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Odd Lomax Chen distribution: An Innovative Statistical Tool for Improving Real Data Modeling and Its Practical Applications

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Abstract: This paper focused on presenting a statistical distribution consisting of Odd Lomax family and Chen distribution called Odd Lomax Chen (LoCH) distribution as a flexible statistical tool for data analysis. It includes results from simulation and an experimental study to test the distribution's ability to represent real data. The paper includes the presentation of some basic functions and graphs such as the CDF, pdf, survival function, and hazard function, showing the effect of changing the parameters on these functions. The results discuss the use of different estimation methods (MLE, LSE, WLSE) and evaluate their performance using Monte Carlo simulation for different sample sizes. The results show that MLE is the most accurate and stable as the sample size increases. The results shows that LoCh has high flexibility in representation data through low and fixed parameters compared to other distributions. The study also includes a practical application on engineering data showing that LoCh outperforms seven other distributions based on criteria such as AIC, BIC, and some statistical tests. The graphs show that LoCh achieves the best fit with real data, accurately capturing the cumulative features and densities, which confirms that LoCh is suitable for real data analysis due to its flexibility and ability to provide accurate estimates, making it a distinct choice compared to other distributions in the fields of applied data analysis.

توزيع Odd Lomax Chen أداة إحصائية مبتكرة لتحسين نمذجة البيانات الحقيقية وتطبيقاتها العملية

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المستخلص

ركزت هذه الورقة على تقديم توزيع إحصائي يتكون من عائلة Odd Lomax وتوزيع Chen يسمى توزيع Odd Lomax Chen (LoCH) كأداة إحصائية مرنة لتحليل البيانات. ويشمل نتائج المحاكاة ودراسة تجريبية لاختبار قدرة التوزيع على تمثيل البيانات الحقيقية. ويتضمن البحث عرض بعض الدوال والرسوم البيانية الأساسية مثل CDF وpdf ودالة البقاء ودالة الخطر، موضحًا تأثير تغيير المعلمات على هذه الدوال. وتناقش النتائج استخدام طرق تقدير مختلفة (MLE وLSE وWLSE) وتقييم أدائها باستخدام محاكاة مونت كارلو لأحجام عينات مختلفة. وتظهر النتائج أن MLE هو الأكثر دقة واستقرارًا مع زيادة حجم العينة. وتظهر النتائج أن LoCh يتمتع بمرونة عالية في تمثيل البيانات من خلال معلمات منخفضة وثابتة مقارنة بالتوزيعات الأخرى. وتتضمن الدراسة أيضًا تطبيقًا عمليًا على البيانات الهندسية يُظهر أن LoCh يتفوق على سبعة توزيعات أخرى بناءً على معايير مثل AIC وBIC وبعض الاختبارات الإحصائية. تظهر الرسوم البيانية أن توزيع LoCh يحقق أفضل توافق مع البيانات الحقيقية، حيث يلتقط بدقة السمات التراكمية والكثافات، مما يؤكد أن توزيع LoCh مناسب لتحليل البيانات الحقيقية بسبب مرونته وقدرته على تقديم تقديرات دقيقة، مما يجعله خيارًا مميزًا مقارنة بالتوزيعات الأخرى في مجالات تحليل البيانات التطبيقية.

الكلمات المفتاحية: عائلة OL-G، توزيع LoCH، الإحصاء المرن، نمذجة البيانات الهندسية، والاختبارات الإحصائية.

1. Introduction:

In recent years, statistical methods have witnessed a remarkable development in developing new distributions for analyzing complex data. Among these methods, the T-X method is one of the most effective tools for generating new families of statistical distributions. This method relies on transforming the basic distributions using specific mathematical functions such as cumulative functions or probability density functions, with the aim of improving real data. The T-X method allows researchers to design distributions that fit specific characteristics of the data, such as concentration, skewness, and kurtosis (Alzaatreh, Lee, & Famoye, 2013). This method has led to the introduction of multiple families of distributions that have proven their efficiency in many practical applications. Among the families resulting from this method are: MKi-G (Al-Babtain, Shakhatreh, Nassar, & Afify, 2020), GME family (Handique, ul Haq, & Subrata, 2020), MT-X (Aslam, Asghar, Hussain, & Shahc, 2020), MOTL-G family (Khaleel, Oguntunde, Al Abbasi, Ibrahim, & AbuJarad, 2020),, logarithmic family (Wang, Feng, & Zahra, 2021), ITL-H family (Hassan, Al-Omari, Hassan, & Alomani , 2022), SHE-G family (Eghwerido, Agu, & Ibidoja, 2022), NOGEE−G family (Odeyale, Gulumbe, Umar, & Aremu, 2023), WEE-X family (Hussain, Hassan, Rashid, & Ahmed, Families of Extended Exponentiated Generalized Distributions and Applications of Medical Data Using Burr III Extended Exponentiated Weibull Distribution, 2023), GOM-G family (Ishaq, Panitanarak, Alfred , Suleiman, & Daud, 2024), and the OL-G family on which the proposed distribution is based has the CDF, and pdf function respectively as follows (Noori, Khalaf, & Khaleel, 2023):

$$F(x) = 1 - \left(1 - \frac{M(x,\phi).\log\left(1 - M(x,\phi)\right)}{\gamma}\right)^{-\alpha} \tag{1}$$

$$f(x) = \frac{\alpha}{\gamma} m(x,\phi) \left(1 - \frac{M(x,\phi).\log\left(1 - M(x,\phi)\right)}{\gamma}\right)^{-(\alpha+1)} \left[\frac{M(x,\phi)}{1 - M(x,\phi)} - \log\left(1 - M(x,\phi)\right)\right], \quad x, \alpha, \gamma, \phi > 0 \tag{2}$$

Where $M(x,\phi)$, $m(x,\phi)$ are CDF, and pdf for any baseline distribution with random variable x and ϕ is parameter of baseline distribution, and shape parameters α, γ for OL-G family.

Despite the great development in the design of statistical distributions using the T-X method, many of new distributions suffer from limitations in their ability to represent data with complex patterns, especially in areas that require high flexibility such as engineering or environmental data analysis. Most previous studies focused on improving specific models (such as Weibull and Exponential) or providing families of distributions based on limited parameters such as skewness or kurtosis. However, the evaluation of the performance of these distributions in accurately representing real data has remained limited, leaving a gap regarding the development of a more comprehensive model that combines interpretive flexibility with ease of practical application.

This study aims to:

- Provide a comprehensive analysis of LoCH distribution by studying its theoretical properties such as density, survival, and hazard.
- Evaluate the performance of LoCH using Monte Carlo simulation and compare it with other distributions based on the T-X method.
- Test the effectiveness of LoCH when applied to real data, by comparing fit criteria such as AIC and BIC, and some statistical tests.
- Highlight the flexibility of LoCH in representing data, which contributes to the development of advanced statistical tools for data analysis in various fields, such as engineering and environmental sciences.

Achieving these goals will enhance our understanding of how to exploit the T-X method to develop more accurate and suitable distributions for complex data analysis.

The paper structure consists of 6 parts. The first part represents the introduction to the study. the second part includes finding the proposed distribution. the third part includes some statistical properties of LoCH distribution. The fourth part includes estimating the distribution parameters using three different methods. The fifth part incudes simulating the distribution parameters with different sample sizes for three estimation methods, then showing which method is better. The last part includes a practical application of real data to clarify the suitability of LoCH distribution.

2. Odd Lomax Chen (LoCH) distribution

Let X be any random variable then the CDF and pdf functions of Chen distribution has form respectively:

$$M(x) = 1 - e^{\lambda\left(1-e^{x^{\delta}}\right)} \tag{3}$$

$$m(x) = \lambda\delta x^{\delta-1} e^{x^{\delta}} e^{\lambda\left(1-e^{x^{\delta}}\right)} \tag{4}$$

Where $x, \lambda, \delta > 0$, and λ, δ are shape parameters of Chen distribution.

The CDF function of LoCH distribution can be finding by substitute equation (3) in equation (1) to get a form:

$$F(x) = 1 - \left(1 - \frac{\left(1 - e^{\lambda\left(1-e^{x^{\delta}}\right)}\right).\log\left(e^{\lambda\left(1-e^{x^{\delta}}\right)}\right)}{\gamma}\right)^{-\alpha} \tag{5}$$

To give a better idea of the shape of LoCH distribution, the CDF function is drawn with different values for the parameters.

