Applications of Spectral Collocation Methods in Fractional Equation

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Abstract:

To build efficient and accurate discrete methods (SM) and apply them to a class of fractional differential equations (FDEs), we present the basic properties of generalized Jacobi functions (GJFs). In particular, it has been shown that GJFs allow us to easily compute stiffness matrices, solving the major single limit of a general class of FDEs.

KEYWORDS: "fractional differential equations; generalized Jacobi functions; spectral method; Petrov-Galerkin method"

1- Introduction

In recent years, (FDEs) has received great attention because of its ability to model certain processes that we cannot describe in a suitable framework using the usual (PDEs). There are two main difficulties to dealing with FDEs:

I. FDs They are non-local operators;

II. FDs Kernel/weight functions involve singularity, and solutions of FDEs are usually weakly singular near the boundary and at Primary time. Hence, a direct extension of usual polynomial numerical methods to FDEs is not efficient because it usually involves dense matrices even for the simplest FDEs and suffers from a low convergence rate due to weak singularities. Thus, one needs to develop non-standard and non-polynomial based numerical methods to effectively deal with the difficulties associated with FDEs.

In this department, Using GJFs, we will focus on some of the SM that were developed to solve FDEs. Some pioneering work has been carried out on using SMs to solve FDEs [16,17]. However, the usual polynomial methods used in these papers are used

Rough estimates do not specifically address the two difficulties mentioned above. A major advance was made in [25] where the authors introduced so-called multifractals, which are eigenfunctions of some fractional SL operator. It results in consistent matrices for some simple model equations, so using them as basis functions greatly simplifies the calculation FDs. Furthermore, one can choose suitable parameters in poly-fractonomials so that its leading term is in fact the leading singular term of the corresponding FDE.

Thus, the SM using poly-fractonomials can also resolve the leading singularity in a FDE. It turns out that the poly-fractonomials introduced in [24] coincide, within certain parameter range, with the GJFs introduced in [10]. In [6], the authors reexamined the GJFs in the context of FDEs and derived optimal approximation results with norms suitable for FDs. In particular, it is shown in [6] that well-constructed SMs using GJFs for some typical FDEs can lead to exponential convergence despite the fact that their solutions are weakly singular.

Thanks to the aforementioned remarkable properties of the poly-fractonomials and GJFs with regards to FDs and FDEs, there is now a significant number of recent work on using the poly-fractonomials and GJFs for different kind of FDEs and singular integral equations, including, for examples:

- spectral-Galerkin or spectral Petrov-Galerkin methods [23,14,27,21,19,15];
- SCM s [26,27,15,14,12,25,13];
- DG spectral-element methods [21].

The aim of this thesis is not to provide an exhaustive review of all developments with regards to SMs for FDEs, rather, it aims to present essential properties of the GJFs, including relations to the FDs and their approximation results in properly weighted SSs, and how one can use them to construct efficient and accurate SMs for FDEs.

In order to illustrate the idea and advantage of SMs using GJFs, we shall consider the following FDE:

$$\begin{cases} \gamma^{C}D_t^{\alpha}u(x,t) - p^{R}D_x^{\beta}u(x,t) - (1-p)\,{}_x^RD^{\beta}u(x,t) = f(x,t) \\ u(x,t)|_{\partial\Lambda} = 0,\ \forall t \in I := (0,T) \\ u(x,0) = u_0(x),\ \forall x \in \Lambda = (a,b) \end{cases} \tag{1}$$

where $0 < \alpha < 1, 1 < \beta < 2, p \in [0,1]$; ${}^CD_t^{\alpha}$, ${}^RD_x^{\beta}$ and ${}_x^RD^{\beta}$ are the CFD operator, left and right- RLFD operator, respectively. More precisely, we shall consider, successively, the following four cases:

a) $\gamma = 0, p = 1$: an IVP and a BVP with one-sided FD;

b) $\gamma = 0, p = 1/2$: Riesz FDE;

c) $\gamma = 0, p \neq 0,1/2,1$: a BVP with general two-sided FDs; and (iv) $\gamma \neq 0$ and $p = 1/2$: a space-time fractional diffusion equation.

The rest of the section is organized as follows. In the next section, we introduce the GJFs corresponding to the one-sided FD, RD and two-sided FDs with different coefficients; summarize their special properties, particularly with regards to FDs, and present approximation results using these GJFs. In next Section, we construct successively efficient SMs using GJFs for the four cases above, and derive their error estimates. We provide some concluding remarks in the last section.

