i+j+k+l=J}^{J} b_{i,j,k,l} x^i y^j z^k w^l$ is the homogeneous polynomial of degree $J = 1,\dots,6$, from equation (27), we have

$$y\frac{\partial g}{\partial x} + z\frac{\partial g}{\partial y} + w\frac{\partial g}{\partial z} - ((1+x)y$$
$$+a_1 w + a_2 z^2 w + z)\frac{\partial g}{\partial w} = K_E\, h.$$

We simplify with collect the equal power of variables, we obtain linear equations of coefficients. We find that

$$g = 2h\tilde{g}(x,y,z,w),$$

where

$$\tilde{g}(x,y,z,w) = \frac{a_2(b_{2,0,0,1} - 2b_{4,0,0,0})z^3}{3} + b_{2,0,2,0}z^2 + (xb_{3,0,1,0} + yb_{2,1,1,0} - 2a_1 b_{4,0,0,0} + b_{2,0,1,0} - 2b_{3,0,1,0})z + b_{2,2,0,0}y^2 + (xb_{3,1,0,0} + b_{2,1,0,0} - 2b_{3,1,0,0} - 2b_{4,0,0,0})y + x^2 b_{4,0,0,0} + (b_{3,0,0,0} - 2b_{4,0,0,0})x + (b_{2,0,0,1} - 2b_{4,0,0,0})w + b_{2,0,0,0} - 2b_{3,0,0,0} + 4b_{4,0,0,0},$$

with the cofactor

$$L = 2wxb_{3,0,1,0} + 2wyb_{2,1,1,0} + 4wzb_{2,0,2,0} + (-2b_{2,0,0,1} + 8b_{4,0,0,0})xy + 2b_{3,1,0,0}xz$$
$$+2b_{3,1,0,0}y^2 + \left(2b_{3,0,1,0} + 4b_{2,2,0,0}\right)yz + 2b_{2,1,1,0}z^2 + w(-2a_1 b_{2,0,0,1} + 2b_{2,0,1,0} - 4b_{3,0,1,0}) + y(-2b_{2,0,0,1} + 2b_{3,0,0,0}) + z(-2b_{2,0,0,1} + 2b_{2,1,0,0} - 4b_{3,1,0,0}).$$

This implies that $\gcd(g,h) = h \neq 1$. This gives that there is no exponential factor of the form $E = e^{g/h}$. So, e^{H^n+g},where g is defined in the proposition, is the only exponential factor of system (1), with the same cofactor

$$L = wx + wy + 2wz + xy + xz + y^2$$
$$+3yz + z^2 + w + y + z.$$

This concludes the proof. ■

Furthermore, in the following we recall some results that we use later on to prove Theorem 2 as well as to show that the first integral $H(x,y,z,w)$ which defines in Theorem 1 is unique. We begin with an analytic differential system.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{28}$$

where $\mathbf{f}: U \subset \mathbb{R}^n \to \mathbb{R}^n$ is C^2 differentiable function with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, and the Jacobian matrix at $\mathbf{x} = \mathbf{0}$ is $D\,\mathbf{f}(\mathbf{0}) = \mathbf{M}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of M. And

$$S = \{(k_1, k_2, \ldots, k_n) \colon \sum_i^n k_i \lambda_i = 0,$$

$$\sum_i^n k_i > 0,\ k_i \in N^*, \leq i \leq n\},$$

where $N^* = N \cup \{0\}$. The dimension of the smallest linear vector space $\hat{S}$ containing S is a maximal number of analytic first integrals of the analytic system (28), more precisely we state the following theorem.

Theorem 4. *(ZHANG X,2003) Let $k \leq n$ be the dimension of $\hat{S}$. Then, an analytic differential system (28) with $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$ has at most k independent analytic first integrals.*

On the other hand, due to (Llibre and Zhang, 2001) we need the following two results concerning with the existence of analytic analytic first integrals.

Theorem 5. *Assume that $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$. If $H_1, \ldots, H_k$ are independent formal first integrals of system (28) for $0 < k < n$, and the matrix M has eigenvalues $\lambda_1, \ldots, \lambda_{n-k}$ which satisfy $\sum_{i=1}^{n-k} k_i \lambda_i \neq 0$, for any $k_1, \ldots, k_{n-k} \in N^*$ with*

$$\sum_{i=1}^{n-k} k_i \geq 1 \tag{29}$$

Then, any other first integrals of system (28) in a neighborhood at the origin are formal series in $H_1, \ldots, H_k$.

