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Properties of [0,1] Truncated Inverse Weibull Inverse Exponential distribution with Real Life Data Application

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Abstract: We propose a new distribution called [0,1] Truncated inverse Weibull inverse exponential distribution with three parameters, which extends the inverse exponential distribution. We derive some mathematical properties of the proposed model including expansion of PDF and the quantile function, moments, generating function, survival, hazard functions entropies. The method of maximum likelihood is used to estimate the parameters of the distribution. We illustrate its potentiality with applications to a real data set which show that the new model provides a better fit than other models considered.

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خصائص توزيع معكوس ويبل معكوس الأسي المبتور [0,1] مع تطبيق بيانات حقيقية

منذر عبدالله خليل انوار ابراهيم امبيرد اية قحطان نافل
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المستخلص

قدمنا توزيعا جديدا بثلاث معلمات يسمى توزيع معكوس ويبل معكوس الأسي المبتور [0,1] والذي يعد توسيعا لتوزيع المعكوس الأسي. العديد من الخواص الإحصائية للتوزيع الجديد مثل توسيع دالة الكثافة و الدالة الكمية و العزوم و الدالة المولدة للعزوم ودالتي الخطر و البقاء و الانتروبيا تم ايجادها. استخدمت طريقة الامكان الاعظم في تقدير معلمات التوزيع الجديد. ولغرض بيان ملاءمة التوزيع تم تطبيق التوزيع الجديد على مجموعة من البيانات الحقيقية والتي أظهرت ملاءمة التوزيع الجديد مقارنة مع بعض التوزيعات الأخرى.

الكلمات المفتاحية: توزيع معكوس الاسي، البتر، العزوم، الانتروبيا، التوسيع.

1. Introduction

New statistical model development is a hotly debated study area in the field of distribution theory. There are several such distributions that are particularly valuable for modeling and predicting real-world phenomena in the literature. For modeling data in a variety of practical fields during the past few decades, including bio-medical analysis, reliability engineering, economics, forecasting, astronomy, demography, and insurance, a number of classical distributions have been extensively used.

Generating new distributions by starting with a baseline distribution and adding one or more extra parameters is one of the most popular research topics in the field of distribution. A generalized distribution may be significant because it has intriguing relationships with other special distributions (via transformations, limits, conditioning, etc.). In some cases, a parametric family may be important because it can be used to model a wide variety of random phenomena. In many cases, a special parametric family of distributions will have one or more distinguished standard members, corresponding to specified values of some of the parameters. Usually, the standard distributions will be mathematically simpler, and often other members of the family can be constructed from the standard distributions by simple transformations on the underlying standard random variable. An incredible variety of special distributions have been studied over the years, and new ones are constantly being added to the literature. Some recent families of distributions are generalized like the new generalized Burr-G

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family of distributions by Nasir et al., (2017), extended Weibull G family by (Korkmaz, 2018), a new uniform distribution with different shaped failure rate by Al Abbasi et al., (2019). Furthermore, the marshall Olkin Topp Leone family by Khaleel et al., (2020). The new distribution (Topp Leone Marshall Olkin-Weibull) by Ahmed et al., (2020), the Marshall Olkin Marshall Olkin family by Al-Noor et al., (2021) and the shifted exponential-G family of distributions by Eghwerido et al., (2022). A new method and technique were investigated and used in 2017 for finding a new family by using [0,1] Truncated. [0,1] Truncated distributions are commonly used in statistical modeling and data analysis due to their ability to model probabilities and proportions that are typically bounded between 0 and 1. These distributions include well-known examples such as the beta distribution, truncated normal distribution, and uniform distribution. The [0,1] truncated family of distributions is a versatile class of probability distributions that has found wide applications in many areas of science and engineering. The rich variety of distributions in this family, each with its unique set of parameters and characteristics, makes it a valuable tool in modeling and analyzing bounded data. Some researchers used this technique to find new family like: (Khalaf and Khaleel, 2022), (Faris and Khaleel, 2022).]. Al Habib et al., (2023) introduced a new family of distributions named [0,1] Truncated Nadarajah-Haghighi family of distributions. The [0,1] Truncated inverse weibull family was very flexibility for modeling and it founded by Khaleel et al., (2022). We will use Khaleel family to find and study new distribution by using the base line inverse exponential distribution.