Due to the space constraint, we shall not present numerical results in this thesis. However, theoretical results presented in this thesis have been validated by ample numerical experiments in the correspondingly cited papers.

2- GJFs and their approximation properties

In this section, we collect some basic relations and approximation properties of GJFs with respect to FDs from [19,20,21]. These results will play essential roles in developing efficient algorithms for FDEs and in deriving the corresponding error estimates in the next section.

2- FIs and FDs

We introduce below the definitions of FIs and FDs. Let $a < b$ and denote $\Lambda = (a, b)$.

Definition (One-sided FIs and FDs [8,22]). For $\in \mathbb{R}^+$, the left and right FIs are respectively defined as

$$
\begin{aligned}
I_x^\rho v(x) &= \frac{1}{\Gamma(\rho)} \int_a^x \frac{v(y)}{(x-y)^{1-\rho}} dy,\ x \in \Lambda, \\
{}_x I^\rho v(x) &= \frac{1}{\Gamma(\rho)} \int_x^b \frac{v(y)}{(y-x)^{1-\rho}} dy,\ x \in \Lambda,
\end{aligned} \tag{2}
$$

where $\Gamma(\cdot)$ is the usual GF.

For $s \in [k-1, k)$ with $k \in \mathbb{N}$, the left-sided RLFD of order s is defined by

$$
{}^R D_x^s v(x) = \frac{1}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_a^x \frac{v(y)}{(x-y)^{s-k+1}} dy,\ x \in \Lambda \tag{3}
$$

and the right-sided RLFD of order s is defined by

$$
{}_x^R D^s v(x) = \frac{(-1)^k}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_x^b \frac{v(y)}{(y-x)^{s-k+1}} dy,\ x \in \Lambda. \tag{4}
$$

For $s \in [k-1, k)$ with $k \in \mathbb{N}$, the left-sided CFDs of order s is defined by

$$
{}^C D_x^s v(x) := \frac{1}{\Gamma(k-s)} \int_a^x \frac{v^{(k)}(y)}{(x-y)^{s-k+1}} dy,\ x \in \Lambda, \tag{5}
$$

and the right-sided CFDs of order s is defined by

$$
{}_x^C D^s v(x) := \frac{(-1)^k}{\Gamma(k-s)} \int_x^b \frac{v^{(k)}(y)}{(y-x)^{s-k+1}} dy,\ x \in \Lambda. \tag{6}
$$

According to [8], we have that for any absolutely integrable function v, and real $s \geq 0$

$$
{}^R D_x^s I_x^s v(x) = v(x),\ {}_x^R D^s\, {}_x I^s v(x) = v(x),\ \text{a.e. in } \Lambda. \tag{7}
$$

The following lemma shows the relationship between the RLFD and CFDs ([8,22]).

3- GJFs for one-sided FDs

We start by considering the one-sided FDs. Unless otherwise specified, we set $\Lambda = (-1,1)$.

JPs and Bateman's FI formula

According to [23], the classical JPs with parameters $\alpha, \beta \in \mathbb{R}$ can be defined by

$$
P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) \tag{8}
$$

where ${}_2F_1(a,b;c;x)$ is the hypergeometric function, and the rising factorial in the Pochhammer symbol, for $a \in \mathbb{R}$ and $j \in \mathbb{N}_0$, is defined by

$$(a)_0 = 1;\ (a)_j := a(a+1)\cdots(a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)},\ \text{for } j \geq 1. \tag{9}$$

For $\alpha, \beta > -1$, the classical JPs are orthogonal with respect to the Jacobi weight function: $\omega^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, namely,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_{n'}^{(\alpha,\beta)}(x)\omega^{(\alpha,\beta)}(x)dx = \gamma_n^{(\alpha,\beta)}\delta_{nn'} \tag{10}$$

where $\delta_{nn'}$ is the Dirac Delta symbol, and the normalization constant is given by

$$\gamma_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\,\Gamma(n+\alpha+\beta+1)}. \tag{11}$$

The following FI formula of hyper geometric functions due to Bateman [4] (also see [3]) plays an important role in the computation of FIs and FDs: for real $c, \rho \geq 0$,

$$\begin{aligned} {}_2F_1(a,b;c+\rho;x) &= \frac{\Gamma(c+\rho)}{\Gamma(c)\Gamma(\rho)} x^{1-(c+\rho)} \int_0^x t^{c-1}(x \\ &- t)^{\rho-1}\, {}_2F_1(a,b;c;t)dt,\ |x| < 1. \end{aligned} \tag{12}$$