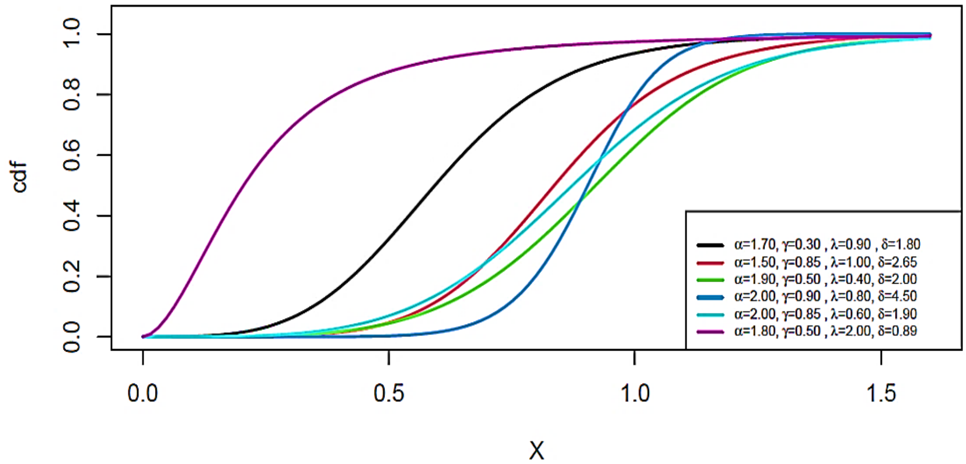


Figure (1): The cdf of LoCH with different value of parameters

Figure 1 shows how the probability of random values falling below a certain value (the cumulative values) changes as the distribution parameters change. The CDF curves indicate that different parameters affect the rate of increase, reflecting the flexibility of the distribution to fit data with different patterns.

While the pdf of LoCH can be finds by substitute equation (3) and (4) in equation (2) to get a form:

$$f(x) = \frac{\alpha}{\gamma}\lambda\delta x^{\delta-1}e^{x^{\delta}}e^{\lambda\left(1-e^{x^{\delta}}\right)}\left(1 - \frac{\left(1-e^{\lambda\left(1-e^{x^{\delta}}\right)}\right).\log\left(e^{\lambda\left(1-e^{x^{\delta}}\right)}\right)}{\gamma}\right)^{-(\alpha+1)}\left[\frac{1-e^{\lambda\left(1-e^{x^{\delta}}\right)}}{e^{\lambda\left(1-e^{x^{\delta}}\right)}} - \log\left(e^{\lambda\left(1-e^{x^{\delta}}\right)}\right)\right] \quad (6)$$

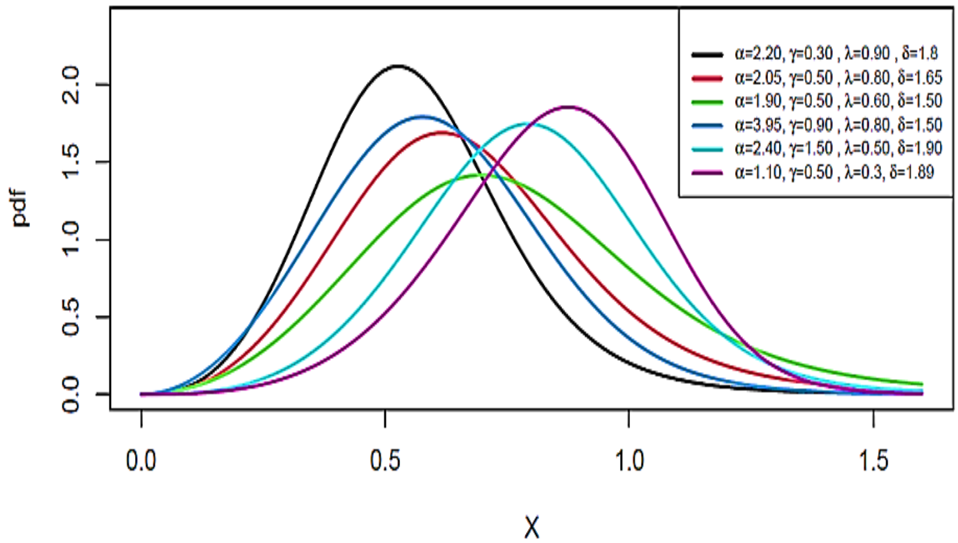


Figure (2): The pdf of LoCH with different value of parameters

This figure shows the probability of random values occurring at certain points. Changes in the shape of the curve reflect the effect of the parameters on the centering and spread of the data. Different peaks indicate the most likely locations of the values.

3. Some Statistical properties of LoCH distribution

3-1. Reliability functions: The survival function $S(x)$ is defined as the probability that an (individual, system, or product) will survive or function efficiently until or after time x. The survival function provides a clear understanding of probability that a particular organism or system will survive for a given period of time, and is useful when comparing the survival of different populations. Its given for LoCH distribution by the formula (Hassan, Almetwally, Khaleel, & Nagy, 2021), (Mahdi, et al., 2024):

$$S(x) = 1 - F(x)$$

$$S(x) = \left(1 - \frac{\left(1 - e^{\lambda\left(1-e^{x^{\delta}}\right)}\right).\, log\left(e^{\lambda\left(1-e^{x^{\delta}}\right)}\right)}{\gamma}\right)^{-\alpha} \qquad (7)$$

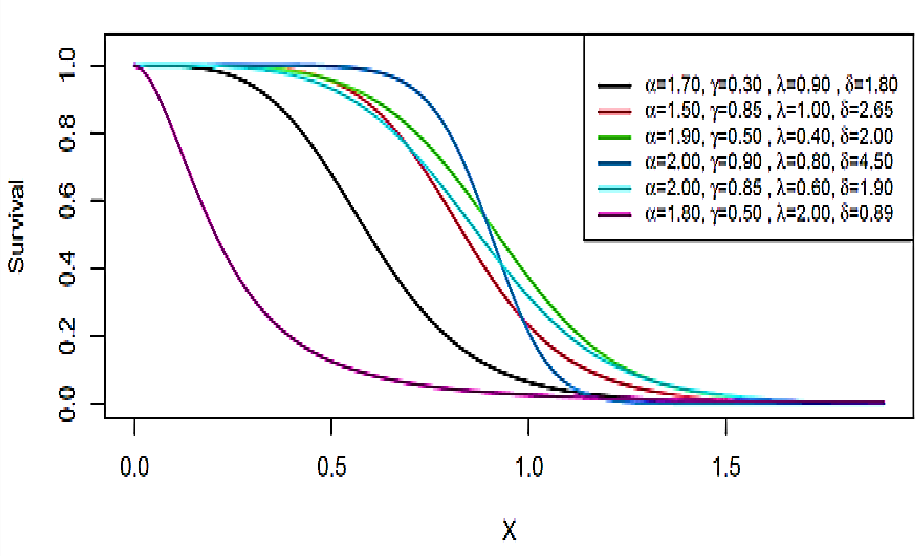


Figure (3): The survival of LoCH with different value of parameters

This figure represents the cumulative probability of values remaining above a certain level. The figure shows the survival function decreasing rapidly or decelerating based on different values of the parameters, enhancing the ability to fit survival data or data with a decreasing nature.