The inverse exponential distribution is a probability distribution that describes the time it takes for a process to reach a certain level. It is often used in reliability analysis and survival analysis, where the goal is to model the time until a failure or event occurs. The distribution has many applications in engineering, finance, and healthcare, where it can be used to estimate the probability of failure or the lifetime of a product or system. The inverse exponential distribution, also known as the Pareto Type II distribution or the Lomax distribution, has a rich history in probability theory and statistics. Many researchers used inverse exponential to find new distributions such as Oguntunde et al., (2018a), Oguntunde et al., (2018b), (Aldahlan, 2019). Godfrey et al., (2020), and many others.

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The paper can be sorted as follows. In Section 2, the pdf, hazard rate function and reliability function of the [0,1] Truncated inverse Weibull inverse exponential ([0,1] TIWIE) distribution are defined. Some statistical properties of the [0,1] TIWIE are derived in Section 3. Estimator based on maximum likelihood (MLE) is provided in Section 4. The proposed model is shown to give a better fit for real data sets as explained in Section 5. Some concluding remarks are given in Section 6.

2. [0,1] TIW Inverse exponential distribution

In this paper, we will employment the [0,1] Truncated Inverse Weibull-G family of distributions which was introduced by Khaleel et al., (2022) where the random variable has [0,1] TIW-G family with the following CDF and PDF respectively:

$$K_{[0,1]TIW-G} = \frac{e^{-\alpha(H(x,\xi))^{-\beta}}}{e^{-\alpha}}, \quad 0 < x < 1 \quad ,\alpha > 0 \ ,\beta > 0 \tag{1}$$

$$k_{[0,1]TIW-G} = \frac{\alpha\beta\, h(x,\xi)\, e^{-\alpha\,[H(x,\xi)]^{-\beta}}\,[H(x,\xi)]^{-\beta-1}}{e^{-\alpha}} \tag{2}$$

Where $H(x,\xi), h(x,\xi)$ are the CDF and PDF for the baseline distribution with vector of parameter ξ.

The inverse exponential distribution we have the following cumulative density function and probability density function respectively:

$$H(x,d) = e^{\frac{-d}{x}} \quad x > 0\,, d > 0 \tag{3}$$

$$h(x,d) = \frac{d}{x^2} e^{\frac{-d}{x}} \tag{4}$$

The equation being referred to is the result of inserting equation (3) for the inverse exponential CDF into equation (1) for the [0,1] Truncated Inverse Weibull-G family of distributions CDF. When we do this, we obtain a new CDF that characterizes the specific distribution we're interested in modeling. To break it down a bit further, the resulting equation can be written as:

$$K_{[0,1]TIWIE} = \frac{e^{-\alpha\left(e^{\frac{-d}{x}}\right)^{-\beta}}}{e^{-\alpha}}, \quad 0 < x < 1 \quad ,\alpha > 0 \ ,\beta > 0 \tag{5}$$

where K([0,1]TIWIE) is the CDF for the new distribution we're modeling, and α and β are the parameters from the [0,1] Truncated Inverse

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Weibull-G family of distributions. By derived Eq(5) we have the pdf of [0,1] Truncated Inverse Weibull Inverse Exponential distribution as follows

$$k_{[0,1]TIWIE} = \frac{\alpha\beta \, \frac{d}{x^2} \, e^{\frac{-d}{x}} \, e^{-\alpha\left[e^{\frac{-d}{x}}\right]^{-\beta}} \left[e^{\frac{-d}{x}}\right]^{-\beta-1}}{e^{-\alpha}} \quad x > 0, \alpha, \beta, d > 0$$

The key difference is that the CDF now includes the inverse exponential distribution with parameter d, which affects the shape of the distribution. The parameter d determines the rate at which the distribution decays to zero as x approaches infinity. It can be written as

$$k_{[0,1]TIWIE} = \frac{\alpha\beta \, \frac{d}{x^2} \, e^{-\alpha\left[e^{\frac{-d}{x}}\right]^{-\beta}} \left[e^{\frac{-d}{x}}\right]^{-\beta}}{e^{-\alpha}} \qquad (6)$$