4- Some important properties of GJFs

For $\alpha, \beta > -1$,

$$\begin{aligned} &\int_{-1}^{1} +_n^{(-\alpha,\beta)}(x)^{+}J_{n'}^{(-\alpha,\beta)}(x)\omega^{(-\alpha,\beta)}(x)dx \\ &= \int_{-1}^{1} -J_n^{(\alpha,-\beta)}(x)^{-}J_{n'}^{(\alpha,-\beta)}(x)\omega^{(\alpha,-\beta)}(x)dx = \gamma_n^{(\alpha,\beta)}\delta_{nn'}, \end{aligned} \tag{13}$$

Similarly, we have that for $\alpha > -1$ and $k \in \mathbb{N}$, and $n, n' \geq k$,

$$\begin{aligned} &\int_{-1}^{1} {}^{+}J_n^{(-\alpha,-k)}(x)^{+}J_{n'}^{(-\alpha,-k)}(x)\omega^{(-\alpha,-k)}(x)dx \\ &= \int_{-1}^{1} -J_n^{(-k,-\alpha)}(x)^{-}J_{n'}^{(-k,-\alpha)}(x)\omega^{(-k,-\alpha)}(x)dx = \gamma_n^{(\alpha,-k)}\delta_{nn'}, \end{aligned} \tag{14}$$

where $\gamma_n^{(\alpha,\beta)}$ and $\gamma_n^{(\alpha,-k)}$ are some constants which can be found in [6].

Approximation by the one-sided GJFs

We show below that approximation by GJFs can lead to truly spectral convergence for functions in properly weighted SSs involving FDs. For simplicity of presentation, we only provide the results $\left\{-J_n^{(\alpha,-\beta)}\right\}$. Similar results can be established for $\left\{\ {}^{+}J_n^{(-\alpha,\beta)}\right\}$[6].

Let $\mathcal{P}_N$ be the set of all algebraic (real-valued) polynomials of degree at most N. Let $\varpi(x) > 0, x \in \Lambda$, be a generic weight function. The weighted space $L^2_{\varpi}(\Lambda)$ is defined as in Adams [2] with the inner product and norm

$$(u,v)_{\varpi} = \int_{\Lambda} u(x)v(x)\varpi(x)dx, \ \| u \|_{\varpi} = (u,u)^{1/2}_{\varpi}.$$

If $\varpi \equiv 1$, we omit the weight function in the notation.

We define the finite-dimensional fractional-polynomial space:

$${}^{-}\mathcal{F}_N^{(\alpha,-\beta)}(\Lambda) = \left\{\phi = (1+x)^{\beta}\psi : \psi \in \mathcal{P}_N\right\} = \operatorname{span}\left\{J_n^{(\alpha,-\beta)} : 0 \le n \le N\right\},$$

By the orthogonality, we can expand any $u \in L^2_{\omega(\alpha,-\beta)}$ as

$$u(x) = \sum_{n=0}^{\infty} \hat{u}_n^{(\alpha,-\beta)-}J_n^{(\alpha,-\beta)}(x) \tag{15}$$

where

$$\hat{u}_n^{(\alpha,-\beta)} = \frac{1}{\gamma_n^{(\alpha,\beta)}}\int_{-1}^{1} u^{-}J_n^{(\alpha,-\beta)}\omega^{(\alpha,-\beta)}dx,$$

and there holds the Parseval identity:

$$\| u \|^2_{\omega(\alpha,-\beta)} = \sum_{n=0}^{\infty} \gamma_n^{(\alpha,\beta)} \left|\hat{u}_n^{(\alpha,-\beta)}\right|^2. \tag{16}$$

Consider the $L^2_{\omega^{(\alpha,-\beta)}}$-orthogonal projection onto ${}^{-}\mathcal{F}_N^{(\alpha,-\beta)}(\Lambda)$:

$$\left({}^{-}{}_N^{(\alpha,-\beta)}u - u, v_N\right)_{\omega(\alpha,-\beta)} = 0, \ \forall v_N \in {}^{-}\mathcal{F}_N^{(\alpha,-\beta)}(\Lambda). \tag{17}$$

Then, it is easy to derive that for any $l \in \mathbb{N}_0$, we have

$$\left({}^{R}D_x^{\beta+l}\left({}^{-}\pi_N^{(\alpha,-\beta)}u - u\right), D^l w_N\right)_{\omega(\alpha+\beta+l,l)} = 0, \ \forall w_N \in \mathcal{P}_N(\Lambda). \tag{18}$$

