On the contraction mapping principle with the Caputo fractional derivative

Munther Salman

Abstract

The Purpose of this manuscript is to study the contraction mapping principle to prove our main result concerning local existence theorem of fractional differential equations in $L_p(a\,,b)$ Space, $1 \le p < \infty$.

Keywords Existence. The contraction mapping principle. Fractional derivative. Fixed point.

Introduction

Although the tools of fractional calculus have been available and applicable to various fields of study, the investigation of the theory of fractional differential equations has only been started quite recently. The concept of fractional calculus (fractional derivatives and fractional integral) is not new. In 1695 *L'Hospital* asked the question as to the meaning of $\frac{d^n y}{dx^n}$ if $n = \frac{1}{2}$; that is " what if n is fractional?". *Leibniz* replied that" $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx:x}$ " It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations. However, in case of fractional-order integration and differentiation, which represent a rapidly growing field both in theory and in applications to real world problems, it is not so. Since the appearance of the idea of differentiation and integration of arbitrary (not necessary integer) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 year. In recent years, fractional differential equations have been investigated by many authors. Rawashdeh [8] used the collocation spline method to approximate the solution of fractional equations. Momani [6] obtained local and global existence and uniqueness solution of the integro-differential equation. In the last years has found use in studies of viscoelastic materials, as well as in many fields of science and engineering including fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory and probability. *Mouffak B*, *Samira H* and *Sotiris K. Ntouyas* [7], studied the existence of solutions for boundary value problems, for fractional differential equation ${}^{\epsilon}D^{q}\mathcal{U}(x) = \varphi(x, \mathcal{U}(x))$, for each $x \in [0, T]$, $q \in (0, 1)$ and satisfying the boundary condition $\xi\mathcal{U}(0) + \omega\mathcal{U}(T) = \epsilon$ where ${}^{\epsilon}D^{q}$ is the Caputo fractional derivative, $\varphi\colon [0, T] \times \mathbb{R} \to \mathbb{R}$, is a continuous function, ξ, ω, ϵ are real constants with $\xi+\omega\neq 0$ by using Schaefer's fixed point theorem. In [10] *Zhenyu G. and Min L.*, studied a system of fractional order differential equations and some sufficient conditions for existence of unique mild solutions for the system is established by Krasnoselskii fixed point theorem. Recently, *Khalida A. and Mouffak B.* [3], it deal with the existence of mild solutions for a class of fractional integro-differential equations with state-dependent delay. Their results are based on the technique of measures of noncompactness and Darbo's fixed point theorem.

Throughout this study the following definitions and propositions which will be used in various places in this work.

Definitions and Formulation of the Main Result

Given $q>0$, let $L(a,b)$ denote to the set of all Lebesgue measurable functionφdefined on measurable set such that:

$$\int_a^b (b-x)^{q-1}\,\varphi(x)dx<\infty,$$

as usual, denote by $\Gamma(q)$ is the Gamma function, and $[q]$ denotes the integer part of q.

Now, we present our main definitions:

Definition 1 (See [1])**.** *Let φ be a lebesgue measurable function defined a.e., on the interval $[a,b]$. The fractional integration of degree $q>0$ for φ is define* :

$$D_q^{-q}\varphi := \frac{1}{\Gamma(q)}\int_a^b (b-x)^{q-1}\,\varphi(x)dx,\ \ for\ \ q>0$$

Provided that this integral (Lebesgue) exists.

Definition 2 (See [4])**.** *The Caputo fractional derivative of order $q>0$ for a function φ is given by*:

$$\left(D_{q+}^{q}\,\varphi\right)(x) := \frac{1}{\Gamma(m-q)}\int_a^x (x-s)^{m-q-1}\,\varphi^{(m)}\big(s,\mathcal{U}(s)\big)dx,$$

Herem $:=[q]+1.$

Auxiliary Results

Let us introduce the following Propositions will be used in the text, first we state Hölder's inequality (See [9]):