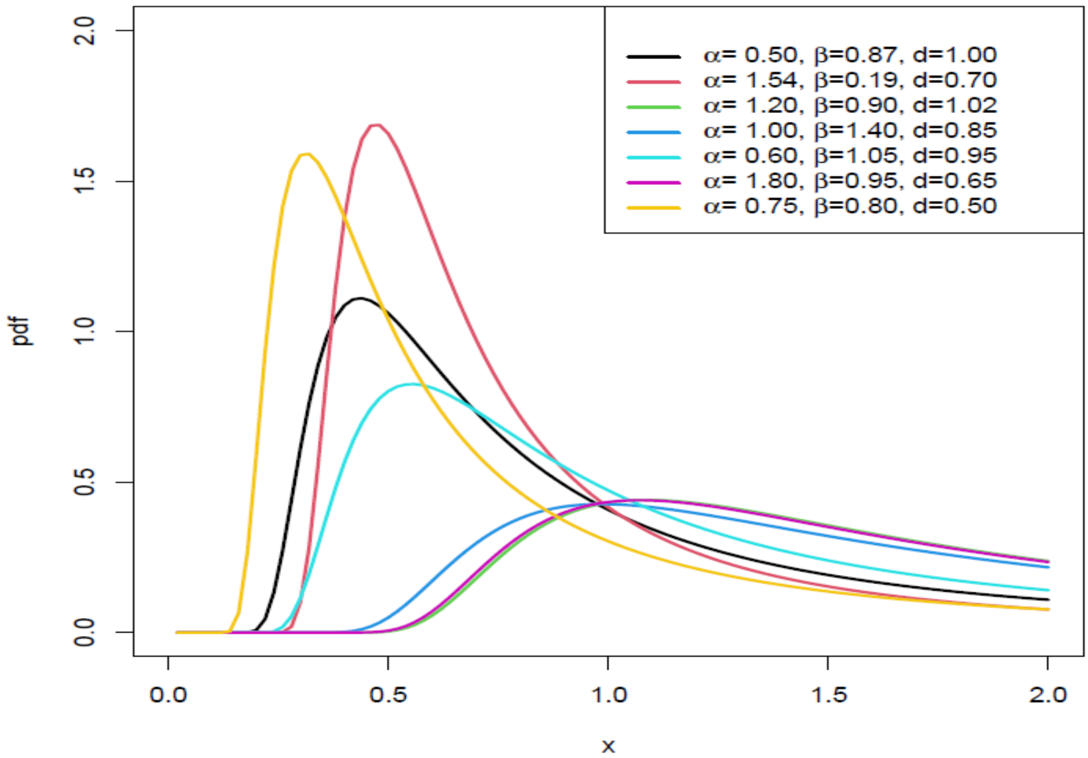

Figure (1): PDF of the new distribution

The Fig (1) can see different shapes like unimodel, right skewed, and platform shapes. The equations (5), (6) are very important to find some essential properties of [0,1] Truncated Inverse Weibull Inverse Exponential distribution ([0,1] TIWIE). The survival and hazard functions can be found from Eqs (5) and (6). The survival and hazard functions for the new distribution are given respectively by the following form:

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$$S_{[0,1]TIWIE}(x) = \frac{k_{[0,1]TIWIE}}{1-K_{[0,1]TIWIE}} = \frac{e^{-\alpha} - e^{-\alpha\left(e^{\frac{-d}{x}}\right)^{-\beta}}}{e^{-\alpha}}$$

$$h_{[0,1]TIWIE}(x) = 1 - K_{[0,1]TIWIE} = \frac{\alpha\beta \frac{d}{x^2} e^{\frac{-d}{x}} e^{-\alpha\left[e^{\frac{-d}{x}}\right]^{-\beta}} \left[e^{\frac{-d}{x}}\right]^{-\beta-1}}{e^{-\alpha} - e^{-\alpha\left(e^{\frac{-d}{x}}\right)^{-\beta}}}$$