The hazard function $h(t)$ is defined as the instantance rate of occurrence of a given event (such as death or failure) at time t, given that the entity has survived to time t. The hazard function gives insight into how the probability of a failure or event chages over time, which helps in predicting the times of greatest risk. The hazard function given for LoCH distribution by the formula (Bhatti, et al., 2019):

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\frac{\alpha}{\gamma}\lambda\delta x^{\delta-1}e^{x^\delta}e^{\lambda\left(1-e^{x^\delta}\right)}\left[\frac{1-e^{\lambda\left(1-e^{x^\delta}\right)}}{e^{\lambda\left(1-e^{x^\delta}\right)}} - log\left(e^{\lambda\left(1-e^{x^\delta}\right)}\right)\right]}{1-\frac{\left(1-e^{\lambda\left(1-e^{x^\delta}\right)}\right).log\left(e^{\lambda\left(1-e^{x^\delta}\right)}\right)}{\gamma}} \quad (8)$$

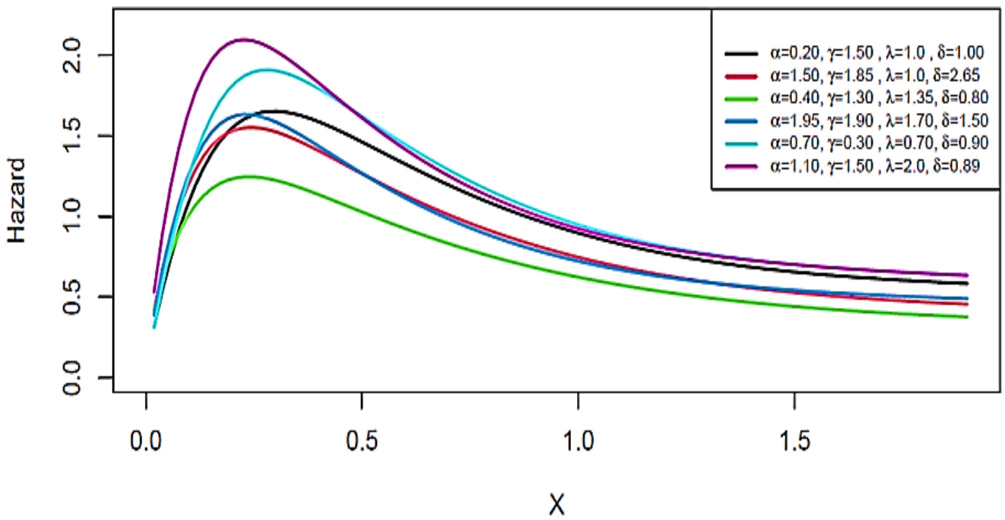


Figure (4): The hazard of LoCH with different value of parameters

From this figure a function reflects the probability of an instantaneous failure or event occurring given the survival to the present moment. The figure highlights the effect of the parameters on the hazard rate, showing constant, increasing or decreasing shapes, making the distribution suitable for risk analysis.

3-2. Useful represent of CDF and pdf functions: The functions in equation (5) and (6) are complex functions and have a degree of difficulty to deal with in proving the statistical properties of LoCH distribution, so the exponential, logarithmic and binomial expansions (Noori, Khalaf, & Khaleel, 2023), are used to simplify the CDF and pdf functions, where the expansion of CDF function is obtained in the form:

$$F(x) = 1 - He^{\omega x^{\delta}} \tag{9}$$

Where $H = \sum_{k=i=r=j=\theta=\rho=0}^{\infty} \frac{\Gamma(\alpha+k)\gamma^{-k}(-1)^{i+j+r+\theta+\omega} d_{k,i}}{K!\rho!\Gamma(\alpha)} \binom{2k+1}{j} \binom{j}{r} \binom{r}{\theta} \binom{\rho}{\omega} \lambda^{\rho} \theta^{\rho}$

And $d_{k,i} = i^{-1} \sum_{m=1}^{i} \frac{[m(k+1)-i]}{m+1}$ for $i \geq 0$ and $d_{k,0} = 1$

And expansion of pdf function has the form:

$$f(x) = \Phi x^{\delta-1} e^{(b+1)x^{\delta}} - \psi x^{\delta-1} e^{(q+1)x^{\delta}} \tag{10}$$

Where

$$\Phi = \sum_{k=i=r=j=z=\tau=\xi=b=0}^{\infty} \frac{(-1)^{i+r+j+z+\tau+b} d_{k,i} \alpha\, \Gamma(\alpha+1+k)\gamma^{-(k+1)}}{k!\,\xi!\,\Gamma(\alpha+1)} \lambda\delta \binom{2k+i}{j}\binom{j}{r}\binom{\xi}{b}\binom{r+z}{\tau}\lambda^{\xi}(\tau+1)^{\xi}$$

$$\psi = \sum_{k=l=s=n=v=\varepsilon=q=0}^{\infty} \frac{\alpha(-1)^{i+s+n+v+q} d_{k+1,i}\Gamma(\alpha+1+k)\gamma^{-(k+1)}}{k!\,\varepsilon!\,\Gamma(\alpha+1)} \lambda\delta \binom{2k+l+1}{s}\binom{s}{n}\binom{\varepsilon}{q}\binom{n}{v}\lambda^{\varepsilon}(v+1)^{\varepsilon}$$

3-3. Quintile function: It is a powerful statistical tool used to understand and analyze the distribution of data. The Quintile function provides an accurate description of distribution by providing values associated with specific parts of distribution (such as percentiles) that are given as the inverse of CDF function ($u = F^{-1}(x)$) (Al Abbasi, et al., 2023), which can be obtained for the LoCH distribution as:

$$x = \left(ln\left\{ 1 - \frac{1}{\lambda} ln\left[1 - \frac{\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}}{\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}} + W_{-1}\left(\left(\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}\right) e^{-\left(\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}\right)} \right)} \right] \right\} \right)^{\frac{1}{\delta}} \quad (11)$$

Where $W_{-1}(x)$ is the secondary branch of Lombard W function, its defined for $-\frac{1}{e} \leq x < 0$, where e is base of natural logarithm $e = 2.718$.

Table 1 shows some value of the Quintile function for different value of parameters.

Table (1): Quintile function values of LoCH for different parameters

u	$(\alpha,\gamma,\lambda,\delta)$				
	(0.8,0.9,0.4,0.7)	(1.6,0.4,0.7,0.5)	(1.5,0.6,0.5,0.6)	(1.7,1.5,1.8,0.7)	(1.7,1.9,1.7,2)
0.1	0.5790286	0.04835048	0.1867966	0.08033148	0.4503836
0.2	0.9152041	0.10029278	0.3306906	0.14078613	0.5476403
0.3	1.2291981	0.15944795	0.4735518	0.20292222	0.6220547
0.4	1.5610559	0.22956062	0.6267960	0.27226625	0.6892102
0.5	1.9449152	0.31628102	0.8012461	0.35460260	0.7558390
0.6	2.4315473	0.42960182	1.0129072	0.45934048	0.8274636
0.7	3.1214674	0.59018537	1.2920461	0.60526802	0.9114012
0.8	4.2385608	0.85108422	1.7133866	0.84048317	1.0222176
0.9	6.3757962	1.42416551	2.5623789	1.34644341	1.2005488

The table shows the relative values of the distribution at specific quintiles (percentile locations). Each row represents a specific location (e.g. 0.1, 0.2...), and shows the effect of different parameters on the location of those quintiles. For example, at $u = 0.1$, the values vary between 0.0483 and 0.579 depending on the parameters. This diversity ensures that the distribution is able to capture variations in the data at different percentile levels.

3-4. Moments functions of LoCH distribution: For any random variable x, then the n^{th} moment function has a form (Al-Habib, Khaleel, & Al-Mofleh, 2023):

$$\mu_n = E(x^n) = \int_{-\infty}^{\infty} x^n f(x)dx \tag{12}$$

Then we can getting the moment function of LoCH from equation (10) and (12) to get a form:

$$\begin{aligned}\mu_n = E(x^n) = \Phi \int_0^{\infty} x^{n+\delta-1} e^{(b+1)x^{\delta}} dx \\ - \psi \int_0^{\infty} x^{n+\delta-1} e^{(q+1)x^{\delta}} dx\end{aligned} \tag{13}$$

Let

$$I_1 = \int_0^\infty x^{n+\delta-1} e^{(b+1)x^\delta} dx \,, I_2 = \int_0^\infty x^{n+\delta-1} e^{(q+1)x^\delta} dx$$

For I_1, let $t = x^\delta \Longrightarrow x = t^{\frac{1}{\delta}} \Longrightarrow dx = \frac{1}{\delta} t^{\frac{1}{\delta}-1} dt$

$$\therefore I_1 = \int_0^\infty \left(t^{\frac{1}{\delta}}\right)^{n+\delta-1} e^{(b+1)t} \frac{1}{\delta} t^{\frac{1}{\delta}-1} dt$$

$$I_1 = \frac{1}{\delta} \int_0^\infty t^{\frac{n}{\delta}-1} e^{(b+1)t} dt$$

Then the finally result of I_1

$$I_1 = \frac{1}{\delta} \Gamma\left(\frac{n}{\delta}\right) (-1)^{-\frac{n}{\delta}} (b+1)^{-\frac{n}{\delta}}$$