Proposition 1

If μ be a measurable space, and φ,ϑ are two measurable functions ,let $q,p\in(1,\infty)$ be such that $\frac{1}{q}+\frac{1}{p}=1$. let $\vartheta\in L_q(\mu)$, $\varphi\in L_p(\mu)$, Then $(\vartheta\varphi)$belongs to $L(\mu)$ and satisfies

$$\int\limits_{\mu} |\vartheta\varphi|\, d\mu \leq \left[\int\limits_{\mu} |\vartheta|^{q} d\mu\right]^{\frac{1}{q}} \left[\int\limits_{\mu} |\varphi|^{p}\, d\mu\right]^{\frac{1}{p}}.$$

We also need:

Proposition 2 (See [1]).

Let $\varphi(x)$ be a continuous function on the interval $[a,b]$ and let $q > 0$, then $D^{-q}\varphi$ exists and it is continuous with respect to x on $[a,b]$.

Proposition 3 (See [5]).

Let $\varphi(x)$ belongs to $L(a,b)$ and let $q > 0$, then for all $x \in [a,b]{}_{a}^{x}D^{-q}\varphi$ exists if $q \geq 1$ and a.e.
if $q < 1$.

Proposition 4 (See [2]).

Given $\varphi(x) \in L(a,b)$ and let $q, \sigma > 0$, we have for all $x \in [a,b]$

$i)$ $\int\limits_{a}^{x} {}_{a}^{s}D^{-q}\,\varphi\, ds := {}_{a}^{x}D^{-(q+1)}\varphi$ *or* ${}_{a}^{x}D^{-1}\,{}_{a}^{s}D^{-q}\varphi := {}_{a}^{x}D^{-(q+1)}\varphi.$

$ii)$ *Let $q \geq 0$, then ${}_{a}^{x}D^{-q}\varphi$ is absolutely continuous in $x \in [a,b]$.*

$iii)$ *If $q \geq 1$. Then* $\frac{d}{dx}\,{}_{a}^{x}D^{-(q+1)}\varphi := {}_{a}^{x}D^{-q}\varphi$, *everywhere and a.e., if $q < 1$.*

$iv)$ *Let $q + \sigma \leq 1$, then ${}_{a}^{x}D^{-(q+\sigma)}\varphi := {}_{a}^{x}D^{-q}\,{}_{a}^{s}D^{-\sigma}\varphi$ almost everywhere.*

Finally, we state Proposition 5 (see [7] for a short proof).

Proposition 5

If φ be a continuous function from $[a,b] \to R$. For $0 < q < 1$, set

$$\mathcal{U}(x) := \frac{1}{\Gamma(q)} \int\limits_{a}^{x} (x-s)^{q-1}\, \varphi\big(s, \mathcal{U}(s)\big) ds - \frac{\omega}{(\xi+\omega)\,\Gamma(q)} \int\limits_{a}^{b} (b-s)^{q-1}\, \varphi\big(s, \mathcal{U}(s)\big) ds + \frac{\epsilon}{\xi+\omega}$$

iff $\mathcal{U}$ be a solution of the fractional boundary value problem $D^{q}\mathcal{U}(x) := \varphi\big(x, \mathcal{U}(x)\big)$, with condition $\xi\, \mathcal{U}(a) + \omega\, \mathcal{U}(b) = \epsilon$, where ξ, ω and ϵ are constant, and $\xi + \omega \neq 0$.

Main Result

In this section we introduce our main result, theorem deals with study the local existence of p-integrable solution in $L_p(a\ ,b)$ space:

Theorem *Given $\varphi\big(x,\mathcal{U}(x)\big)$ the right-hand side of the fractional differential equation $D^q\mathcal{U}(x) := \varphi\big(x,\mathcal{U}(x)\big)$ satisfy the Lipschitz condition in $\mathcal{U}$ with Lipschitz constant $\mathcal{C}$, such that $|\varphi(x,\mathcal{U}_1)-\varphi(x,\mathcal{U}_2)| \leq \mathcal{C}|\mathcal{U}_1-\mathcal{U}_2|$, on the domain $\mathfrak{D}$, whence: $\mathfrak{D} := \{(x,\mathcal{U}) : |x-x_0| \leq a,$ $|\mathcal{U}-\mathcal{U}_0| \leq b\}$ and it is p-integrable on (a,b) . There exists a p-integrable solution for the fractional differential equation $D^q\mathcal{U}(x) := \varphi\big(x,\mathcal{U}(x)\big)$ with the boundary condition$\xi\ \mathcal{U}(a)+\omega\ \mathcal{U}(b)=\epsilon$, where ξ , ω and ϵ are constant, and $\xi+\omega \neq 0$.*

