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SOME POWERFUL TECHNIQUES FOR SOLVING NONLINEAR VOLTERRA-FREDHOLM MODEL

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Abstract. In this paper, we plan to provide numerical schemes for solving the system of nonlinear Volterra-Fredholm integral equations (V-FIEs), by applying the quick and innovative methods to our problem-solving. We attempt to talk about a few numerical topics, such examination of uniqueness. Lastly, the suggested approaches' applicability and correctness are evaluated, and comparisons are made with a few numerical instances.

1. Introduction

This work aims to address the non-homogeneous V-FIE of the second sort, which takes the following form:

$$\Xi(\beta, \Upsilon) = \Phi(\beta, \Upsilon) + \int_0^{\beta}\int_{\Omega} \theta(\beta, \Upsilon, \xi, t, \Xi(\xi, t))d\xi dt, \tag{1.1}$$

where $\Xi(\beta, \Upsilon)$ is an unknown function, the functions $\Phi(\beta, \Upsilon)$ and $\theta(\beta, \Upsilon, \xi, t, \Xi(\xi, t))$ are analytic on $\mathbf{D} = \Omega \times [0, T]$ and where $T > 0$, and Ω is a closed subset of $\mathbb{R}^n$, $n = 1, 2, 3$.

These kinds of equations come up in a number of physical and biological issues, the theory of parabolic boundary value problems, and the mathematical modelling of the spatiotemporal evolution of epidemic models. [23] contains in-depth analyses and descriptions of these models. The following writers solve the two-dimensional V-FIEs: The V-FIEs and by [20] and [**?**], respectively, are solved in 1986 using the time collocation technique. [17] and [10] employ methods for solving V-FIEs that are based on the ADM. Two-dimensional V-FIE are solved by [3] using the homotopy perturbation technique, and by [2] using the two-dimensional Legendre Wavelets method. We direct the reader to the following publications for further information on numerical solutions to two-dimensional V-FIEs [15, 18, 27]. Different kinds of differential and integral equations can be approximatedly solved using power series expansion and its characteristics [21].

Numerous methods, including matrix-based approach [14], HPM [9] and MHPM [7], spline collocation method [5], and iterative method [24], have been used recently for the nonlinear computation of the two-dimensional V-FIEs. A HAM for solving nonlinear V-FIEs of the first sort was suggested by Behzadi [4].

The linear VFIEs were solved by Shekarabi et al. [22] using the two-dimensional Bernstein operational matrices technique, and the V-FIEs were solved by Dastjerdi et al. [8] using the radial basis function approximation. Additionally, [25] suggests using the Taylor polynomial

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approach to approximate the solution of the V-FIEs. Conversely, Paripour and Kamyar [19] used novel basis functions to solve nonlinear V-FIEs directly, whilst Bhrawy et al. [6] employed the Legendre-Gauss-Lobatto collocation technique to approximate the multi-dimensional Fredholm integral equation solution.

In this work, we use some approximation techniques HAM and VIM to solve a broad type of nonlinear V-FIEs. Additionally, we shall examine a few fresh uniqueness findings for the solutions. Using a number of test issues as examples, a successful conclusion is presented, backed up by the tabulated results of the given instances, demonstrating the approaches' versatility and competence.

2. Description of the Techniques

The development of more sophisticated and effective techniques for solving V-FIE has been the focus of certain potent techniques, such the HAM [1, 12, 13, 16], VIM [10, 11, 13, 20, 21, 24, 26]. In this part, we will go over each of these techniques:

2.1. **HAM Technique.** The solution provided in HAM is primarily local. We split the interval I into sub-intervals in order to extend this solution over the interval $I = [0, T]$ to $I_j = [\beta_{j-1}, \beta_j)$, $j = 1, 2, 3, \cdots, p$, where $0 \leq \beta_0 < \beta_1 < \cdots \leq \beta_p = T$. We solve the equation (1.1) in each subinterval I_j. Let $\Xi_1(\beta)$ be solution of equation (1.1) in the subinterval I_1. For, $2 \leq i \leq p$, $\Xi_i(\beta)$ is solution of equation (1.1) in the subinterval I with initial conditions by obtaining the initial conditions from the interval I_{i-1}.