By same way for I_2, we get:

$$I_2 = \frac{1}{\delta} \Gamma\left(\frac{n}{\delta}\right) (-1)^{-\frac{n}{\delta}} (q+1)^{-\frac{n}{\delta}}$$

And finally result:

$$\mu_n = E(x^n) = \frac{\Gamma\left(\frac{n}{\delta}\right)}{\delta} \left[\Phi (b+1)^{-\frac{n}{\delta}} - \psi (q+1)^{-\frac{n}{\delta}}\right], b, q > -1, n > 0 \tag{14}$$

The first 4-moments has forms:

$$\mu_1 = \frac{\Gamma\left(\frac{1}{\delta}\right)}{\delta} \left[\Phi (b+1)^{-\frac{1}{\delta}} - \psi (q+1)^{-\frac{1}{\delta}}\right] \tag{15}$$

$$\mu_2 = \frac{\Gamma\left(\frac{2}{\delta}\right)}{\delta} \left[\Phi (b+1)^{-\frac{2}{\delta}} - \psi (q+1)^{-\frac{2}{\delta}}\right] \tag{16}$$

$$\mu_3 = \frac{\Gamma\left(\frac{3}{\delta}\right)}{\delta} \left[\Phi (b+1)^{-\frac{3}{\delta}} - \psi (q+1)^{-\frac{3}{\delta}}\right] \tag{17}$$

$$\mu_4 = \frac{\Gamma\left(\frac{4}{\delta}\right)}{\delta} \left[\Phi (b+1)^{-\frac{4}{\delta}} - \psi (q+1)^{-\frac{4}{\delta}}\right] \tag{18}$$

Then the variance of LoCH has a form (Noori, Khalaf, & Khaleel, 2024):

$$\sigma^2 = \frac{\Gamma\left(\frac{2}{\delta}\right)}{\delta}\left[\Phi(b+1)^{-\frac{2}{\delta}} - \psi(q+1)^{-\frac{2}{\delta}}\right] - \left[\frac{\Gamma\left(\frac{1}{\delta}\right)}{\delta}\left[\Phi(b+1)^{-\frac{1}{\delta}} - \psi(q+1)^{-\frac{1}{\delta}}\right]\right]^2 \quad (19)$$

The skewness (S_K) and kurtosis (K_U) respectively has a forms (Hussain, Hassan, Rashid, & Ahmed, Families of Extended Exponentiated Generalized Distributions and Applications of Medical Data Using Burr III Extended Exponentiated Weibull Distribution, 2023):

$$S_K = \frac{\frac{\Gamma\left(\frac{3}{\delta}\right)}{\delta}\left[\Phi(b+1)^{-\frac{3}{\delta}} - \psi(q+1)^{-\frac{3}{\delta}}\right]}{\left[\frac{\Gamma\left(\frac{2}{\delta}\right)}{\delta}\left[\Phi(b+1)^{-\frac{2}{\delta}} - \psi(q+1)^{-\frac{2}{\delta}}\right]\right]^{\frac{3}{2}}} \quad (20)$$

$$K_U = \frac{\frac{\Gamma\left(\frac{4}{\delta}\right)}{\delta}\left[\Phi(b+1)^{-\frac{4}{\delta}} - \psi(q+1)^{-\frac{4}{\delta}}\right]}{\left[\frac{\Gamma\left(\frac{2}{\delta}\right)}{\delta}\left[\Phi(b+1)^{-\frac{2}{\delta}} - \psi(q+1)^{-\frac{2}{\delta}}\right]\right]^{2}} - 3 \quad (21)$$

Table 2 shows a set of values of the first four moments for different value of parameters with variance, skewness, and kurtosis of LoCH distribution.

Table (2): First four Moment functions values of LoCH for different parameters

α	γ	λ	δ	μ_1	μ_2	μ_3	μ_4	σ^2	S_k	K_U
0.8	0.6	0.4	0.1	0.048051	0.022919	0.014996	0.011136	0.02061	4.321811	21.20028
			0.2	0.101247	0.04805	0.031092	0.022919	0.037799	2.951925	9.926728
		0.6	0.3	0.141093	0.066203	0.042177	0.030766	0.046296	2.476074	7.019619
			0.4	0.188431	0.091145	0.058053	0.042177	0.055638	2.109726	5.077127
	0.9	0.8	0.5	0.226526	0.113511	0.072774	0.052891	0.062197	1.902923	4.10491
			0.6	0.264928	0.137715	0.08902	0.064745	0.067528	1.741866	3.413821
		1	0.7	0.294447	0.155997	0.100989	0.073256	0.069299	1.639077	3.01029
			0.8	0.329333	0.181601	0.119251	0.086907	0.073141	1.540938	2.635215

The table contains basic values such as mean, standard deviation, skewness, and kurtosis. The differences in values between different parameters show the effect of distribution on the properties of the statistical data. For example: at a parameter of 0.8, the mean is 0.329, indicating that relatively large values are expected.

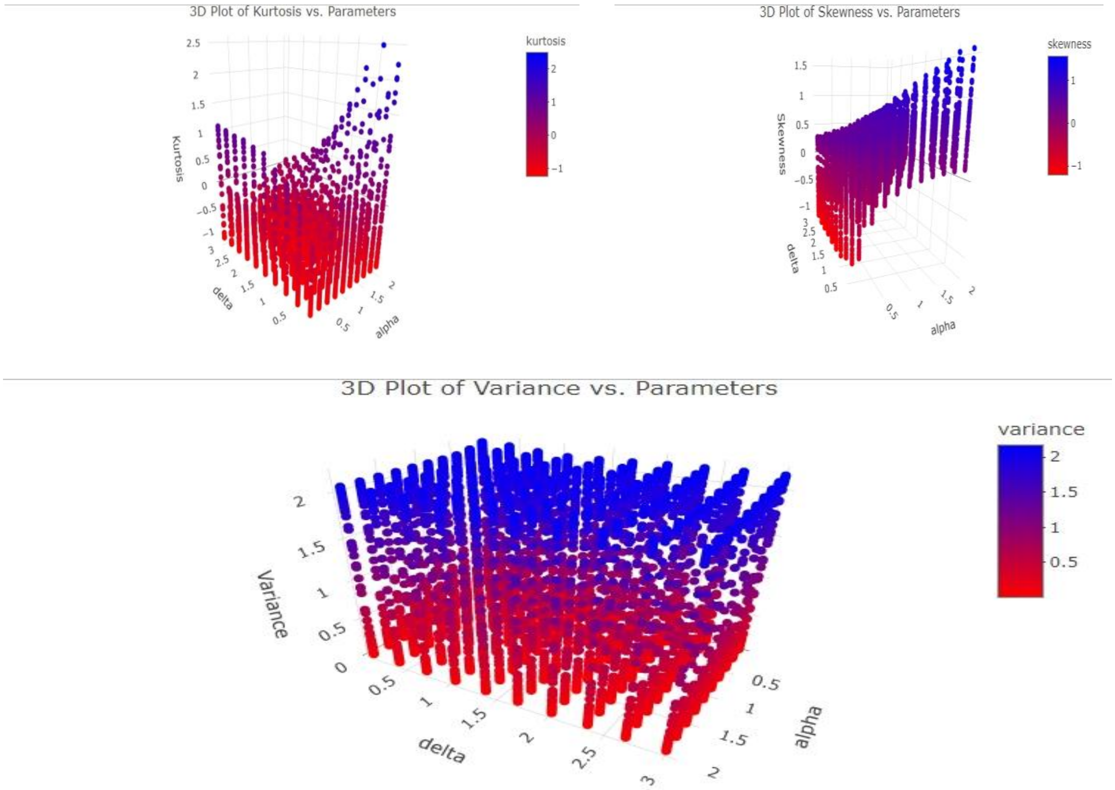


Figure 5. 3-D plot of Skewness, Kurtoses, and Variance for LoCH distribution

3-5. Moments Generating function of LoCH distribution: The moment generating function (M.G.F) of LoCH distribution can be obtained by using the moment distribution function in equation (14) and exponential expansion to obtain its equation in the form (Habib , Khaleel, Al-Mofleh, Oguntunde, & Adeyeye, 2024):

$$\mu_x(y) = \sum_{n=0}^{\infty} \frac{y^n}{n!}\left[\frac{\Gamma\left(\frac{n}{\delta}\right)}{\delta}\left[\Phi(b+1)^{-\frac{n}{\delta}} - \psi(q+1)^{-\frac{n}{\delta}}\right]\right] \qquad (22)$$