Proof First assume that Ψ be a mapping on $L_p(a,b)$, given by

$$\Psi_\vartheta(x) := \frac{1}{\Gamma(q)}\int_a^x (x-s)^{q-1}\,\varphi\big(s,\vartheta(s)\big)ds - \frac{\omega}{(\xi+\omega)\,\Gamma(q)}\int_a^b (b-s)^{q-1}\,\varphi\big(s,\vartheta(s)\big)ds + \frac{\epsilon}{\xi+\omega}$$

and set

$$G(x) := \frac{1}{\Gamma(q)}\int_a^x (x-s)^{q-1}\,\varphi\big(s,\vartheta(s)\big)ds - \frac{\omega}{(\xi+\omega)\,\Gamma(q)}\int_a^b (b-s)^{q-1}\,\varphi\big(s,\vartheta(s)\big)ds + \frac{\epsilon}{\xi+\omega}$$

It guarantees that $G(x)$ is continuouson the interval $[a,\ b]$ and it is measurable. Indeed, by virtue of Proposition 2, the first and the second terms on the right hand side of equation $G(x)$ is continuous for $a \leq x \leq b$ and for all $q > 0$. we are going to write

$$|G(x)|^p := \left|\frac{1}{\Gamma(q)}\int_a^x (x-s)^{q-1}\,\varphi\big(s,\vartheta(s)\big)ds - \frac{\omega}{(\xi+\omega)\,\Gamma(q)}\int_a^b (b-s)^{q-1}\,\varphi\big(s,\vartheta(s)\big)ds + \frac{\epsilon}{\xi+\omega}\right|^p$$

Note that $|\varphi+\vartheta| \leq |\varphi|+|\vartheta|$ implies $|\varphi+\vartheta|^p \leq (|\varphi|+|\vartheta|)^p$, since $a+b \leq 2\max(a,b)$.

Therefore $|\varphi+\vartheta|^p \leq 2^p(|\varphi|^p+|\vartheta|^p)$, So,

$$|G(x)|^p \leq 2^p \left|\frac{1}{\Gamma(q)}\int_a^x (x-s)^{q-1}\,\varphi\big(s,\vartheta(s)\big)ds\right|^p$$

$$+2^p \left|\frac{\omega}{(\xi+\omega)\,\Gamma(q)}\int_a^b (b-s)^{q-1}\,\varphi\big(s,\vartheta(s)\big)ds - \frac{\epsilon}{\xi+\omega}\right|^p$$

It is readily seen that

$$\left|\frac{\omega}{(\xi+\omega)\,\Gamma(q)}\int_a^b (b-s)^{q-1}\,\varphi\big(s,\vartheta(s)\big)ds - \frac{\epsilon}{\xi+\omega}\right|^p \quad \text{is Lebsegue integrable.}$$

Thus, we set

$$(F(x))^p := \left(\frac{1}{\Gamma(q)}\int_a^x (x-s)^{q-1}\,\varphi\big(s,\vartheta(s)\big)ds\right)^p$$

