INTEGRAL EQUATIONS

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1. Abstract

It is knowledgeable that, the Integral Equations are considered significantly as important topics regarding to (Pure and Applied Mathematics). This importance comes to the prominence because these equations are taken a critical and a vital role of physical and chemical problems.

Moreover, a lot of ordinary differential equation (ODE) and partial differential equation (PDE) are possibly to be converted into (Integral Equations). Henceforth, we are going to present some of mathematical principle concepts in this domain (Integral Equations). In turns, these mathematical concepts will aid us to understand their brands and features, accordingly.

Keywords: Integral Equations, Linearity, Homogeneity, Solution of An Integral Equation.

2. Introduction

Rahman (2007 : 1) states that, an integral equation is defined as an equation in which the unknown function $u(x)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems.

Wazwaz (2011 : 65) accepts that, many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations.

Rahman (2005 : 62) explains that, the most standard type of integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt \qquad (2.1)$$

where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a known function, of two variables x and t, called the kernel or the nucleus of the integral equation. The unknown function $u(x)$ that will be determined appears inside the integral sign.

In many other cases, the unknown function $u(x)$ appears inside and outside the integral sign. The functions $f(x)$ and $K(x,t)$ are given in advance. It is to be noted that, the limits of integration $g(x)$ and $h(x)$ may be both variables, constants, or mixed.

3. Classification of integral equations

An integral equation can be classified as a linear or nonlinear integral equation. In the previous section, we have noticed that the differential equation can be equivalently represented by the integral equation. Therefore, there is a good relationship between these two equations.

It is evident that, Polyanin and Manzhirov (2008 : 519) state that, the most frequently used integral equations fall under two major classes, namely Volterra and Fredholm integral equations. Of course, we have to classify them as homogeneous or nonhomogeneous; and also linear or nonlinear. In some practical problems, we come across singular equations also.

In this study, we shall distinguish four major types of integral equations – the two main classes and two related types of integral equations. In particular, the four types are given below:

•Volterra integral equations
•Fredholm integral equations
•Integro-differential equations
• Singular integral equations

We shall outline these equations using basic definitions and properties of each type.

3.1 Volterra integral equations

Henceforth, Ali and Abdelwahid (2013 :202) see that, the most standard form of Volterra linear integral equations is of the form

$$\phi(x)u(x) = f(x) + \lambda\int_a^x K(x,t)u(t)dt \qquad (3.1)$$

where the limits of integration are function of x and the unknown function $u(x)$ appears linearly under the integral sign. If the function $\varphi(x) = 1$, then equation (3.1) simply becomes

$$u(x) = f(x) + \lambda\int_a^x K(x,t)u(t)dt \qquad (3.2)$$

and this equation is known as the Volterra integral equation of the second kind; whereas if $\varphi(x) = 0$, then equation (3.1) becomes

$$f(x) = \lambda\int_a^x K(x,t)u(t)dt \qquad (3.3)$$

which is known as the Volterra equation of the first kind.

Examples of the Volterra integral equation of the second kind are

$$u(x) = 1 - \int_0^x u(t)dt, \qquad (3.4)$$

and

$$u(x) = x + \int_0^x (x-t)u(t)dt. \qquad (3.5)$$

However, examples of the Volterra integral equations of the first kind are

$$xe^{-x} = x + \int_0^x e^{x-t}u(t)dt, \qquad (3.6)$$

and

$$5x^2 + x^3 = x + \int_0^x (5+3x-3t)u(t)dt. \qquad (3.7)$$

3.2 Fredholm integral equations

However, Hosseini (2006 : 1685) explains that, for Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$ may appear only inside integral equation in the form:

$$f(x) = \lambda \int_a^b K(x,t)u(t)dt \qquad (3.8)$$

This is called Fredholm integral equation of the first kind. However, for Fredholm integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (3.9)$$

Examples of the two kinds are given by

$$\frac{\sin x - x\cos x}{x^2} = \int_0^1 \sin(xt)u(t)dt \qquad (3.10)$$

and

$$u(x) = x + \frac{1}{2}\int_{-1}^1 (x-t)u(t)dt \qquad (3.11)$$

respectively.

3.3 Singular integral equations

Usman and Zubair (2013 : 123) state that, Volterra integral equations of the first kind is

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt \qquad (3.12)$$

or of the second kind is

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt \qquad (3.13)$$

are called singular if one of the limits of integration $g(x), h(x)$ or both are infinite. Moreover, Maleknejad and Mahmoudi (2003 : 641) explain that, the previous two equations are called singular if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration.

Here, Davies (2002 : 97) maintains that, the general form of the singular integral equations of the first kind is

$$f(x) = \lambda \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \ 0 < \alpha < 1 \qquad (3.14)$$

and the second kind is

$$u(x) = f(x) + \lambda \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \ 0 < \alpha < 1 \qquad (3.15)$$

The last two standard forms are called generalized Abel's integral equation and weakly singular integral equations respectively.

For $\alpha = \frac{1}{2}$ the equation:

$$f(x) = \lambda \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, \qquad (3.15)$$

is called the Abel's singular integral equation. It is to be noted that, the kernel in each equation becomes infinity at the upper limit $t = x$. Examples of Abel's integral equation, generalized Abel's integral equation, and the weakly singular integral equation are given by

$$\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, \qquad (3.16)$$

$$x^3 = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt, \qquad (3.17)$$

$$u(x) = 1 + \sqrt{x} + \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt, \qquad (3.18)$$

respectively.