$$\Xi_i(\beta_{i-1}) = \Xi_{i-1}(\beta_{i-1}), \quad \text{for} \quad i = 2, \cdots, p.$$

The solution of equation (1.1) is given by

$$\Xi = \sum_{i=1}^{p} \chi I_i \Xi_i, \tag{2.1}$$

where

$$\chi_m = \begin{cases} 1 & m > 1 \\ 0 & m \leq 1. \end{cases}$$

For equation (1.1), first we choose

$$L[\varphi_i(\beta; p)] = \frac{\partial^n \varphi_i(\beta; p)}{\partial \beta^n}, \quad n = 0, 1, 2, \cdots$$

and

$$\begin{aligned} H_i(\beta, \Upsilon) &= 1 \\ \Xi_{i0}(\beta, \Upsilon) &= \sum_{j=0}^{m} \Xi_{(i-1)_j}(\beta, \Upsilon), \quad \text{for} \quad i = 2, \cdots. \end{aligned}$$

We use HAM to equation (1.1) to confirm its applicability and promise in solving Volterra-Fredholm integral equations. Initially, we select

$$L[(\beta, \Upsilon; p)] = \varphi(\beta, \Upsilon; p)$$

and

$$H(\beta, \Upsilon) = 1.$$

A nonlinear operator is defined as follows:

$$N[\varphi(\beta, \Upsilon; p)] = \varphi(\beta, \Upsilon; p) + \Phi(\beta, \Upsilon) + \int_0^{\beta} \int_{\Omega} \theta(\beta, \Upsilon, \xi, t, \varphi(\xi, t; p)) d\xi dt, \tag{2.2}$$

The zero order deformation equation is:

$$(1 - p)L[(\beta, \Upsilon; p) - \Xi_0(\beta, \Upsilon)] = p\hbar N[\varphi(\beta, \Upsilon; p)]. \quad (2.3)$$

Differentiating equation (2.3), we have the so-called m^{th}-order deformation equation for $m \geq 1$ after dividing them by $m!$, setting $p = 0$, and rearranging m-times with regard to the embedding parameter p:

$$\Xi_m(\beta, \Upsilon) = \chi_m \Xi_{m-1}(\beta, \Upsilon) + h R_m(\vec{\Xi}_{m-1}), \quad (2.4)$$

where

$$R_m(\vec{\Xi}_{m-1}) = \Xi_{m-1}(\beta, \Upsilon) - \int_0^{\beta}\int_{\Omega} \frac{\partial^{m-1}\theta(\beta, \Upsilon, \xi, t, \varphi(\xi, t; p))}{\partial p^{m-1}}\bigg|_{p=0} d\xi dt - (1 - \chi_m)\Phi(\beta, \Upsilon). \quad (2.5)$$

Now, we have:

$$\Xi(\beta, \Upsilon) = \Xi_0(\beta, \Upsilon) + \Xi_1(\beta, \Upsilon) + \Xi_2(\beta, \Upsilon) + \cdots .$$

2.2. **Variational Iteration Method (VIM).** Consider the integral equation given in Eq.(1.1):

$$\Xi(\beta, \Upsilon) = \varphi(\beta, \Upsilon) + \int_0^{\beta}\int_{\Omega} \theta(\beta, \Upsilon, \xi, t, \Xi(\xi, t)) d\xi dt, \quad (2.6)$$

with $\Omega = [0, b]$,

For the integral equation (1.1), let $w(\beta)$ be a function such that $w'(\beta) = \Xi(\beta)$, noting that $\Xi(\beta)$ is continuous. Then for Eq. (1.1) first we take the partial derivative with respect to β. We have

$$\frac{\partial \Xi}{\partial \beta} - \frac{\partial \varphi}{\partial \beta} - \int_0^b \theta(\beta, \Upsilon, \xi, t, \Xi(\xi, \beta)) d\xi - \int_0^{\beta}\int_0^b \frac{\partial \theta}{\partial \beta} d\xi dt = 0 \quad (2.7)$$