It follows by virtue of Hölder's inequality and $\varphi\big(x,\mathcal{U}(x)\big), (x-s)^{q-1} \in L_p(a,b)$ yields

$$\big(F(x)\big)^p = \left(\frac{1}{\Gamma(q)}\int_a^x (x-s)^{q-1}\,\varphi\big(s,\mathcal{U}(s)\big)ds\right)^p$$

$$\leq \left(\left[\frac{1}{\Gamma(q)}\int_a^x (x-s)^{\mathfrak{q}(q-1)}\,ds\right]^{\frac{1}{\mathfrak{q}}}\left[\int_a^x \varphi^p\big(s,\mathcal{U}(s)\big)ds\right]^{\frac{1}{p}}\right)^p$$

$$\leq \left(\frac{1}{\Gamma(q)}\int_a^x (x-s)^{\mathfrak{q}(q-1)}\,ds\right)^{\frac{p}{\mathfrak{q}}}\int_a^x \varphi^p\big(s,\mathcal{U}(s)\big)ds$$

$$\leq \frac{(x-a)^{pq-1}}{\left(\frac{pq-1}{p-1}\right)^{p-1}}\int_a^x \varphi^p\big(s,\mathcal{U}(s)\big)ds$$

For all $x \in [a,b]$, by virtue of Proposition 4.i, yields

$$\int_a^x \left[\int_a^s \varphi^p\big(\theta,\mathcal{U}(\theta)\big)d\theta\right]ds = \int_a^x \left[\int_a^s (s-\theta)^{1-1}\varphi^p\big(\theta,\mathcal{U}(\theta)\big)d\theta\right]ds$$

$$= \int_a^x \left[{}_a^xD^{-1} \varphi^p\big(s, \mathcal{U}(s)\big) \right] ds = {}_a^xD^{-2} \varphi^p$$

By virtue of Definition 1, we get

$$\int_a^x \left[\int_a^s \varphi^p\big(\theta, \mathcal{U}(\theta)\big) d\theta \right] ds = \frac{1}{\Gamma(2)} \int_a^x (x-s) \varphi^p\big(s, \mathcal{U}(s)\big) ds$$

For $a \leq x$

$\leq b,$ we get by virtue of Proposition 3 that $\int_a^x \varphi^p$ Lebesgue integrable

If, on the other hand, that there exists a Lebesgue integrable function $\Omega(x)$ on the interval $[a,b]$ such that $|\varphi(x)| \leq \Omega(x)$almost everywhere on $[a,b]$, here $\varphi(x)$ is measurable. Thus, $\varphi(x)$ Lebesgue integrable function.This, in turn, implies that $(F(x))^p$ Lebesgue integrable. Then we conclude that $|G(x)|^p$ is Lebesgue integrable which, in turn, yields Ψ maps every function $\vartheta \in L_p(a,b)$ into a function which belongs to $L_p(a,b)$. We only need to prove Ψ is a contraction mapping. To this end, Taking $\vartheta_1, \vartheta_2 \in L_p(a,b)$, we obtain

$$\left\| \Psi\big(\vartheta_1(x)\big) - \Psi\big(\vartheta_2(x)\big) \right\|_p^p$$

$$= \left\| \frac{1}{\Gamma(q)} \int_a^x (x-s)^{q-1} \left(\varphi(s, \vartheta_1(s)) - \varphi\big(s, \vartheta_2(s)\big)\right) ds \right.$$
$$\left. - \frac{\omega}{(\xi+\omega)\,\Gamma(q)} \int_a^b (b-s)^{q-1} \left(\varphi(s, \vartheta_1(s)) - \varphi\big(s, \vartheta_2(s)\big)\right) ds \right\|_p^p$$

$$= \int_a^b \left| \frac{1}{\Gamma(q)} \int_a^x (x-s)^{q-1} \left(\varphi(s, \vartheta_1(s)) - \varphi\big(s, \vartheta_2(s)\big)\right) ds \right.$$
$$\left. - \frac{\omega}{(\xi+\omega)\,\Gamma(q)} \int_a^b (b-s)^{q-1} \left(\varphi(s, \vartheta_1(s)) - \varphi\big(s, \vartheta_2(s)\big)\right) ds \right| dx$$

$$\leq \left(\frac{2}{\Gamma(q)}\right)^p \int_a^b \left(\int_a^x (x-s)^{q-1} \left|\varphi\big(s, \vartheta_1(s)\big) - \varphi\big(s, \vartheta_2(s)\big)\right| ds \right)^p dx$$

$$+\left(\frac{2\omega}{(\xi+\omega)\,\Gamma(q)}\right)^{p}\int_{a}^{b}\left(\int_{a}^{b}(b-s)^{q-1}\,|\varphi(s,\vartheta_1(s))-\varphi(s,\vartheta_2(s))|ds\right)^{p}dx$$