3-6. Incomplete Moments: For any random variable x, then the Incomplete moment function given by a form (Akarawak, et al., 2023):

$$\mu_r(y)=\int_0^y x^r f(x)dx$$

Substituting equation (10) in above equation to get a form:

$$\mu_r(y)=\Phi\int_0^y x^{n+\delta-1}e^{(b+1)x^\delta}dx-\psi\int_0^y x^{n+\delta-1}e^{(q+1)x^\delta}dx$$

By same way in prove the moment function we get:

$$\int_0^y x^{n+\delta-1}e^{(b+1)x^\delta}dx=\frac{1}{\delta(b+1)^{\frac{n}{\delta}}}\Gamma\left(\frac{n}{\delta},(b+1)y^\delta\right)$$

$$\int_0^y x^{n+\delta-1}e^{(q+1)x^\delta}dx=\frac{1}{\delta(q+1)^{\frac{n}{\delta}}}\Gamma\left(\frac{n}{\delta},(q+1)y^\delta\right)$$

Then the final form is:

$$\mu_r(y)=\frac{\Phi}{\delta(b+1)^{\frac{n}{\delta}}}\Gamma\left(\frac{n}{\delta},(b+1)y^\delta\right)-\frac{\psi}{\delta(q+1)^{\frac{n}{\delta}}}\Gamma\left(\frac{n}{\delta},(q+1)y^\delta\right) \tag{23}$$

3-7. Entropy

1. **Rényi Entropy:** The Rényi's Entropy function is a generalization of the Shannon Entropy concept and is used to measure the amount of information in probability distributions and expresses a flexible concept that allows changing the sensitivity of the measurements towards low or high probabilities through the rank of parameter. Its expressed by the equation (Khubaz, Abdal-Hameed, Mohamood, & Khaleel, 2023):

$$I_R(\beta)=\frac{1}{1-\beta}log\int_0^\infty f^\beta(x)dx$$

Then the expansion $f^\beta(x)$ of LoCH distribution has a form:

$$f^\beta=Hx^{\beta(\delta-1)}e^{(q+\beta)x^\delta} \tag{24}$$

Where

$$H=\sum_{j=i=k=l=s=z=p=q=0}^{\infty}\frac{(-1)^{i+k+l+z+q}d_{j,i}d_{k,l}\Gamma(\beta(\alpha+1)+j)\Gamma(\beta-k+s)\lambda^{p+\beta}\alpha^\beta\delta^\beta(z+\beta)^p}{j!\gamma^{j+\beta}\Gamma\big(\beta(\alpha+1)\big)s!p!\Gamma(\beta-k)}\binom{i+2j+\beta+l+s}{z}\binom{p}{q}$$

And $d_{j,i} = i^{-1} \sum_{r=1}^{j} \frac{[r(j+1)-i]}{r+1}$ for $i \geq 0$ and $d_{j,0} = 1$

$d_{k,l} = l^{-1} \sum_{c=1}^{l} \frac{[c(k+1)-l]}{c+1}$ for $l \geq 0$ and $d_{k,0} = 1$

By substitute equation (24) in Rényi's Entropy function we have a form:

$$I_R(\beta) = \frac{1}{1-\beta} log \left\{ H \int_0^\infty x^{\beta(\delta-1)} e^{(q+\beta)x^\delta} dx \right\}$$

The finally result is:

$$I_R(\beta) = \frac{1}{1-\beta} log \left\{ \frac{H\Gamma\left(\frac{\beta+1}{\delta}\right)}{\delta(q+\beta)^{\frac{\beta+1}{\delta}}} \right\} \tag{25}$$

2. **Arimoto Entropy:** Arimoto entropy (AE) is calculated using the following equation (Khaoula, Seddik-Ameur, Abd El-Baset, & Khaleel, 2022):

$$AE(\beta) = \frac{\beta}{1-\beta} \left(\left[\int_0^\infty f^\beta(x) dx \right]^{\frac{1}{\beta}} - 1 \right)$$

Form Eq(28) we get:

$$AE(\beta) = \frac{\beta}{1-\beta} \left(\left[\frac{H\Gamma\left(\frac{\beta+1}{\delta}\right)}{\delta(q+\beta)^{\frac{\beta+1}{\delta}}} \right]^{\frac{1}{\beta}} - 1 \right) \tag{26}$$

3. **Havrda and Charvat Entropy:**

$$HC(\beta) = \frac{1}{2^{1-\beta}-1} \left(\left[\int_0^\infty f^\beta(x) dx \right]^{\frac{1}{\beta}} - 1 \right)$$

The final result has a Form:

$$HC(\beta) = \frac{1}{2^{1-\beta}-1} \left(\left[\frac{H\Gamma\left(\frac{\beta+1}{\delta}\right)}{\delta(q+\beta)^{\frac{\beta+1}{\delta}}} \right]^{\frac{1}{\beta}} - 1 \right) \tag{27}$$

4. **Tsallis Entropy**

$$T(\beta)=\frac{1}{\beta-1}\left(1-\int_{0}^{\infty}f^{\beta}(x)dx\right)$$

Form Eq(28) we get:

$$HC(\beta)=\frac{1}{\beta-1}\left(1-\left[\frac{H\Gamma\left(\frac{\beta+1}{\delta}\right)}{\delta(q+\beta)^{\frac{\beta+1}{\delta}}}\right]^{\frac{1}{\beta}}\right) \tag{28}$$

4. Estimation

4-1. maximum likelihood estimation: the technique of Maximal likelihood estimation is used to calculate the LoCH distribution parameters. For the random sample $x_1, x_2, \dots, x_m$. The LoCH distribution pdf is adhered to (Noori & khaleel, 2024), (Khalaf & khaleel, 2024):

$$L(\vartheta, x_i)=\prod_{i=1}^{m}\frac{\alpha}{\gamma}\lambda\delta x_i^{\delta-1}e^{x_i^{\delta}}e^{\lambda\left(1-e^{x_i^{\delta}}\right)}\left(1-\frac{\left(1-e^{\lambda\left(1-e^{x_i^{\delta}}\right)}\right).log\left(e^{\lambda\left(1-e^{x_i^{\delta}}\right)}\right)}{\gamma}\right)^{-(\alpha+1)}\left[\frac{1-e^{\lambda\left(1-e^{x_i^{\delta}}\right)}}{e^{\lambda\left(1-e^{x_i^{\delta}}\right)}}-log\left(e^{\lambda\left(1-e^{x_i^{\delta}}\right)}\right)\right]$$

we compute the log- likelihood:

$$L=mlog(\alpha)+mlog(\lambda)+mlog(\delta)-mlog(\gamma)+(\delta-1)\sum_{i=1}^{m}log\,x_i+\sum_{i=1}^{m}x_i^{\delta} \tag{29}$$

$$+\sum_{i=1}^{m}\lambda\left(1-e^{x_i^{\delta}}\right) - (\alpha + 1)\sum_{i=1}^{m} log\left(1 - \frac{\left(1-e^{\lambda\left(1-e^{x_i^{\delta}}\right)}\right).log\left(e^{\lambda\left(1-e^{x_i^{\delta}}\right)}\right)}{\gamma}\right)$$

$$+\sum_{i=1}^{m} log\left[\frac{1-e^{\lambda\left(1-e^{x_i^{\delta}}\right)}}{e^{\lambda\left(1-e^{x_i^{\delta}}\right)}} - log\left(e^{\lambda\left(1-e^{x_i^{\delta}}\right)}\right)\right]$$

4-2. Least square estimation: The following formula can be used to estimate a parameter using the least square estimation (LSE) method (Noori & khaleel, 2024):

$$\varphi(\vartheta) = \sum_{i=1}^{m}\left[1-\left(1-\frac{\left(1-e^{\lambda\left(1-e^{x^{\delta}}\right)}\right).log\left(e^{\lambda\left(1-e^{x^{\delta}}\right)}\right)}{\gamma}\right)^{-\alpha} - \frac{1}{m+1}\right]^2 \tag{30}$$

4-3. Weighted Least square estimation: The following formula can be used to estimate a parameter using the weighted least square estimation (WLSE) method (Noori & khaleel, 2024):

$$\phi(\vartheta) = \sum_{i=1}^{m} \frac{(m+1)^2(m+2)}{i(m-i+1)} \left[1 - \left(1 - \frac{\left(1 - e^{\lambda\left(1-e^{x^{\delta}}\right)}\right) . \log\left(e^{\lambda\left(1-e^{x^{\delta}}\right)}\right)}{\gamma} \right)^{-\alpha} - \frac{i}{m+1} \right]^2 \tag{31}$$

Estimates of the parameters for the three previously described methods may be obtained by finding the partial derivative of four parameters and setting it to zero. Computer technologies such as the R language are used since it is difficult to find these values in numerical solutions.