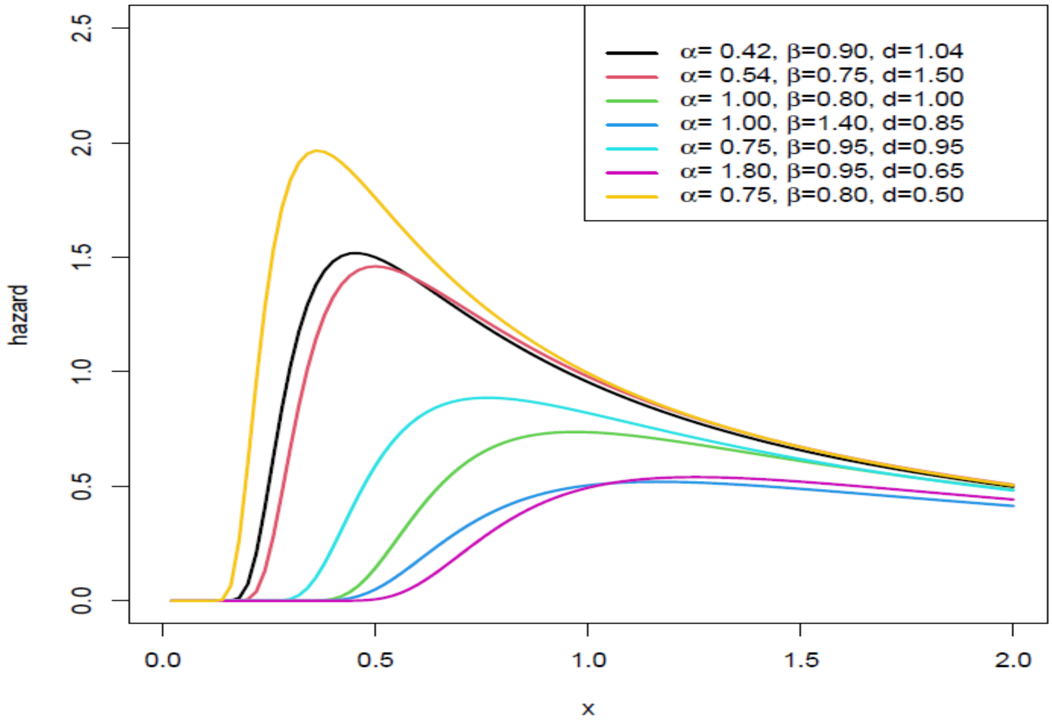

Figure (2): Hazard function for the new distribution with select parameters

Fig (2) has only unimodel shape with different scale.

3. Some statistical properties for [0,1] TIWIE distribution:

3.1 Expansion PDF for [0,1] TIWIE:

The expansion for the PDF is very important for the properties that obtained by the integral such as moments, moment generating function. Equation (6) can be rewritten by the following form:

$$k_{[0,1]TIWIE} = \frac{\alpha\beta \frac{d}{x^2} e^{-\alpha\left[e^{\frac{-d}{x}}\right]^{-\beta}} \left[e^{\frac{-d}{x}}\right]^{-\beta}}{e^{-\alpha}}, \qquad x > 0, \alpha, \beta, d > 0$$

By use the expansion of exponential we get Abdullah et al., (2019), and Oguntunde et al., (2019)

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$$e^{-\alpha\left[e^{\frac{-d}{x}}\right]^{-\beta}} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\alpha^i e^{\frac{di\beta}{x}} \tag{7}$$

Substitute (8) in (6) we get that

$$k_{[0,1]TIWIE} = \frac{\alpha\beta\frac{d}{x^2}\left[e^{\frac{-d}{x}}\right]^{-\beta} \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\alpha^i e^{\frac{di\beta}{x}}}{e^{-\alpha}} \tag{8}$$

Let $\varphi_i = \frac{\alpha\beta \ \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\alpha^i}{e^{-\alpha}}$

Then after some algebra steps we can write equation (8) as

$$k_{[0,1]TIWIE} = \varphi_i \frac{d}{x^2}\ e^{\frac{d\beta(i+1)}{x}} \tag{9}$$

Equation (9) is very important in this section for find many properties later.