To describe the projection error, we define

$${}^{-}\mathcal{B}^m_{\alpha,\beta}(\Lambda) := \left\{u \in L^2_{\omega^{(\alpha,-\beta)}}(\Lambda) : {}^{R}D_x^{\beta+l}u \in L^2_{\omega^{(\alpha+\beta+l,l)}}(\Lambda) \text{ for } 0 \le l \le m\right\}. \tag{19}$$

GJFs for RDs

The RDs include both the left- and right-sided FDs so the one-sided GJFs defined above are not suitable. Instead, we define a new class of GJFs:

$$\mathcal{J}_n^{-\mu,-\nu}(x) = (1-x)^{\mu}(1+x)^{\nu}P_n^{(\mu,\nu)}(x), \ \mu,\nu > -1. \tag{20}$$

It can be derived from (10) that the GJFs $\mathcal{J}_n^{-\mu,-\nu}(x)$ are mutually orthogonal:

$$\int_{-1}^{1} \mathcal{J}_n^{-\mu,-\nu}(x)\mathcal{J}_m^{-\mu,-\nu}(x)\omega^{(-\mu,-\nu)}(x) = \gamma_n^{(\mu,\nu)}\delta_{mn} \tag{21}$$

and

$$\mathbb{F}_N^{-\mu,-\nu}(\Lambda) := \left\{\mathcal{J}_n^{-\mu,-\nu}(x) : n = 0,1,\cdots,N\right\}. \tag{22}$$

GJFs for two-sided FDs with different coefficients

For $1 < \beta < 2, 0 < \mu, \nu < \beta, \mu + \nu = \beta, 0 \leq p \leq 1$ and $p \neq 1/2$, we define the two-sided FI operator

$$\mathcal{I}_p^{\mu,\nu,\rho} := C_{\beta,p}\left(p I_x^{\rho} + (1-p)_x I^{\rho}\right), \tag{23}$$

where

$$C_{\beta,p} := C(\beta,\mu,\nu) = \frac{\sin(\pi\mu) + \sin(\pi\nu)}{\sin(\pi\beta)}.$$

Then for $s \in (k-1, k)$, we define the two-sided FD operator

$$\mathcal{D}_p^{\mu,\nu,s} := \frac{d^k}{dx^k}\mathcal{I}_p^{\mu,\nu,k-s}. \tag{24}$$

It turns out that suitable basis functions for dealing with two-sided FDEs with different coefficients are of the form $\mathcal{J}_n^{-\mu,-\nu}(x)$ where (μ,ν) are determined as follows ([9,20]) :

Theorem Given (p,β) such that $0 \leq p \leq 1$ and $1 < \beta < 2$, let (μ,ν) be determined from

$$\mu + \nu = \beta, \;\; p\sin(\pi\mu) = (1-p)\sin(\pi\nu), \tag{25}$$

then for $n = 0,1,2,\cdots$, it holds that

$$\mathcal{I}_p^{\mu,\nu,2-\beta}\mathcal{J}_n^{-\mu,-\nu}(x) = 4\frac{\Gamma(n+\beta-1)}{n!}P_{n+2}^{(\nu-2,\mu-2)}(x) \tag{26}$$

and for $k = 1,2,\cdots, n+2$,

$$\mathcal{D}_p^{\mu,\nu,k+\beta-2}\mathcal{J}_n^{-\mu,-\nu}(x) = \frac{\Gamma(n+k+\beta-1)}{2^{k-2}n!}P_{n+2-k}^{(\nu-2+k,\mu-2+k)}(x). \tag{27}$$

In particular, for $k = 2$,

$$\mathcal{D}_p^{\mu,\nu,\beta}\mathcal{J}_n^{-\mu,-\nu}(x) = \frac{\Gamma(n+\beta+1)}{n!}P_n^{(\nu,\mu)}(x). \tag{28}$$

For the sake of simplicity, we shall denote $\mathcal{I}_p^{\mu,\nu,\rho}$ and $\mathcal{D}_p^{\mu,\nu,\rho}$ by $\mathcal{I}_p^{\rho}$ and $\mathcal{D}_p^{\rho}$, respectively.

5- Spectral method for fractional differential equation based generalized Jacobi function

In this section, we present SMs using the GJFs defined in the last section to solve several typical FDEs.

We consider first a fractional IVP, followed by a fractional BVP with one-sided FD.