Consider

$$-\int_0^b \theta(\beta, \Upsilon, \xi, t, \Xi(\xi, \beta)) d\xi - \int_0^{\beta}\int_0^b \frac{\partial \theta}{\partial \beta} d\xi dt,$$

as a restricted variation, we use the VIM in direction Υ. Then we have the following iteration sequence:

$$\Xi_{n+1}(\beta, \Upsilon) = \Xi_n(\beta, \Upsilon) + \int_0^{\beta} \lambda \Big[\frac{\partial \Xi_n}{\partial \tau}(\tau, \Upsilon) - \frac{\partial \varphi}{\partial \tau}(\tau, \Upsilon) - \int_0^b \theta(\tau, \Upsilon, \xi, t, \Xi_n(\xi, \tau)) d\xi - \int_a^{\tau}\int_0^b \frac{\partial \theta}{\partial \tau} d\xi dt \Big] d\tau. \quad (2.8)$$

Taking the variation with respect to the independent variable Ξ_n and noticing that $\delta\Xi_n(0) = 0$, we get

$$\delta\Xi_{n+1} = \delta\Xi_n + \lambda\delta\Xi_n|_{\tau=\Upsilon} \int_0^x \lambda' \delta\Xi_n d\tau = 0 \quad (2.9)$$

Then we apply the following stationary conditions:

$$1 + \lambda(\tau)|_{\tau=\Upsilon} = 0, \quad \lambda'(\tau)|_{\tau=\Upsilon} = 0,$$

The general Lagrange multiplier, therefore, can be readily identified:

$$\lambda = -1,$$

and as a result, we obtain the following iteration formula:

$$\Xi_{n+1}(\beta, Y) = \Xi_n(\beta, Y) - \int_0^{\beta} \Big[\frac{\partial \Xi_n}{\partial \tau}(\tau, y) - \frac{\partial \varphi}{\partial \tau}(\tau, Y) - \int_0^{b} \theta(\tau, Y, \xi, t, \Xi_n(\xi, \tau))d\xi - \int_a^{\tau}\int_0^{b} \frac{\partial \theta}{\partial \tau} d\xi dt \Big] d\tau. \tag{2.10}$$

Consequently, the approximate solution is given by

$$\lim_{n \longrightarrow \infty} \Xi_n(\beta, Y) = \Xi(\beta, Y).$$

3. Uniqueness Results

Assume that the space of all continuous functions is $\mathbb{S}$. In $\mathbf{D}$, $\varphi : \mathbf{D} \longrightarrow \mathbb{R}^n$ satisfied

$$\|\varphi(\beta, Y)\| = O(exp(\mu(\beta + \|Y\|))), \ (\beta, Y) \in \mathbf{D}, \ \mu > 0. \tag{3.1}$$

In this space $\mathbb{S}$ we define

$$|\varphi| = \sup_D [\|\varphi(\beta, Y)\| exp(-\mu(\beta + \|Y\|))]. \tag{3.2}$$

With the aforementioned norm, it is obvious that $\mathbb{S}$ is a Banach space. We see that a constant $M > 0$ exists from (3.1) such that:

$$\|\varphi(\beta, Y)\| = M(exp(\mu(\beta + \|Y\|))),$$

we get

$$|\varphi| \le M. \tag{3.3}$$

Our principal finding is the following theorem, which provides adequate conditions for the uniqueness and existence of equation (1.1) solutions. Before starting and proving the main results, we provide the following theories:

(H1): A continuous nonnegative function exists $h(\beta, Y, \xi, t)$ defined on D^2 such that

$$\|\theta(\beta, Y, \xi, t, \Xi_1) - \theta(\beta, Y, \xi, t, \Xi_2)\| \le h(\beta, Y, \xi, t)\|\Xi_1 - \Xi_2\|$$

and

$$\int_0^{\beta}\int_{\Omega} h(\beta, Y, \xi, t) exp(\mu(\xi + \|t\|))d\xi dt \le Q \exp(\mu(\beta + \|Y\|)),$$

where $(\beta, Y, \xi, t, \Xi_i) \in D^2 \times \mathbb{R}^n$, $i = 1, 2$ and $Q > 0$.