We note that $\varphi(x,\mathcal{U})$ satisfy the Lipschitz condition on $\mathfrak{D}$ with respect to $\mathcal{U}$. Hence,

$$\|\Psi(\vartheta_1(x))-\Psi(\vartheta_2(x))\|_p^p$$

$$\leq\left(\frac{2\mathcal{C}}{\Gamma(q)}\right)^{p}\int_{a}^{b}\left(\int_{a}^{x}(x-s)^{q-1}\,|\vartheta_1(s)-\vartheta_2(s)|ds\right)^{p}dx$$

$$+\left(\frac{2\mathcal{C}\omega}{(\xi+\omega)\,\Gamma(q)}\right)^{p}\int_{a}^{b}\left(\int_{a}^{b}(b-s)^{q-1}\,|\vartheta_1(s)-\vartheta_2(s)|ds\right)^{p}dx$$

We used the fact that $\vartheta_1,\vartheta_2\in L_p(a,b)$ and L_p is a linear space, we clearly have, $|\vartheta_1\vartheta_2|,(x-s)^{q-1},(b-s)^{q-1}\in L_p(a,b)$. Once again, by virtue of Hölder's inequality. we get

$$\|\Psi(\vartheta_1(x))-\Psi(\vartheta_2(x))\|_p^p$$

$$\leq\left(\frac{2\mathcal{C}}{\Gamma(q)}\right)^{p}\int_{a}^{b}\left(\left[\int_{a}^{x}(x-s)^{\mathscr{q}(q-1)}\,ds\right]^{\frac{1}{\mathscr{q}}}\left[\int_{a}^{x}|\vartheta_1(s)-\vartheta_2(s)|^{p}\,ds\right]^{\frac{1}{p}}\right)^{p}dx$$

$$+\left(\frac{2\mathcal{C}\omega}{(\xi+\omega)\,\Gamma(q)}\right)^{p}\int_{a}^{b}\left(\left[\int_{a}^{b}(b-s)^{\mathscr{q}(q-1)}\,ds\right]^{\frac{1}{\mathscr{q}}}\left[\int_{a}^{b}|\vartheta_1(s)-\vartheta_2(s)|^{p}\,ds\right]^{\frac{1}{p}}\right)^{p}dx$$

$$\leq\left(\frac{2\mathcal{C}}{\Gamma(q)}\right)^{p}\int_{a}^{b}\left[\frac{(x-a)^{pq-1}}{\left(\frac{pq-1}{p-1}\right)^{p-1}}\int_{a}^{x}|\vartheta_1-\vartheta_2|^{p}\,ds\right]dx$$

$$+\left(\frac{2\mathcal{C}\omega}{(\xi+\omega)\,\Gamma(q)}\right)^{p}\int_{a}^{b}\left[\frac{(b-a)^{pq-1}}{\left(\frac{pq-1}{p-1}\right)^{p-1}}\int_{a}^{b}|\vartheta_1-\vartheta_2|^{p}\,ds\right]dx$$

$$\leq\left(\frac{2\mathcal{C}}{\Gamma(q)}\right)^{p}\left[\frac{p-1}{pq-1}\right]^{p-1}\int_{a}^{b}(x-a)^{pq-1}\,\Phi(x)dx$$

$$+\left(\frac{2\mathcal{C}\omega}{(\xi+\omega)\,\Gamma(q)}\right)^{p}\left[\frac{p-1}{pq-1}\right]^{p-1}\int_{a}^{b}\left[(b-a)^{pq-1}\int_{a}^{b}|\vartheta_1-\vartheta_2|^{p}ds\right]dx$$