As the first example, we consider the fractional IVP of order $s \in (0,1)$:

$${}^{C}D_t^s u(t) = f(t),\; t \in I := (0,T);\; u(0) = u_0. \tag{29}$$

For the non-homogeneous ICs $u(0) = u_0$, we first decompose the solution $u(t)$ into two parts as

$$u(t) = u^h(t) + u_0, \tag{30}$$

with $u^h(0) = 0$. By definition, ${}^{C}D_t^s u_0 = 0$, we then derive from (35) that the equation (29) is equivalent to the following equation with RLFD:

$${}^{R}D_t^s u^h(t) = f(t),\; t \in I := (0,T);\; u^h(0) = 0. \tag{31}$$

A Petrov-Galerkin scheme for (31) is: find $u_N^h \in {}^{-}\mathcal{F}_N^{(-s,-s)}(I)$ such that

$$\left({}^{R}D_t^s u_N^h, v_N\right) = (f, v_N),\ \forall v_N \in \mathcal{P}_N(I). \tag{32}$$

We expand f(t) as

$$f(t) = \sum_{n=0}^{\infty} \tilde{f}_n \tilde{P}_n(t), \tag{33}$$

where $\tilde{P}_n(t) := P_n^{(0,0)}(x(t)), x(t) = (2t - T)/T$ is the shifted LPof degree n on I, and write

$$u_N^h(t) = \sum_{n=0}^{N} \tilde{u}_n^{(s)} - \tilde{J}_n^{(-s,-s)}(t) \in {}^{-}\mathcal{F}_N^{(-s,-s)}(I). \tag{34}$$

Taking $v_N = \tilde{P}_l(t)$ in (32), we obtain from the orthogonality of LPs that

$$\tilde{u}_n^{(s)} = \frac{n!}{\Gamma(n+s+1)} \tilde{f}_n,\ 0 \le n \le N. \tag{35}$$

Therefore, we obtain the numerical solution u_N^h without solving any algebraic equation. Hence, the method is very efficient. As for the error estimate, we have the following result [6]:

Theorem Let u^h and u_N^h be the solution of (31) and (32), respectively. Then

$$\| \ {}^{R}D_t^s\left(u^h - u_N^h\right)\| \le cN^{-m}\|f^{(m)}\|_{\omega^{(m-1,m-1)}}. \tag{36}$$

One sided fractional boundary value problem (BVPs)

Now we consider a one-sided fractional BVP

$${}_{x}^{R}D^{\nu} u(x) = f(x),\ x \in \Lambda = (-1,1);\ u(\pm 1) = 0, \tag{37}$$

where $\nu \in (1,2)$

Let $s = \nu - 1$ and introduce the solution and test function spaces:

$$\begin{aligned} U &:= \left\{u \in L^2_{\omega(-s,-1)}(\Lambda) : {}_{x}^{R}D^s u \in L^2_{\omega(0,s-1)}(\Lambda)\right\}; \\ V &:= \left\{v \in L^2_{\omega(-1,-s)}(\Lambda) : Dv \in L^2_{\omega^{(0,1-s)}}(\Lambda)\right\}, \end{aligned} \tag{38}$$

equipped with the norms

$$\begin{aligned} \| u \|_U &= \left(\| u \|^2_{\omega(-s,-1)} + \| {}_{x}^{R}D^s u\|^2_{\omega(0,s-1)}\right)^{1/2}; \\ \| v \|_V &= \left(\| v \|^2_{\omega(-1,-s)} + \| Dv \|^2_{\omega(0,1-s)}\right)^{1/2}. \end{aligned} \tag{39}$$

For $u \in U$ and $v \in V$, we write

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} \hat{u}_n +_n^{(-s,-1)}(x) = (1-x)^s(1+x) \sum_{n=1}^{\infty} \tilde{u}_n P_{n-1}^{(s,1)}(x), \\ v(x) &= \sum_{n=1}^{\infty} \hat{v}_n J_n^{(-1,-s)}(x) = (1-x)(1+x)^s \sum_{n=1}^{\infty} \tilde{v}_n P_{n-1}^{(1,s)}(x). \end{aligned} \tag{40}$$

With the above setup, we can build in the homogenous BCs and also perform FI by parts. Hence, a weak form of (37) is to find $u \in U$ such that

$$a(u,v) := ({}_{x}^{R}D^{s}u, Dv) = (f,v),\ \forall v \in V. \tag{41}$$

Let ${}^{+}\mathcal{F}_{N}^{(-s,-1)}(\Lambda)$ and ${}^{-}\mathcal{F}_{N}^{(-1,-s)}(\Lambda)$ be the finite-dimensional spaces as defined in the previous section. Then the GJF-Petrov-Galerkin scheme for (41) is to find $u_N \in {}^{+}\mathcal{F}_{N}^{(-s,-1)}(\Lambda)$ such that

$$a(u_N, v_N) = ({}_{x}^{R}D^{s}u_N, Dv_N) = (f, v_N),\ \forall v_N \in {}^{-}\mathcal{F}_{N}^{(-1,-s)}(\Lambda). \tag{42}$$

Fractional BVPs with RDs

We consider first the so called Riesz FDEs, followed by a more general case with a 0th-order term.