5. **Simulation Study:** Simulation is an important tool for evaluation the performance of distribution. Therefore, the performance of LoCH distribution estimators is evaluated for three methods represented by MLE, LSE, and WLSE (Sharqa , Ahsan-ul-Haq, Zafar, & Khaleel, 2022) through Monte Carlo simulation using the R package. The method of sitting the parameters compare the performance of estimation method is summarized as follows:

1. Sitting the real parameters and generating the data:

❖ Determining the real parameters: specific values are chosen for the LoCH distribution parameters.

❖ Generating random samples using these parameters, N=1000 random samples are generating of LoCH with different sizes (N=50,100,150,200). Each sample represents simulation data that can be used to estimate the parameters.

2. Applying the estimation methods to each sample.
3. Evaluating the performance using statistical measures, i.e. after estimating the parameters for each sample, following measures are calculated: arithmetic mean it is close to the true value if the method is unbiased, mean square error (MSE), root mean square error (RMSE), and bias for each sample,

   We create N=1000 samples for the real parameter values listed in Table 3.

Table (3): Monte Carlo simulations conducted for the LoCH

$\alpha = 0.4\ \gamma = 0.5, \lambda = 0.7, \delta = 0.8$					
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**
50	Mean	$\hat{\alpha}$	0.37938020	0.44657203	0.41759698
		$\hat{\gamma}$	0.5858387	0.48239862	0.51339787
		$\hat{\lambda}$	0.78292219	0.72393239	0.74381783
		$\hat{\delta}$	0.85121816	0.79099952	0.77871211
	MSE	$\hat{\alpha}$	0.02425856	0.03968336	0.02445980
		$\hat{\gamma}$	0.7066821	0.06555375	0.12370679
		$\hat{\lambda}$	0.05646757	0.01720128	0.02023997
		$\hat{\delta}$	0.11447882	0.02804492	0.04222493
	RMSE	$\hat{\alpha}$	0.15575161	0.19920682	0.15639630
		$\hat{\gamma}$	0.8406438	0.25603466	0.35171976
		$\hat{\lambda}$	0.23762907	0.13115364	0.14226724
		$\hat{\delta}$	0.33834719	0.16746617	0.20548706
	Bias	$\hat{\alpha}$	0.02061980	0.04657203	0.01759698
		$\hat{\gamma}$	0.0858387	0.01760138	0.01339787
		$\hat{\lambda}$	0.08292219	0.02393239	0.04381783
		$\hat{\delta}$	0.05121816	0.00900048	0.02128789
100	Mean	$\hat{\alpha}$	0.395364854	0.42790681	0.41870735
		$\hat{\gamma}$	0.6491893	0.47238101	0.507268431
		$\hat{\lambda}$	0.74604473	0.71333543	0.72166861
		$\hat{\delta}$	0.83189856	0.81375987	0.807360632
	MSE	$\hat{\alpha}$	0.017285362	0.01553941	0.01091465
		$\hat{\gamma}$	0.5662532	0.06191147	0.116063758
		$\hat{\lambda}$	0.03638033	0.01293739	0.01417245
		$\hat{\delta}$	0.10625167	0.02767893	0.026763091
	RMSE	$\hat{\alpha}$	0.131473808	0.12465719	0.10447318
		$\hat{\gamma}$	0.7524980	0.24882016	0.340681314
		$\hat{\lambda}$	0.19073628	0.11374264	0.11904810
		$\hat{\delta}$	0.32596268	0.16636987	0.163594289
	Bias	$\hat{\alpha}$	0.004635146	0.02790681	0.01870735
		$\hat{\gamma}$	0.1491893	0.02761899	0.007268431
		$\hat{\lambda}$	0.04604473	0.01333543	0.02166861
		$\hat{\delta}$	0.03189856	0.01375987	0.007360632
150	Mean	$\hat{\alpha}$	0.404875045	0.423189731	0.411556058
		$\hat{\gamma}$	0.7502296	0.47905358	0.47527937
		$\hat{\lambda}$	0.71278461	0.71391721	0.722781410

$\alpha = 0.4\ \gamma = 0.5, \lambda = 0.7, \delta = 0.8$					
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**
		$\hat{\delta}$	0.82480347	0.802689724	0.81106298
	MSE	$\hat{\alpha}$	0.015775509	0.009793611	0.008273837
		$\hat{\gamma}$	0.7599438	0.06800600	0.07753536
		$\hat{\lambda}$	0.03140722	0.00672382	0.009740447
		$\hat{\delta}$	0.05352820	0.017982134	0.02300739
	RMSE	$\hat{\alpha}$	0.125600594	0.098962673	0.090960634
		$\hat{\gamma}$	0.8717476	0.26077960	0.27845172
		$\hat{\lambda}$	0.17722081	0.08199890	0.098693703
		$\hat{\delta}$	0.23136161	0.134097480	0.15168186
	Bias	$\hat{\alpha}$	0.004875045	0.023189731	0.011556058
		$\hat{\gamma}$	0.2502296	0.02094642	0.02472063
		$\hat{\lambda}$	0.01278461	0.01391721	0.022781410
		$\hat{\delta}$	0.02480347	0.002689724	0.01106298
200	Mean	$\hat{\alpha}$	0.41317553	0.427202049	0.410643594
		$\hat{\gamma}$	0.6867661	0.48662555	0.48443907
		$\hat{\lambda}$	0.72460591	0.709339912	0.714312057
		$\hat{\delta}$	0.802654291	0.793460569	0.81054514
	MSE	$\hat{\alpha}$	0.01180115	0.007366778	0.004922737
		$\hat{\gamma}$	0.5440034	0.02418551	0.05493195
		$\hat{\lambda}$	0.02810601	0.005715931	0.009442916
		$\hat{\delta}$	0.047296399	0.016517607	0.01229515
	RMSE	$\hat{\alpha}$	0.1086330	0.085829938	0.070162219
		$\hat{\gamma}$	0.7375658	0.15551693	0.23437566
		$\hat{\lambda}$	0.16764848	0.075603777	0.097174668
		$\hat{\delta}$	0.217477353	0.128520843	0.11088349
	Bias	$\hat{\alpha}$	0.01317553	0.027202049	0.010643594
		$\hat{\gamma}$	0.1867661	0.01337445	0.01556093
		$\hat{\lambda}$	0.02460591	0.009339912	0.014312057
		$\hat{\delta}$	0.002654291	0.006539431	0.01054514

Results by sample size:

- At n=50, the average MLE is closest to the true value among the three methods. MSE=0.02425856 and RMSE=0.15575161 show that MLE outperforms in accuracy. Bias is low (e.g. 0.02061980). While LSE and WLSE show poorer accuracy performance compared to MLE as RMSE and bias are relatively higher.

❖ At n=200, all methods improve with increasing sample size. However, MLE is still the best, achieving the lowest values for MSE (0.01180115) and RMSE (0.1086330). Bias approaches zero in MLE.

As sample size increases, all methods improve, but MLE is consistently the most accurate, showing clear superiority at all sample sizes. The superiority of MLE is due to its optimal use of the statistical information inherent in the distribution, making it more accurate and stable, especially as the sample size increases. Whereas LSE/WLSE relies on geometric parameters that may not reflect the full probabilistic properties, leading to less efficient estimates in complex models such as LoCH.