Where $\Phi(x)=\int_{a}^{x}|\vartheta_1-\vartheta_2|^{p}\,ds,$

Applying simple calculations yields

$$\left\|\Psi(\vartheta_1(x))-\Psi(\vartheta_2(x))\right\|_{p}^{p}$$

$$\leq\left(\frac{2\mathcal{C}}{\Gamma(q)}\right)^{p}\left[\frac{p-1}{pq-1}\right]^{p-1}\left(\frac{(b-a)^{pq}}{pq}\int_{a}^{b}|\vartheta_1-\vartheta_2|^{p}dx-\int_{a}^{b}\frac{(x-a)^{pq}}{pq}\Phi'(x)dx\right)$$

$$+\left(\frac{2\mathcal{C}\omega}{(\xi+\omega)\,\Gamma(q)}\right)^{p}\left[\frac{p-1}{pq-1}\right]^{p-1}(b-a)^{pq-1}\int_{a}^{b}\left[\int_{a}^{b}|\vartheta_1-\vartheta_2|^{p}\,ds\right]dx$$

It is readily seen that $\int_{a}^{b}\frac{(x-a)^{pq}}{pq}\Phi'(x)dx\geq 0$, implies

$$\left\|\Psi(\vartheta_1(x))-\Psi(\vartheta_2(x))\right\|_{p}^{p}\leq\left(\frac{2\mathcal{C}}{\Gamma(q)}\right)^{p}\left[\frac{p-1}{pq-1}\right]^{p-1}\frac{(b-a)^{pq}}{pq}\|\vartheta_1-\vartheta_2\|_{p}^{p}$$

$$+\left(\frac{2\mathcal{C}\omega}{(\xi+\omega)\,\Gamma(q)}\right)^{p}\left[\frac{p-1}{pq-1}\right]^{p-1}(b-a)^{pq}\|\vartheta_1-\vartheta_2\|_{p}^{p}$$

Finally, we have

$$\left\|\Psi(\vartheta_1(x))-\Psi(\vartheta_2(x))\right\|_{p}$$

$$\leq\frac{2\mathcal{C}}{\Gamma(q)}\left(\left[\frac{p-1}{pq-1}\right]^{p-1}(b-a)^{pq}\left[\frac{1}{pq}+\left(\frac{\omega}{(\xi+\omega)}\right)^{p}\right]\right)^{\frac{1}{p}}\|\vartheta_1-\vartheta_2\|_{p}$$

so if we denote $\frac{2\mathcal{C}}{\Gamma(q)}(b-a)^q\left[\frac{p-1}{pq-1}\right]^{1-\frac{1}{p}}\sqrt[p]{\frac{1}{pq}+\left(\frac{\omega}{(\xi+\omega)}\right)^p}$ simply by λ, then the above inequality reads

$$\left\|\Psi\big(\vartheta_1(x)\big)-\Psi\big(\vartheta_2(x)\big)\right\|_p \le \lambda\,\|\vartheta_1-\vartheta_2\|_p$$

we observe that Ψ is a contraction mappingand thus it has only one fixed point, $\Psi\,\mathcal{U}(x)=\mathcal{U}(x)$.

This concludes our proof.

□

For the sake of completeness, we consider explicit example of fractional differential equations for the above theorem.

Example

$$Given\ \ D^{0.7}\mathcal{U}(x) = \frac{\mathrm{e}^{2x}\mathcal{U}^2}{55.5(1+x^2)}\ with\ boundary\ condition\ \xi\mathcal{U}(0)+\omega\mathcal{U}(1) = \epsilon, and\ the\ space$$

$L_p(a,b)$, *Choosing* $a=0,\ b=1,\ \epsilon=1,\ \xi=1,\ \omega=1,\ p=2,$

Next, for all $(x,\mathcal{U}_1),(x,\mathcal{U}_2)\in\mathfrak{D}$, *we have*:

$$|\varphi(x,\mathcal{U}_1)-\varphi(x,\mathcal{U}_2)| = \frac{\mathrm{e}^{2x}\mathcal{U}_1^2}{55.5(1+x^2)}-\frac{\mathrm{e}^{2x}\mathcal{U}_2^2}{55.5(1+x^2)} \le \frac{1}{15}|\mathcal{U}_1-\mathcal{U}_2|$$