3.2 Quantile function

The quantile function is also known as the inverse cdf, as it is the inverse of the cdf. It is a useful tool in statistics and data analysis, as it allows us to calculate the value of a random variable at a certain probability level. For example, the 90th percentile of a distribution is the value that is exceeded by 90% of the observations. The quantile function is commonly used in fields such as finance, economics, and engineering to analyze and model data. The quantile function of [0,1]TIWIE can by obtained by inverted equation (5) as follows:

$$x = \frac{-d}{\ln\left\{\left(\frac{1}{-\alpha}\ln(u\ e^{-\alpha})\right)^{\frac{-1}{\beta}}\right\}} \tag{10}$$

Eq (10) very important for analyzing and understanding the distribution of data, allowing us to identify outliers, simulation study and generate the random number when $u \sim U(0,1)$. Also, understand the spread and central tendency of the data, and make predictions about future observations. It is widely used in fields such as finance, economics, and engineering for modeling and hypothesis testing. The median of [0,1]TIWIE distribution can be written as

$$Median\ (x) = \frac{-d}{\ln\left\{\left(\frac{1}{-\alpha}\ln(0.5\ e^{-\alpha})\right)^{\frac{-1}{\beta}}\right\}} \tag{11}$$

3.3 Moments

The moments of a probability distribution are statistical measures that summarize the properties of the distribution. They provide insight into the central tendency, variability, and shape of the distribution, and can be used to compare different distributions and make inferences about the underlying processes AbuJarad et al., (2020). The moments for [0,1]TIWIE distribution can be obtained as

$$E(x^t) = \int_0^\infty x^t k_{[0,1]TIWIE}\, dx = \varphi_i d \int_0^\infty x^{t-2}\ e^{\frac{d\beta(i+1)}{x}}\, dx$$

$$E(x^t) = \varphi_i d(d\beta(i+1))(-d\beta(i+1))^{t-2}\Gamma(1-t) \qquad (12)$$

where the $\Gamma(a) = \int_0^\infty x^{a-1}\ e^{-x}\, dx$ is denoted by gamma function. By using the moment Eq (12) and the formula $E(e^{rx}) = \sum_{t=0}^{\infty} \frac{r^t}{t!} E(x^t)$, the moment generating function (MGF) is given by Ahmed et al.,(2021):

$$E(e^{rx}) = \sum_{t=0}^{\infty} \frac{r^t}{t!}\ \varphi_i d(d\beta(i+1))(-d\beta(i+1))^{t-2}\Gamma(1-t)$$

Taking the first derivative of the MGF with respect to t and evaluating at t=0, we obtain the first moment of the distribution. Taking the second derivative of the MGF with respect to t and evaluating at t=0, we obtain the second moment of the distribution.

3.4 Rényi entropy

The Rényi entropy is a measure of entropy or randomness that is based on the Rényi information divergence. For a probability distribution, the Rényi entropy is defined as a function of a parameter alpha, which determines the degree of sensitivity to rare events. Specifically, the alpha-th Rényi entropy of a probability distribution with probability mass function (PMF) or probability density function (PDF) f(x) is given by:

$$I_R(c) = \frac{1}{1-c}\log\left\{\int_{-\infty}^{\infty} f^c(x)dx\right\},\ c \neq 1\,, c > 0$$

So for for [0,1]TIWIE distribution will be given as follows

$$I_R(c) = \frac{1}{1-c}\log\left\{\int_0^{\infty} \left[\varphi_i \frac{d}{x^2}\ e^{\frac{di\beta+d\beta}{x}}\right]^c dx\right\},\ c \neq 1\,, c > 0$$

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The Rényi entropy has applications in various fields including information theory, physics, and statistics, and is particularly useful when the distribution is heavy-tailed or has power-law behavior.

3.5 Shannon Entropy

The Rényi entropy reduces to the Shannon entropy as (c) approaches 1, and is related to other entropy measures such as the Tsallis entropy and the Kullback-Leibler divergence. The Shannon entropy is a measure of the average amount of information needed to specify the outcome of a random variable drawn from the distribution. It is maximized when the distribution is uniform (all outcomes are equally likely), and minimized when the distribution is concentrated on a single outcome. The Shannon entropy has applications in various fields including information theory, data compression, cryptography, and statistical inference.

$$\eta_x = E\left(-\log \varphi_i \frac{d}{x^2}\ e^{\frac{di\beta+d\beta}{x}}\right) \tag{13}$$

3.6 Delta Entropy

Delta entropy is a measure of the uncertainty or randomness in a continuous probability distribution that is based on the Rényi entropy. The delta entropy of a continuous probability distribution with probability density function (PDF) f(x) is defined as:

$$H(\delta) = \frac{1}{1-\delta}\log\left\{1 - \int_0^\infty \left[\varphi_i \frac{d}{x^2}\ e^{\frac{di\beta+d\beta}{x}}\right]^\delta dx\right\} \tag{14}$$

The delta entropy can be interpreted as a measure of the number of independent "bits" needed to specify a random variable drawn from the distribution, relative to a reference distribution with the same mean. It is zero when the distribution is a delta function (all probability mass or density is concentrated on a single point), and is maximized when the distribution is uniform over its support.