Riesz FDEs

We consider the Riesz fractional equation of order $2\alpha \in (2k-1, 2k)$ with $\in \mathbb{N}$:

$$\begin{aligned} (-1)^{k}D^{2\alpha}u(x) &= f(x),\ x \in \Lambda, \\ u^{(l)}(\pm 1) &= 0,\ l = 0,1,\cdots,k-1. \end{aligned} \tag{43}$$

A Petrov-Galerkin SM for (43) is: Find $u_N \in \mathbb{F}_{N}^{-\alpha,-\alpha}$ such that

$$(-1)^{k}(D^{2\alpha}u_N, v_N)_{\omega^{(\alpha,\alpha)}} = (f, v_N)_{\omega^{(\alpha,\alpha)}},\ \forall v_N \in P_N$$

The solution to this discrete problem can be found directly as follows. Given

$$f(x) = \sum_{m=0}^{\infty} f_m P_m^{(\alpha,\alpha)}(x) \tag{44}$$

and write

$$u_N(x) = \sum_{n=0}^{N} \hat{u}_n \mathcal{J}_n^{-\alpha,-\alpha}(x). \tag{45}$$

Plugging the above in (43), using the orthogonality of $\left\{P_m^{(\alpha,\alpha)}\right\}$ in $L^2_{\omega(\alpha,\alpha)}(\Lambda)$, we find

$$\hat{u}_n = f_n \Big/ \left(2\cos(\pi\alpha)\frac{\Gamma(n+1+2\alpha)}{n!}\right),\ \forall 0 \le n \le N \tag{46}$$

A more general case:

In the previous examples, we have developed optimal SMs using GJFs in the sense that:

- the numerical solution can be determined directly without solving any algebraic equation; and
- the error converges faster than any algebraic rate as long as the right-hand side function f is smooth, despite the fact that the solution is weakly singular at the endpoint(s).

However, it is not possible to construct such optimal SMs for more general FDEs. Nevertheless, using proper GJFs still allows us to:

a) deal with FDs efficiently, and
b) resolve the leading singular term in the solution. Consider for example

$$\rho u(x) - D^{2\alpha}u(x) = f,\ x \in \Lambda = (-1,1) \qquad (47)$$
$$u(\pm 1) = 0$$

where $\rho > 0$ and $2\alpha \in (1,2)$. A GJF spectral Galerkin approximation to (47) is: find $u_N \in \mathbb{F}_N^{-\alpha,-\alpha}(\Lambda)$ such that

$$a(u_N, v_N) := \rho(u_N, v_N) - (D^{2\alpha}u_N, v_N) = (f, v_N),\ \forall v_N \in \mathbb{F}_N^{-\alpha,-\alpha}(\Lambda). \qquad (48)$$

Setting

$$u_N(x) = \sum_{n=0}^{N} \tilde{u}_n \mathcal{J}_n^{-\alpha,-\alpha}(x) \qquad (49)$$

Setting

$$\begin{aligned} &u_N(x) = \sum_{n=0}^{N} \tilde{u}_n \mathcal{J}_n^{-\alpha,-\alpha}(x), U = (\tilde{u}_0, \tilde{u}_1, \cdots, \tilde{u}_N) \\ &m_{jk} = \left(\mathcal{J}_k^{-\alpha,-\alpha}(x), \mathcal{J}_j^{-\alpha,-\alpha}(x)\right),\ s_{jk} = -\left(D^{2\alpha}\mathcal{J}_k^{-\alpha,-\alpha}(x), \mathcal{J}_j^{-\alpha,-\alpha}(x)\right), \\ &M = (m_{jk}), S = (s_{jk}), \tilde{f}_j = \left(f, \mathcal{J}_j^{-\alpha,-\alpha}(x)\right),\ F = \left(\tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_N\right)^T \end{aligned} \qquad (50)$$

Then, (48) reduces to the following matrix system:

$$(\rho M + S)U = F.$$

S is a diagonal matrix

$$s_{jk} = \frac{2^{2\alpha+1}\Gamma(k+\alpha+1)^2}{(k!)^2(2k+2\alpha+1)}\delta_{jk}. \qquad (51)$$