3.4 Integro-differential equations

Collins (2006 : 4) argues that, in the early 1900, Vito Volterra studied the phenomenon of population growth, and new types of equations have been developed and termed as the integro-differential equations. In this type of equations, the unknown function $u(x)$ appears as the combination of the ordinary derivative and under the integral sign. Examples can be cited as follows:

$$u''(x) = f(x) + \lambda\int_0^x (x-t)u(t)dt,\ u(0)=0,\ \ u''(0)=1 \qquad (3.19)$$

$$u'(x) = f(x) + \lambda\int_0^1 (xt)u(t)dt,\ u(0)=1. \qquad (3.20)$$

Equations (3.19) is of Volterra type integro-differential equations, whereas equation (3.20) Fredholm type integro-differential equations. These terminologies were concluded because of the presence of indefinite and definite integrals.

4. Linearity and Homogeneity

Integral equations and integro-differential equations fall into two other types of classifications according to linearity and homogeneity concepts. These two concepts play a major role in the structure of the solutions. In what follows, we highlight the definitions of these concepts accordingly.

4.1 Linearity Concept

It is understood that, Wazwaz (2002 : 405) sees that, if the exponent of the unknown function $u(x)$ inside the integral sign is one, the integral equation or the integro-differential equation is called linear. If the unknown function $u(x)$ has exponent other than one, or if the equation contains nonlinear functions of $u(x)$, such as e^u, $\cos u, \ln(1+u)$, the integral equation or the integro-differential equation is called nonlinear. To explain this concept, we consider the equations:

$$u(x) = 1 - \int_0^x (x-t)u(t)dt, \qquad (4.1)$$

$$u(x) = 1 - \int_0^1 (x-t)u(t)dt, \qquad (4.2)$$

$$u(x) = 1 + \int_0^x (1+x-t)u^4(t)dt, \qquad (4.3)$$

and

$$u'(x) = 1 + \int_0^1 xte^{u(t)}dt, \quad u(0)=1 \qquad (4.4)$$

The first two examples are linear Volterra and Fredholm integral equations respectively, whereas the last two are nonlinear Volterra integral equation and nonlinear Fredholm integro-differential equation respectively.

It is knowledgably that, Almazmumy and Hendi (2012 : 228) explain that, it is important to point out that linear equations, except Fredholm integral equations of the first kind, give a unique solution if such a solution exists.

However, solution of nonlinear equation may not be unique. Nonlinear equations usually give more than one solution and it is not usually easy to handle. Both linear and nonlinear integral equations of any kind can be investigated by using traditional and new methods.

4.2 Homogeneity Concept

Porter and Stirling (2004 : 93) state that, Integral equations and integro-differential equations of the second kind are classified as homogeneous or inhomogeneous, if the function $f(x)$ in the second kind of Volterra or Fredholm integral equations or integro-differential equations is identically zero, the equation is called homogeneous. Otherwise it is called inhomogeneous. Notice that this property holds for equations of the second kind only. To clarify this concept we consider the following equations:

$$u(x) = \sin x + \int_0^x xtu(t)dt, \qquad (4.5)$$

$$u(x) = x + \int_0^1 (x-t)^2 u(t)dt, \qquad (4.6)$$

$$u(x) = \int_0^x (1+x-t)u^4(t)dt, \qquad (4.7)$$

and

$$u''(x) = \int_0^x xtu(t)dt, \quad u(0)=1, \quad u'(0)=0 \qquad (4.8)$$

The first two equations are inhomogeneous because $f(x) = sinx$ and $f(x) = x$, whereas the last two equations are homogeneous because $f(x) = 0$ for each equation. We usually use specific approaches for homogeneous equations, and other methods are used for inhomogeneous equations.

5. Solution of an Integral Equation

Wazwaz (2009 : 51) accepts that, a solution of a differential or an integral equation arises in any of the following two types:

5.1 Exact solution

The solution is called exact if it can be expressed in a closed form, such as a polynomial, exponential function, trigonometric function or the combination of two or more of these elementary functions. Examples of exact solutions are as follows:

$$u(x) = x + e^{x},$$
$$u(x) = \sin x + e^{2x}, \tag{4.9}$$
$$u(x) = 1 + \cosh x + \tan x,$$

and many others.

5.2 Series solution

For concrete problems, sometimes we cannot obtain exact solutions. In this case, we determine the solution in a series form that may converge to exact solution if such a solution exists. Other series may not give exact solution, and in this case the obtained series can be used for numerical purposes. The more terms that we determine the higher accuracy level that we can achieve.

A solution of an integral equation or integro-differential equation is a function $u(x)$ that satisfies the given equation. In other words, the obtained solution $u(x)$ must satisfy both sides of the examined equation.

6. Conclusion

It is obvious that, the variable mathematical domains are witnessed a critical importance for a lot of researchers, but the field of (Integral Equations) have taken the most important scope in last decades, because this domain is intermingled with other important sciences like physics or chemistry for instance.

According to this study, we have produced some important and principle concepts of (Integral Equations). We categorized them into many levels. Moreover; we produce each category according to its proper level.

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ملخص

من المعروف بأن المعادلات التكاملية ذات اهمية كبيرة سواء في الرياضيات البحتة او التطبيقية وتأتي هذه الاهمية لكون هذه المعادلات تدخل في حل الكثير من المسائل الفيزيائية والكيميائية.

بالاضافة الى ذلك فان الكثير من المعادلات التفاضلية العادية والمعادلات التفاضلية الجزئية من الممكن تحويلها الى معادلات تكاملية لغرض حلها بطريقة اسهل ولتلك الاهمية سنقدم بعض المفاهيم الاساسية في هذا الحقل (المعادلات التكاملية). هذه المفاهيم ستساعد على فهم انواع هذه المعادلات ومميزاتها.