(H2): A constant $N > 0$ exists such that:

$$\|\Phi(\beta, Y)\| + \int_0^{\beta}\int_{\Omega} \|\theta(\beta, Y, \xi, t, 0)\| d\xi dt \le N \exp(\mu(\beta + \|Y\|)).$$

Theorem 3.1. *Assuming the validity of (H1) and (H2), a unique solution to Eq.*(1.1) *exists if* $0 < Q < 1$.

Proof. Let the right half of equation (1.1) define the operator $T : \mathbf{S} \longrightarrow \mathbf{S}$. It appears that $T(u)$ is continuous in $\mathbf{D}$ and $T(\Xi(\beta, Y)) \in \mathbb{R}^n$ for $\Xi \in \mathbf{S}$ and $(\beta, t) \in \mathbf{D}$.

Using assumptions (H1) and (H2), we first demonstrate the satisfaction of (3.1).

$$\begin{aligned}
\|T(\Xi(\beta,\Upsilon))\| &\le \int_0^\beta\int_\Omega \|\theta(\beta,\Upsilon,\xi,t,\Xi(\xi,t)) - \theta(\beta,\Upsilon,\xi,t,0)\|\,d\xi dt \\
&\quad + \|\Phi(\beta,\Upsilon)\| + \int_0^\beta\int_\Omega \|\theta(\beta,\Upsilon,\xi,t,0)\|\,d\xi dt \\
&\le |\Xi| \int_0^\beta\int_\Omega h(\beta,\Upsilon,\xi,t)\exp(\mu(\xi+\|t\|))d\xi dt \\
&\quad + N\exp(\mu(\xi+\|\Upsilon\|)) \\
&\le [MQ+N]\exp(\mu(\xi+\|\Upsilon\|)).
\end{aligned}$$

$\therefore\ T(\Xi) \in \mathbf{S}$.

We shall also show that $T(\Xi)$ is a contraction map in the second place. Assuming $\Xi_1, \Xi_2 \in \mathbf{S}$, the following may be deduced from (H1):

$$\begin{aligned}
&\|T(\Xi_1(\beta,\Upsilon)) - T(\Xi_2(\beta,\Upsilon))\| \\
&\le \int_0^\beta\int_\Omega \|\theta(\beta,\Upsilon,\xi,t,\Xi_1(\xi,t)) - \theta(\beta,\Upsilon,\xi,t,\Xi_2(\xi,t))\|\,d\xi dt \\
&\le |\Xi_1 - \Xi_2| \int_0^\beta\int_\Omega h(\beta,\Upsilon,\xi,t)\exp(\mu(\xi+\|t\|))d\xi dt \\
&\le Q|\Xi_1 - \Xi_2|\exp(\mu(\xi+\|\Upsilon\|)).
\end{aligned}$$

Consequently, we have

$$|T(\Xi_1) - T(\Xi_2)| \le Q|\Xi_1 - \Xi_2|.$$

In other words, T is a contraction map. According to the Banach contraction principle, **S** contains a single fixed point Ξ for T.

4. Numerical Examples

The semi-analytical methods for solving nonlinear V-FIEs based on ADM, MADM, and HAM are presented in this section.

Example 1. Consider the nonlinear V-FIE (1.1) with:

$\Phi(\beta,t) = e^{-\Upsilon}\ \cos(\beta) + \Upsilon\cos(\beta) + \frac{1}{2}\cos(\beta-2)\sin(2)$,

$\theta(\beta,\Upsilon,\xi,t,\Xi(\xi,t)) = -cos(\beta-\xi)e^{-(\Upsilon-t)}\Xi(\xi,t), \quad (\beta,\Upsilon) \in [0,2]\times\Omega.$

The exact solution is $\Xi(\beta,\Upsilon) = cos(\beta)e^{-\Upsilon}$, with $\Omega = [0,2]$. We applied the methods presented in this paper:

TABLE 1. Numerical Results of the Example 1.