The mass matrix M is full but its entries can be evaluated exactly ([5]):

$$\begin{aligned} m_{jk} &= \int_{-1}^{1} (1-x^2)^\alpha P_j^{(\alpha,\alpha)}(x)(1-x^2)^\alpha P_k^{(\alpha,\alpha)}(x)dx \\ &= \frac{(-1)^{\frac{j-k}{2}}(j+k)!}{2^{j+k}j!\,k!} \frac{(-j-\alpha)_{\frac{j+k}{2}}(-k-\alpha)^2_{\frac{j+k}{2}}}{\left(2\alpha+\frac{3}{2}\right)_{\frac{j+k}{2}}\frac{j+k}{2}!} \frac{\sqrt{\pi}\Gamma(2\alpha+1)}{\Gamma\left(2\alpha+\frac{3}{2}\right)}, \end{aligned} \qquad (52)$$

where the Pochhammer symbol $(a)_\nu = \frac{\Gamma(a+\nu)}{\Gamma(a)}$.

Two-sided FDEs with different coefficients

We consider the two-sided FDE

$$\begin{aligned} &\mathcal{D}_p^\beta u(x) := C_{\beta,p}\left(p\,{}^R D_x^\beta u(x) + (1-p)\,{}_x^R D^\beta u(x)\right) = f(x),\ x \in \Lambda, \\ &u(\pm 1) = 0, \end{aligned} \qquad (53)$$

where $1 < \beta < 2, 0 \le p \le 1, C_{\beta,p}$, and f(x) is a given function. A Petrov-Galerkin SM for (53) is: Find $u_N \in \mathbb{F}_N^{-\mu,-\nu}$ such that

$$\left(\mathcal{D}_p^\beta u_N, v_N\right)_{\omega^{(\nu,\mu)}} = (f, v_N)_{\omega^{(\nu,\mu)}},\ \forall v_N \in \mathcal{P}_N. \qquad (54)$$

The solution to this discrete problem can be found directly as follows. We expand f(x) as

$$f(x) = \sum_{m=0}^{\infty} f_m P_m^{(\nu,\mu)}(x), \tag{55}$$

and write

$$u_N(x) = \sum_{n=0}^{N} \hat{u}_n \mathcal{J}_n^{-\mu,-\nu}(x). \tag{56}$$

Plugging (56) and (55) using the orthogonality of $\left\{P_m^{(\nu,\mu)}\right\}$ in $L^2_{\omega^{(\nu,\mu)}}(\Lambda)$, we find

$$\hat{u}_n = f_n / \left(\frac{\Gamma(n+1+\beta)}{n!}\right), \ \forall 0 \le n \le N. \tag{57}$$

Space-time FDEs

As the last example, we consider

$$\begin{aligned} &{}^{C}D_t^{\alpha} u(x,t) - D^{2\beta} u(x,t) = f(x,t), \ \forall (x,t) \in Q = \Lambda \times I \\ &u(x,t)|_{\partial\Lambda} = 0, \ \forall t \in I := (0 < T) \\ &u(x,0) = u_0(x) \ \forall x \in \Lambda \end{aligned} \tag{58}$$

where $\alpha \in (0,1), 2\beta \in (1,2)$. To deal with the non-homogeneous IC, we first decompose the solution $u(x,t)$ into two parts as

$$u(x,t) = u^h(x,t) + u_0(x) \tag{59}$$

with $u^h(x,0) = 0$. Hence, the equation (58) is equivalent to the following with RLFD:

$$\begin{aligned} {}^{R}D_t^{\alpha} u^h(x,t) - D^{2\beta} u^h(x,t) &= g(x,t), \ \forall (x,t) \in Q \\ u^h(x,0) &= 0, \ \forall x \in \Lambda, \\ u^h(x,t)\big|_{\partial\Lambda} &= 0, \ \forall t \in I, \end{aligned} \tag{60}$$

where

$$g(x,t) = f(x,t) + D^{2\beta} u_0(x).$$

We consider the following weak formulation of (60): find $u \in X$, such that

$$\mathcal{A}\left(u^h, v\right) := \left({}^{R}D_t^{\alpha} u^h, v\right)_Q - \left(D^{2\beta} u^h, v\right)_Q = (g, v)_Q, \ \forall v \in Y, \tag{61}$$

and describe a space-time SM for solving the above equation.

For the time variable, we shall use the shifted fractional polynomial space.