6. Application: To demonstrate the effectiveness of the proposed distribution, a practical application is carried out on real data where the effectiveness of LoCH distribution in fitting the data is proven accurately. The application shows the benefits of LoCH and its excellent fit to the data. Table 4 presents a comparison between LoCH and seven other distribution represented by distribution as {[0,1] Tracked Exponential Exponential Chen (TEECH), Beta Chen (BeCH), Kumaraswamy Chen (KuCH), Weibull Chen (WeCH), Exponential generalized Chen (EGCH), Log-Gamma Chen (LGamCH), and Chen (CH)}.

This comparison uses eight measures. Two statistical used in this analysis are Kolmogorov-Smirnov (KS), Anderson-Darling (A) (Khalaf, Hameed, Moudher , Khaleel, & Abdullah, 2022), (Afify , Yousof, & Nadarajah, 2017). the analysis also takes into account Cramer-von-Mises (W), and P-values, in addition to information criteria AIC, BIC, HQIC, and CAIC (Oguntunde, Khaleel, Okagbue, & Odetunmibi, 2019), (Chipepa, Oluyede, & Makubate, 2019). Furthermore, it uses the probability value obtained through the KS test. These measures are commonly used to evaluate the accuracy of the fit.

The data used in application part is a civil engineering data 85 times call (Benjamin & Cornell, 2014)

Table 4 presents the LoCH distribution with the lowest of criteria's. these values have been compared to non-overlapping distribution values. Furthermore, statistical tests in Table 5, provide compelling evidence that the LoCH distribution closely matches the observed patterns in the data.

Table (4): comparison by information criteria for data

Dist.	-2L	AIC	CAIC	BIC	HQIC
LoCH	133.3925	274.785	275.285	284.5556	278.715
TEECH	137.6853	283.3706	283.8706	293.1412	287.3006
BeCH	140.1528	288.3055	288.8055	298.0761	292.2355
KuCH	140.3978	288.7955	289.2955	298.5661	292.7255
EGCH	133.5386	275.0772	275.5772	284.8478	279.0072
WeCH	140.689	289.3779	289.8779	299.1485	293.3079
LGamCH	141.202	290.404	290.904	300.1746	294.334
CH	146.542	297.2482	297.3945	302.1335	299.2132

It is shown that the LoCH distribution has the lowest values for the criteria used, indicating its high fit to the data compared to other distributions such as TEECH and KuCH. For example: AIC=274.785 for LoCH distribution vs. AIC=283.3706 for TEECH distribution. The differences enhance the reliability of LoCH as a suitable model.

Table (5): comparison by statistical tests for data

Dist.	W	A	K-S	p-value
LoCH	0.1056066	0.5695775	0.125007	0.1403315
TEECH	0.2235378	1.308495	0.1772736	0.009568662
BeCH	0.2479046	1.4524	0.1552976	0.03314692
KuCH	0.2412264	1.411637	0.1514618	0.04048695
EGCH	0.1231543	0.6694166	0.1293808	0.1161648
WeCH	0.2856196	1.679977	0.1716906	0.01332579
LGamCH	27.10933	168.7643	0.9911261	0
CH	0.2009055	1.163272	0.1827173	0.006857527

The lowest values of W (0.1056066) and A (0.5695775), indicating a better fit to the data. K−S=0.125007 with P−value=0.1403315 supports that LoCH fits the data well. While the other distributions: TEECH, BeCH, and KuCH have higher values of W, A, K−S and lower P−values (e.g. P−value=0.009568662 for TEECH), indicating a poorer fit to the data compared to LoCH. Some distributions like LGamCH show very poor results (e.g. P−value=0).

Table (6): Estimated parameters of distributions by MLE method

Dist.	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\delta}$
LoCH	0.45806798	3.25510248	0.03150105	0.81590381
TEECH	0.761884109	1.462397203	0.004459669	0.927046476
BeCH	1.130835152	0.924297773	0.007349301	0.891723768
KuCH	1.128775015	0.925191180	0.007839694	0.885030520
EGCH	0.1803128	3.6123019	0.2610463	0.7428958
WeCH	1.099985482	1.551014376	0.008944646	0.895722816
LGamCH	1.5459694	0.1653997	0.1633739	0.7827582
CH	---	---	0.01141574	0.83160502

The results of the table show that these values obtained for the LoCH model are highly flexible in representing data by combining parameters that control shape, spread, and centering. Where LoCH is distinguished from other distributions by low and consistent parameter values, which enhances its efficiency and flexibility.

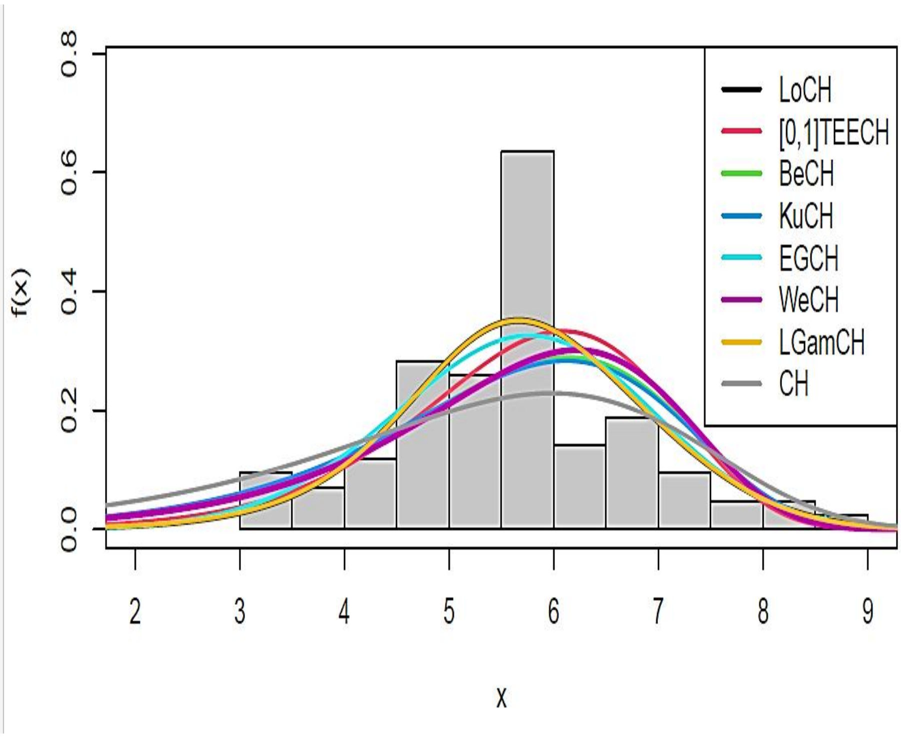
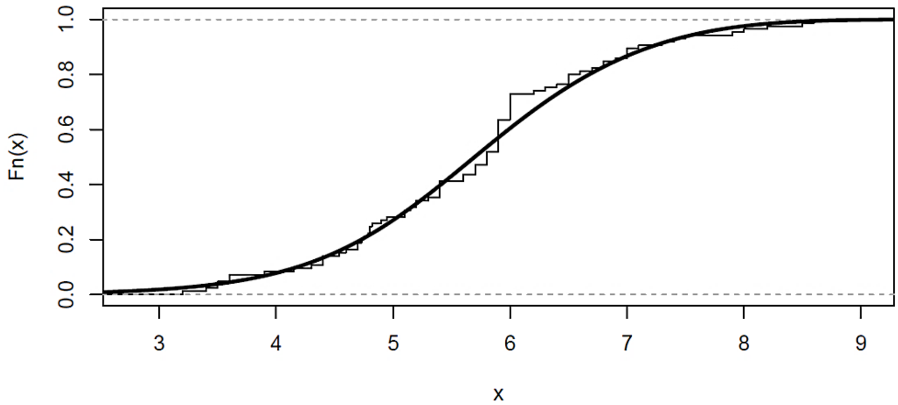
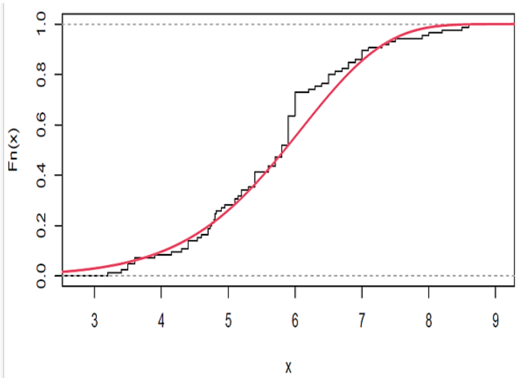