Theorem 6.
*Assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of M satisfy $\lambda_1 = 0$ and $\sum_{i=2}^{n} k_i \lambda_i \neq 0$,
for any $k_2, \ldots, k_n \in N^*$ with $\sum_{i=2}^{n} k_i \geq 1$.
Then, for $n > 2$, system (27) has a formal first integral in its neighborhood of origin if and only if the origin is not an isolated unique point. In particular, if the isolated singular point is the origin, then system (27) has no analytic first integrals around the origin.*

We also need the following facts before we prove Theorem 2: Since system (1) has the line of singularities $E_x(x,0,0,0)$. The eigenvalues at E_x are

$$\lambda_1 = 0,$$

$$\lambda_2 = \frac{1}{6}\left(s_1 + 12\sqrt{s}\right)^{\frac{1}{3}} - \frac{2}{3}\frac{3 - a_1^2}{\left(s_1 + 12\sqrt{s}\right)^{\frac{1}{3}}} - \frac{a_1}{3},$$

$$\lambda_{3,4} = \frac{-1}{12}\left(s_1 + 12\sqrt{s}\right)^{\frac{1}{3}} - \frac{1}{3}\frac{3 - a_1^2}{\left(s_1 + 12\sqrt{s}\right)^{\frac{1}{3}}} - \frac{a_1}{3},$$

$$\pm i\frac{\sqrt{3}}{12}\left(\frac{\left(s_1 + 12\sqrt{s}\right)^{\frac{2}{3}} - 4\left(a_1^2 - 3\right)}{\left(s_1 + 12\sqrt{s}\right)^{\frac{1}{3}}}\right)$$

Where

$$s = 81x^2 + (12a_1^3 - 54a_1 + 162)x$$
$$+ 12a_1^3 - 3a_1^2 - 54a_1 + 93,$$

and

$$s_1 = 36a_1 - 8a_1^3 - 108x - 108.$$

For $x = -\frac{2a_1^3}{27} + \frac{a_1}{3} - 1$, at the singular point E_x, we obtain

$$\lambda_2^2 = \frac{a_1^2}{9},\ \lambda_2\lambda_3 = \frac{a_1}{9}\frac{a_1\sqrt{a_1^2 - 3} + \sqrt{3}\left(a_1^2 - 3\right)}{\sqrt{a_1^2 - 3}},$$

and

$$\lambda_2\lambda_4 = \frac{a_1}{9}\frac{a_1\sqrt{a_1^2 - 3} - \sqrt{3}\left(a_1^2 - 3\right)}{\sqrt{a_1^2 - 3}}.$$

We now consider two cases,

Case1: if $a_1 \neq \pm\sqrt{3}$. Therefore

$$k_2\lambda_2 + k_3\lambda_3 + k_4\lambda_4 = \frac{1}{\lambda_2}(k_2\lambda_2^2$$
$$+ k_3\lambda_2\lambda_3\ k_4\lambda_4\lambda_2)$$

$$= \frac{1}{\lambda_2}\frac{a_1}{9}\frac{1}{\sqrt{a_1^2 - 3}}\left((k_2 + k_3 + k_4)a_1\sqrt{a_1^2 - 3}\right.$$
$$\left.\sqrt{a_1^2 - 3} + \sqrt{3}\left(a_1^2 - 3\right)(k_3 - k_4)\right) = 0. \quad (29)$$

For all $k_2, k_3, k_4 \in N$. Then, to solve equation (29) is equivalent to solve the equations

$k_3 - k_4 = 0$ and $k_2 + k_3 + k_4 = 0$.

This implies that $k_3 = k_4$ and $\frac{k_2}{k_3} = -2$.

Therefore, equation (29) has no solution for $k_2, k_3, k_4 \in N^*$ with $k_2 + k_3 + k_4 \geq 1$ such that $a_1 \neq \sqrt{3}$.