Let $x(t) = 2t/T - 1$. We denote $\tilde{P}_n^{(\alpha,\beta)}(t) = P_n^{(\alpha,\beta)}(x(t)), -\tilde{J}_n^{(-\alpha,-\alpha)}(t) := t^{\alpha} \tilde{P}_n^{(-\alpha,\alpha)}(t)$, and

$$^{-}\mathcal{F}_N^{(-\alpha,-\alpha)}(I) = \mathrm{span}\left\{ {}^{-}\tilde{J}_n^{(-\alpha,-\alpha)}(t) : 0 \le n \le N \right\}. \tag{62}$$

For the space variable, we shall use $\mathbb{F}_M^{-\beta,-\beta}$ defined in (60). Then, the Petrov-Galerkin method for (60) is: find $u_L^h(x,t) := u_{MN}^h \in \mathbb{F}_M^{-\beta,-\beta} \otimes \mathcal{F}_N^{(-\alpha,-\alpha)}$ such that

$$\mathcal{A}\left(u_L^h, v\right) := \left({}^{R}D_t^{\alpha} u_L^h, v\right)_Q - \left(D^{2\beta} u_L^h, v\right)_Q = (g, v)_Q,\ \forall v \in \mathbb{F}_M^{-\beta,-\beta} \otimes \mathcal{P}_N. \quad (63)$$

We first describe an efficient algorithm for solving (63) similar to the one used in [21]. We write

$$u_L^h(x,t) = \sum_{m=0}^{M} \sum_{n=0}^{N} \tilde{u}_{mn}^h \mathcal{J}_m^{-\beta,-\beta}(x)^{-}\tilde{J}_n^{(-\alpha,-\alpha)}(t). \quad (64)$$

For the test function, we take $v_L = \mathcal{J}_p^{-\beta,-\beta}(x) L_q^{(\alpha)}(t)$ where $L_q^{(\alpha)}(t) := \kappa_{q,\alpha} \tilde{P}_q^{(0,0)}(t)$ with $\kappa_{q,\alpha} = \frac{q!(2q+1)}{T \cdot \Gamma(q+\alpha+1)}$. Substituting the above in into (63), we obtain

$$\sum_{m=0}^{M} \sum_{n=0}^{N} \tilde{u}_{mn}^h \left\{ \left(\mathcal{J}_m^{-\beta,-\beta}, \mathcal{J}_p^{-\beta,-\beta}\right)\left({}^{R}D_t^{\alpha -}\tilde{J}_n^{(-\alpha,-\alpha)}, L_q^{(\alpha)}\right) \right.$$
$$\left. \left(D^{2\beta}\mathcal{J}_m^{-\beta,-\beta}, \mathcal{J}_p^{-\beta,-\beta}\right)\left(-\tilde{J}_n^{(-\alpha,-\alpha)}, L_q^{(\alpha)}\right) \right\} \quad (65)$$
$$= \left(g, \mathcal{J}_p^{-\beta,-\beta} L_q^{(\alpha)}\right)_Q.$$

6- Concluding remarks

We presented in this article essential properties of the GJFs and their application to a class of FDEs. In particular, we showed that (i) by using suitable GJFs, the non-local fractional operators become local operators in the space spanned by GJFs; (ii) for simple FDEs, the SMs using GJFs can lead to exponential convergence rate despite the non-smoothness of the solution in usual SSs; and (iii) for more general FDEs, a suitable SM using GJFs is still very efficient as the non-local fractional stiffness matrices can be easily computed, and furthermore, it is also more accurate than using a usual polynomial based method as the GJFs include the leading singular term of the underlying FDEs.

We only consider 1 dimensional FDEs in this thesis. For multi-dimensional fractional PDEs with only FD in time, one can couple the GJF SM in time with a usual spatial approximation to construct a space-time Petrov-Galerkin method. It can still be efficiently solved by using the matrix diagonalization method as in the last subsection, we refer to [1] for more detail. As for FDEs with multi-dimensional fractional operators in space, one has to construct appropriate numerical methods with respect to the specific definitions of fractional operator. In particular, the GJFs for Riesz equation in one-dimension can be extended to deal with fractional Laplacian by the integral definition on the multidimensional balls [19]. On the other hand, for fractional Laplacian defined through the spectral decomposition of the Laplacian operator, one can use the Caffarelli-Silvestre extension to cast the fractional Laplacian equation in d-dimension into an extended problem in $d+1$-dimension with regular derivatives and a weakly singular weight in the extended direction. Then, one can construct efficient and accurate SM in the extended direction to couple with any consistent approximation in space, for more detail, we refer to [7]. For other types of multidimensional fractional PDEs, we refer to a recent review paper [18] for a nice presentation on different definitions of fractional Laplacian and their numerical treatments.

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