4. Estimation parameters

In this section we will used the maximum likelihood method to estimate the parameters for [0,1] TIWIE distribution. Let $x_1, x_2, \dots, x_n$ be a sample random of size n each distributed according to the [0,1] TIWIE distribution with parameters $\alpha, \beta,$ and d then the likelihood function $L(\alpha, \beta, d)$ will be:

$$L(\alpha, d, \beta) = \prod_{i=1}^{n} \left(\frac{\alpha\beta \dfrac{d}{{x_i}^2}\, e^{-\alpha\left[e^{\frac{-d}{x_i}}\right]^{-\beta}} \left[e^{\frac{-d}{x_i}}\right]^{-\beta}}{e^{-\alpha}} \right) \tag{15}$$

Now

$$ll = nlog(\alpha) + nlog(\beta) + nlog(d) - 2\sum_{i=1}^{n} log(x_i) - \beta \sum_{i=1}^{n} (\frac{d}{x_i}) - \alpha \sum_{i=1}^{n} \left(\frac{-d}{x_i}\right)^{-\beta} \tag{16}$$

Taking the first derivative of the log likelihood function is a common technique used in maximum likelihood estimation to find the values of the parameters that maximize the likelihood of observing the data Abdal-hameed et al., (2018).

$$\frac{\partial ll}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \left(\frac{-d}{x_i}\right)^{-\beta}$$

$$\frac{\partial ll}{\partial \beta} = \frac{n}{\beta} - \alpha \left(\sum_{i=1}^{n} \left(\frac{-d}{x_i}\right)^{-\beta} ln\left(\frac{-d}{x_i}\right) \right)$$

$$\frac{\partial}{\partial d} = -\alpha \sum_{i=1}^{n} \left(- \frac{\left(\frac{-d}{x_i}\right)^{-\beta} \beta}{d} \right) - (1+\beta) \sum_{i=1}^{n} \left(\frac{1}{x_i}\right)$$

The resulting equation, known as the score function, can be set to zero the equations $\frac{\partial ll}{\partial \alpha}, \frac{\partial ll}{\partial \beta}, \frac{\partial ll}{\partial d}$ and solved for the parameter values that optimize the likelihood. Solving it algebraically is very difficult, so we use numerically iterative methods such as Newton-Raphson type algorithms using R software package “Newdistns” we can get the estimators of the parameters as in next section (Abdal, 2020), and (Mohammad, 2020).

5. Application

This particular section presents the applications to real life data to illustrate the flexibility of the [0,1]TIWIE distribution defined in Section 2.

The MLEs of the model parameters are determined and some goodness-of-fit statistics for this distribution are compared with other competitive models such as Beta inverse exponential, Kumaraswamy inverse exponential, Weibull inverse exponential, Gompertz inverse exponential, inverse exponential. The model selection is carried out based upon the value of the log-likelihood function evaluated at the MLEs (- $\ell\ell$), Akaike Information Criterion, AIC, Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) and Hannan Quin Information Criterion (HQIC) (Ibrahim and Khaleel, 2018), and (Ahmedd, 2020). The data set is the mean of spend money in iraq governs from 2003 to 2020 (222.8, 124.0, 297.3, 71.44, 293.1, 112.7, 120.5, 148.5, 111.8, 110.4, 117.3, 103.5, 93.1, 116.1, 99.0, 109.2, 114.2, 120.1)