Now, if the singular point E_x where $x = -\frac{2\,a_1^3}{27} + \frac{a_1}{3} - 1$ translates into the origin of the coordinates, we are in the assumption of Theorems 7 and 8 with n = 4, k = 1 and

$$H_1 = H(x, y, z, w) = x + y + a_1 z + w,$$
$$+\frac{1}{2}x^2 + \frac{1}{3}a_2 z^3$$

defined in Theorem 1. Hence, any other first integrals of system (1) in a neighborhood of E_x are formal series in the variable H_1. As a result, all analytic first integrals in a neighborhood of E_x are functions of the variable H_1.

Case2: if $a_1 = \mp\sqrt{3}$, we only consider $a_1 = \sqrt{3}$, the prove whre $a_1 = -\sqrt{3}$ is similar. Then, eigenvalues of system (1) at $E_x(x, 0,0,0)$ become

$\lambda_1 = 0, \lambda_2 = \frac{1}{3}\,(s_2)^{\frac{1}{3}} - \frac{1}{\sqrt{3}}$,

$\lambda_{3,4} = \frac{-1}{6}\,(s_2)^{\frac{1}{3}} - \frac{1}{\sqrt{3}} \mp i\frac{\sqrt{3}}{6}\,(s_2)^{\frac{1}{3}}$,

where $s_2 = 3\sqrt{3} - 27\,x - 27$.

For $x_{a_1=\sqrt{3}} = -\frac{2\,a_1^3}{27} + \frac{a_1}{3} - 1 = \frac{\sqrt{3}}{9} - 1$, at E_x, we obtain $\lambda_2 = \lambda_3 = \lambda_3 = -\frac{1}{\sqrt{3}}$. Therefore

$k_2\,\lambda_2 + k_3\lambda_3 + k_4\lambda_4 = -\frac{1}{\sqrt{3}}.(k_2 + k_3 + k_4) = 0$.

For all $k_2, k_3, k_4 \in N^*$. Then, to solve the above equation is equivalent to solve the equations

$k_2 + k_3 + k_4 = 0$. Therefore, the above equation has no solution for $k_2, k_3, k_4 \in N^*$ with $k_2 + k_3 + k_4 \geq 1$ such that $a_1 = \sqrt{3}$.

Now, if the singular point E_x where $x = \frac{\sqrt{3}}{9} - 1$ translates into the origin of the coordinates, we are in the assumption of Theorems 7 and 8 with n = 4, k = 1 and

$$H_1 = H(x, y, z, w) = x + y + a_1 z$$
$$+w + \frac{1}{2}x^2 + \frac{1}{3}a_2 z^3,$$

defined in Theorem 1 for $a_1 = \mp\sqrt{3}$. Hence, any other first integrals of system (1) in a neighborhood of E_x are formal series in the variable H. As a result, all analytic first integrals in a neighborhood of E_x are functions of the variable H.

Now we turn to prove Theorem 2 using two Propositions 2,3 and the above explanation .

Proof. (Proof of Theorem 2:) It follows from and Theorem 3 that system (1) has a first integral of Darboux type if and only if there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that equation (14) is satisfied proposition 2 there is no invariant algebraic surfaces. And, from Proposition 3, we have only one exponential factor

$e^g = e^{\frac{a_2 z^3}{3} + (x+y+z)^2 - (x+y-a_1-1)z - (y-2)x + w + 2y + 1}$.

Using Theorem 3 equation (14), we have $\lambda_i \mathrm{K_i} + \mu_j L_j = 0$. This gives that $\mu_j L_j = 0$, since L_j not equal to zero, then $\mu_j = 0$. This implies that for any non zero λ in R, (say $\lambda = 1$) $H(x, y, z, w) = x + y + a_1 z + w + \frac{1}{2}x^2 + \frac{1}{3}a_2 z^3$ is the only Darboux first integral which is the minimal polynomial first integral, the proof of the Theorem follows. ■

3. CONCLUSIONS

In this work, a novel four-dimensional dynamical memristive system has considered, originally this system suggested in (Prousalis et al., 2017). They addressed Lyapunov exponents, phase pictures, and bifurcation diagrams via standard methods of dynamical system theory. Here, we found that the system has the minimal polynomial first integral, we also proved that the system has no a first integral in Darboux type.

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