(β,Υ)	Exact	HAM	VIM
(0.0625,0.0625)	0.937578	0.937577	0.937577
(0.125,0.125)	0.875611	0.875598	0.875597
(0.25,0.25)	0.754589	0.754488	0.754378
(O.5,0.5)	0.532280	0.531273	0.529861
(1,1)	0.198766	0.188564	0.176687

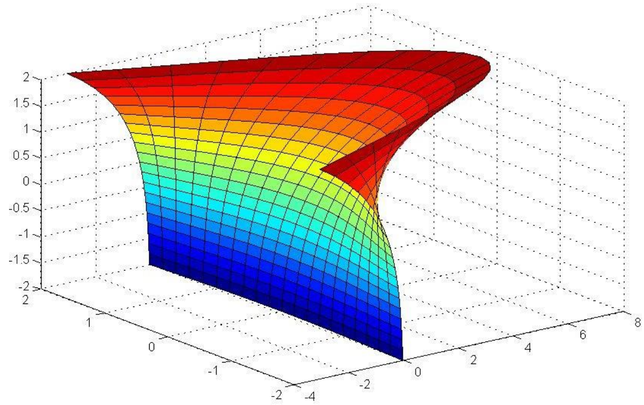

FIGURE 1. Numerical Results of the Example 1.

Example 2. Consider the nonlinear V-FIE (1.1) with:

$$\Phi(\beta, t) = \frac{\beta \Upsilon^2}{8(1+\Upsilon^2)(1+\Upsilon)} - \log\left(1 + \frac{\beta \Upsilon}{1+\Upsilon^2}\right),$$

$$\theta(\beta, \Upsilon, \xi, t, \Xi(\xi, t)) = \frac{\beta(1-\xi^2)}{(1+\Upsilon)(1+t^2)}(1 - e^{(-\Xi(\xi,t))}), \qquad (\beta, \Upsilon) \in [0, 1] \times \Omega.$$

The exact solution is $\Xi(\beta, \Upsilon) = -\log(\frac{1+\beta\Upsilon}{(1+\Upsilon^2)})$, with $\Omega = [0, 1]$. We applied the methods presented in this paper:

TABLE 2. Numerical Results of the Example 2.

(β, Υ)	Exact	HAM	VIM
(0.03125,0.03125)	-0.000975134	-0.000975134	-0.000975131
(0.0625,0.0625)	-0.003883492	-0.003883457	-0.003883387
(0.125,0.125)	-0.015267453	-0.015266645	-0.015266523
(0.25,0.25)	-0.057158421	-0.057157346	-0.057146473
(O.5,0.5)	-0.182321014	-0.182214013	-0.182132611
(1,1)	-0.405465006	-0.404802854	-0.404406365

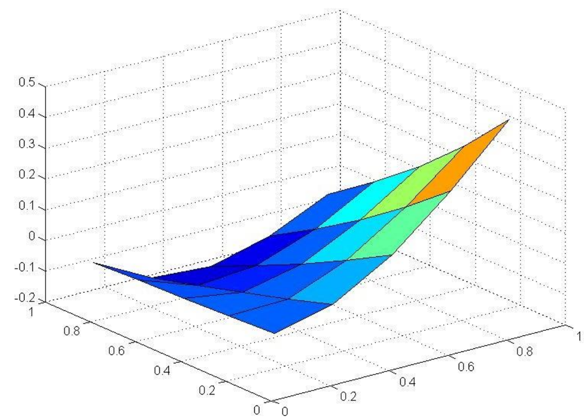

FIGURE 2. Numerical Results of the Example 2.

5. CONCLUSION

In this study, the V-FIEs are solved using the HAM and VIM. To show the method's validity and usefulness, we explained the procedures, applied them to two test problems, and compared the outcomes with the precise solutions. Furthermore, a minimal amount of iterations are required to achieve a desirable outcome. This assertion is supported by the provided numerical examples. The precise answer and the estimated solutions of the illustrated examples-which used HAM, and VIM with varying iterations and terms-are contrasted in the above tables.

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