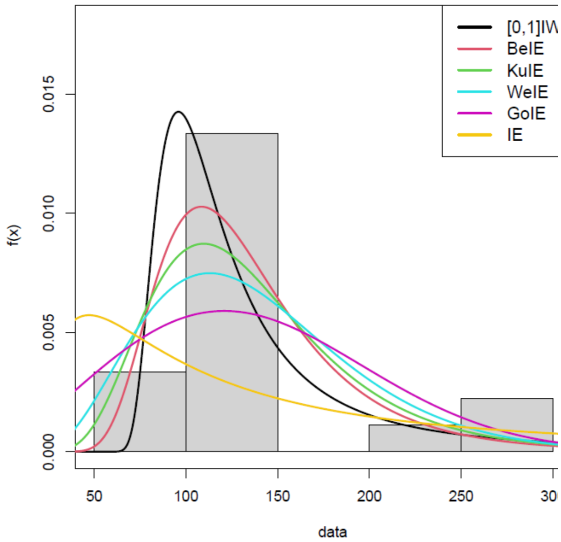
Table (1): The criteria value for the compares distributions

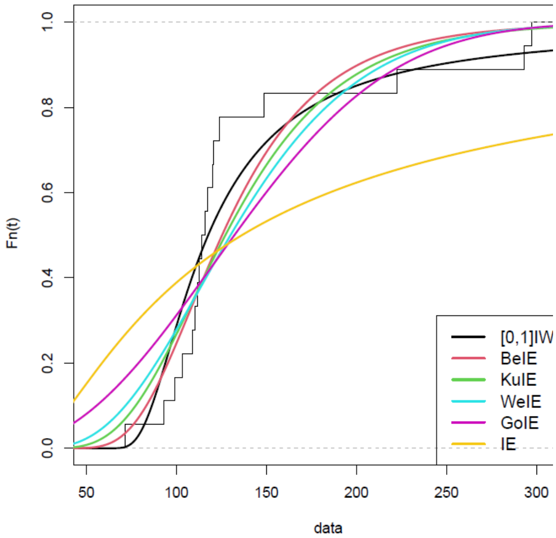
Distributions	Par. Est	- $\ell\ell$	AIC	CAIC	BIC	HQIC
[0,1]TIWIE	$\hat{\alpha} = 19.4866$ $\hat{\beta} = 0.02809$ $\hat{d} = 19.5967$	92.9	191.96	193.67	194.63	192.32
BeIE	$\hat{\alpha} = 31.8386$ $\hat{\beta} = 8.88630$ $\hat{d} = 30.1290$	93.5	193.12	194.84	195.80	193.49
KuIE	$\hat{\alpha} = 34.1189$ $\hat{\beta} = 12.1188$ $\hat{d} = 10.7757$	94.7	195.47	197.18	198.14	195.83
WeIE	$\hat{\alpha} = 3.75340$ $\hat{\beta} = 1.26881$ $\hat{d} = 49.480$	96.2	198.49	200.20	201.16	198.85
GoIE	$\hat{\alpha} = 0.00608$ $\hat{\beta} = 2.32637$ $\hat{d} = 12.5773$	98.8	203.72	205.43	206.39	204.08
IE	$\hat{d} = 94.5239$	106.8	215.69	215.94	216.58	215.81

Table (2): the K-S, P-Value, W, A values for the compares distributions

Distribution	K-S	P-value	W	A
TIWIE	0.2195	0.3045	0.1880568	1.06762
BeIE	0.2910	0.0761	0.2856323	1.505123

Distribution	K-S	P-value	W	A
KuIE	0.3042	0.0563	0.3317712	1.750802
WeIE	0.3218	0.0369	0.3906388	2.061793
GoIE	0.3260	0.0332	0.4705427	2.484804
IE	0.3111	0.0478	0.2881713	1.518362





(a) **(b)**

Figure (3)**: (a)** Densities for the new distribution with data set**, (b)** CDF for the new model with data set

Figure 3 displays the estimated pdf (a) and cumulative distribution function (b) of the [0,1] TIWIE distribution in comparison with other competing distributions for the first dataset. Tables 3 and 4 reveal that the [0,1] TIWIE distribution exhibits the best fit and is superior to the other lifetime distributions in modeling the studied datasets. The findings are further supported by analyzing density, and cumulative distribution fit of the distributions for the real lifetime data sets.

6. Conclusion

In this paper, we proposed a new extension for the inverse exponential distribution the new model with three parameters is more flexibility with respect to the inverse exponential distribution. We derived some statistical properties for the new model such as expansion for PDF, quantile function, moments, moment generating function, entropies. In addition, we estimated the parameters for the new model by using the maximum likelihood method and finally we provided that the new distribution are fitted to the dataset if it's compared with other distributions.

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