Hence,λandwe can applying simple calculations yields

$$\lambda=\frac{2\mathcal{C}}{\Gamma(q)}(b-a)^q\left[\frac{p-1}{pq-1}\right]^{1-\frac{1}{p}}\sqrt[p]{\frac{1}{pq}+\left(\frac{\omega}{(\xi+\omega)}\right)^p}<1$$

And

$$\mathcal{U}(x)=\frac{1}{\Gamma(q)}\int_0^x(x-s)^{-0.3}\frac{\mathrm{e}^{2s}\,\mathcal{U}(s)^2}{55.5(1+s^2)}ds - \frac{1}{2\Gamma(q)}\int_0^1(1-s)^{-0.3}\frac{\mathrm{e}^{2s}\,\mathcal{U}(s)^2}{55.5(1+s^2)}ds+\frac{1}{2}$$

llowsfrom the above Theorem. The function $\mathcal{U}(x)$is the only solution for

$$D^{0.7}\mathcal{U}(x) = \frac{\mathrm{e}^{2x}\mathcal{U}^2}{55.5(1+x^2)}\ \text{ which satisfies }\ \mathcal{U}(0)+\mathcal{U}(1)=1.$$

References

1. Barrett, J. H.: ‘‘*Differential Equations of Noninteger Order*’’ Canad. J. Math. **6**, 529–541 (1954)
2. Bassam, A. M.: ‘‘*Some Existence Theorems on Differential Equations of Generalized Order*’’ (presented to mathematical association of America – Texas section) , April, 10 (1964)

3. Khalida Aissani, Mouffak Benchohra: ‘‘*Fractional Integro-Differential Equations with State-Dependent Delay*’’ Advances in Dynamical Systems and Applications Vol. 9 No. 1, 17–30 (2014)
4. Kilbas, A.A., Hari, M.S., and Juan, J. T.: ‘‘*Theory and Applications of Fractional differential Equations*’’ North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam (2006)
5. Miller, S.K., Ross, B.: ‘‘*An Introduction to the Fractional Calculus and Fractional Differential Equation*’’ A Wiley–Interscience publication, John Wiley &Sons, New York, USA., (1993)
6. Momani, S.M.: ‘‘*Local and global uniqueness theorems on Differential Equations of Non-integer order via Bihari's and Gronwall's inequalities*’’ Revista Twcnica j., 23(1) , 66–69 (2000)
7. Mouffak Benchohra, Samira Hamani, Sotiris K. Ntouyas.: ‘‘*Boundary Value Problems for Differential Equations with Fractional Order*’’ Surveys Mathematics and its Applications Vol. **3**, 1–12 (2008)
8. Rawashdeh, E.A.: ‘‘*Numerical solution of fractional integro-differential equations by collocation method*’’ Applied Mathematics and Computation, 176, 1–6 (2006)
9. Royden, H.L.: ‘‘*Real Analysis*’’ Prentice-Hall of India Private Limited, New Delhi-110001 (2005)
10. Zhenyu Guo, Min Liu: ‘‘*Mild Solutions For A System of Fractional Order differential Equations With Nonlocal Conditions*’’ International Journal of Pure and Applied Mathematics Vol. 80 No. 3, 305–313 (2012)

م.م. منذر سلمان جميغ اللامي
وزارة التربية - المديرية العامة للتربية في محافظة بابل - ثانوية النجوم للبنين

الخلاصة

الغرض من هذا البحث هو دراسة مبدأ التطبيق الانكماشي لإثبات النتائج الرئيسية المتعلقة بنظرية الوجود المحلية للمعادلات التفاضلية الكسرية في فضاء $L_p(a\,,b)$ حيث $1 \le p < \infty$.

كلمات البحث الوجود. مبدأ التطبيق الانكماشي. المشتقة الكسرية. النقطة الثابتة.