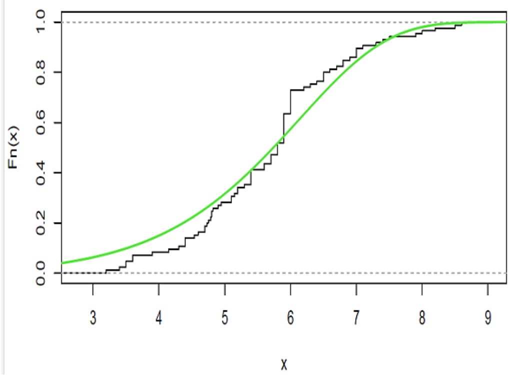
Figure (5): Fitted densities for Data



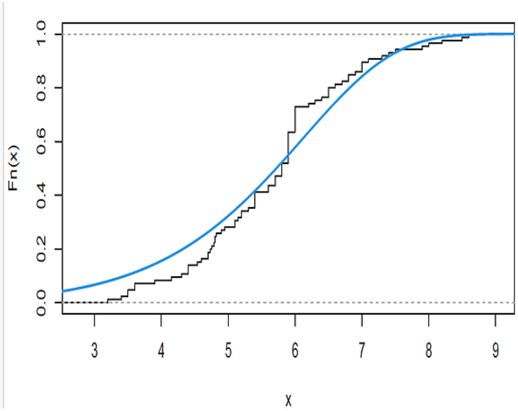
LoCH



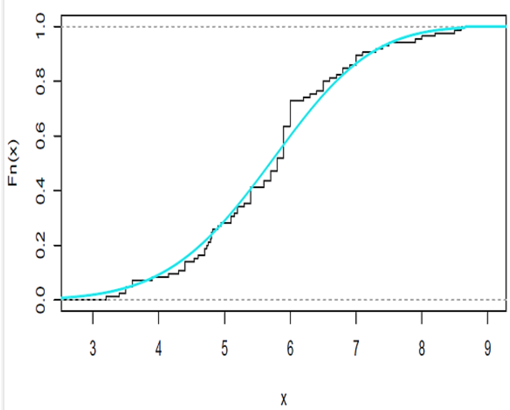
TEECH



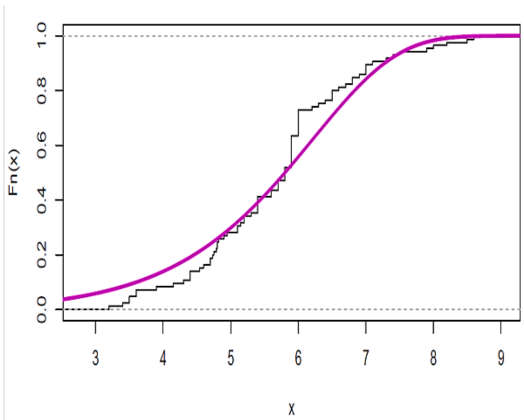
BeCH

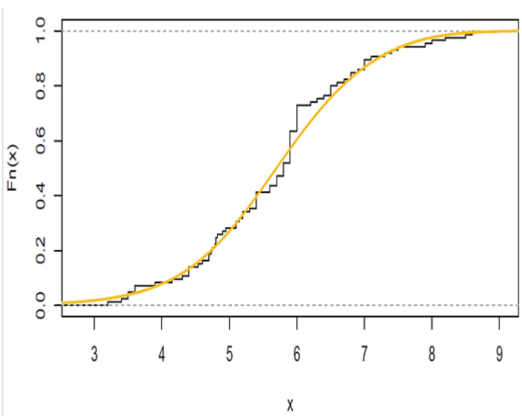


KuCH

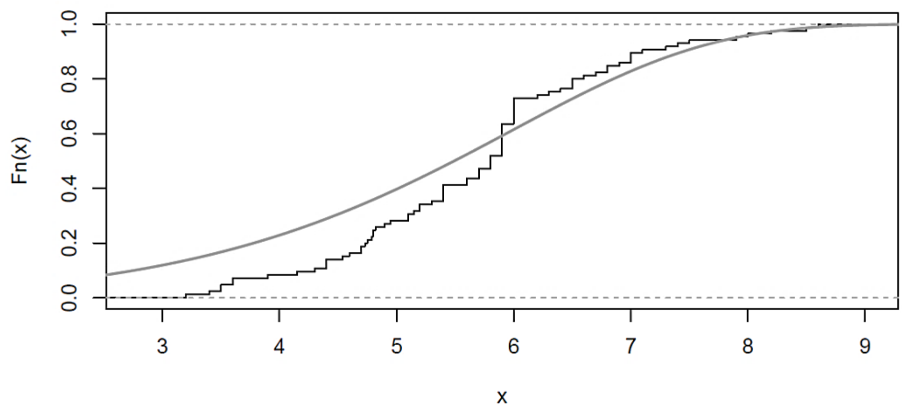


EGCH





WeCH **LGamCH**



CH

Figure (6): Fitted empirical CDF for Data

Figure 5 shows a comparison between different distributions (such as LoCH, TEECH, BeCH, KuCH, etc.) in terms of their ability to represent the probability density of the data. The LoCH distribution curve shows a better fit to the shape of the real data, which highlights the accuracy of its fit. Other distributions such as BeCH and KuCH show curves that differ from the real density, which means that they are less able to represent the data. Therefore, LoCH shows a more consistent shape with the data compared to the other distributions. This is shown by the concentration of density near the regions where the highest probabilities are located, indicating a high accuracy in the estimate.

In Figure 6, the LoCH distribution appears to achieve the best fit to the real data, which proves its ability to accurately represent the data. The figure highlights that LoCH is able to capture the cumulative features of the data better than competing distributions.

Entropy can be used to show the flexibility of LoCH, but it's not a substitute for AIC and BIC, but its powerful tool for assessing the quality of a theoretical distribution from the perspective of uncertainty and shape properties. In the case of distribution such as LoCH, it can be used to confirm its superiority in representing complex data, especially when AIC and BIC criteria are not decisive.

In this aspect, the data of civil engineering data 85 times call were dealt with to show the efficiency of the distribution. As for the mechanism for dealing with extreme values, it is as follows:

- The original Chen distribution is known for its ability to modeling heavy-tails data, especially in reliability analysis. When combined with the Odd

Lomax family, the tail flexibility is increased thanks to the additional α, allowing the distribution to respect extreme values more accurately. For example, in figure (2) the curves show sharp changes in the tail when α, and γ are modified, reflecting the distribution's ability to adapt to values far from the center.

- The hazard function in LoCH shows a variable behavior (increasing, decreasing, or constant) based on the parameters. This helps in the identifying periods of high risk, which may be associated with the presence of extreme values.
- The use of infinite series (such as the exponential and logarithmic expansion in equations 9 and 10) allows LoCH to capture unusual behaviors of the data, including extreme values.

Conclusion: The LoCH distribution shows high flexibility and adaptability to data, making it suitable for real data analysis. It achieves the lowest values in fit criteria such as AIC and BIC compared to other distributions. The distribution has proven to be a good fit to the data through statistical tests such as Kolmogorov-Smirnov and Anderson-Darling.

The MLE method shows greater accuracy compared to LSE and WLSE, especially with increasing sample size. The bias is low, and the estimated values are very close to the true values, which enhances the reliability of LoCH.

The LoCH distribution outperformed other distributions such as TEECH, KuCH and WeCH based on fit criteria (AIC, CAIC, HQIC, BIC). Other distributions showed poor performance, especially in adapting to extremes or concentrated densities of data. LOCH has shown high efficiency when applied to real data in the field of civil engineering.

The distribution provided the best fit compared to other distributions, which indicates its usefulness in real applications. The LoCH distribution has low and constant parameters that allow it to adapt to different forms of data. The distribution has shown a high agreement with experimental data in densities and cumulative functions.

The superiority of LoCH is not limited to the criterial of suitability, but extends to analytical flexibility (ability to adapt to diverse data formats), estimation efficiency (stability of parameters with increasing sample size), accuracy in statistical tests (quantitative and qualitative superiority) and practically (better fit to complex real-data).

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