Stability of Homogenous Systems with Constant Coefficients

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Abstract

Stability of different equations, especially for Homogeneous systems with constant coefficient, is a very important task of applied mathematics, since it has a wide range of application in different fields of real and physical life problems. Since the process of finding the general solution to a dynamical system is almost impossible for linear system with constant coefficient.

1- <u>Phase plane, critical points:[3]</u>

We continue our discussion of homogeneous linear system

$$Y' = Ay \qquad \text{……..(1)}$$

now assuming that the (n × n) matrix A=[ajk] is constant, that is, its entries do not depend on t. we wish to solve (1), for this we remember that a single equation $Y' = Ky$ has the solution $Y = ce^{kt}$. A ccordingly, we try

$$Y = Xe^{\lambda t} \qquad \text{……..(2)}$$

by substitution into (1) we get

$$Y' = \lambda X e^{\lambda t} = Ay = Axe^{\lambda t}$$

$\lambda x = Axe^{\lambda t}$ [dividing by e^{λ} , we are left with the eigenvalue problem]

$$Ax = \lambda x \qquad \text{……..(3)}$$

Thus the nontrivial solution of (1) are of form (2), where λ is an eigen ralue of A and X is acorresponding eigenvector.

Let us further assume that A has a basis of n eigenvectors $X^{(1)}$….., $X^{(n)}$ corresponding to eigen values λ_1, λ_2, ….., λ_n (which may all be different or some of which – or all – may be equal). Then the corresponding solutions (2) are

$$Y^{(1)} = X^{(1)} e^{\lambda_1 t}, \ldots\ldots\ldots, Y^{(n)} = X^{(n)} e^{\lambda_n t} \qquad \ldots\ldots..(4)$$

$$w(Y^{(1)}, \ldots.., y^{(n)}) = \begin{vmatrix} X_1^{(1)} e^{\lambda_1 t} & \ldots\ldots\ldots & X_1^{(n)} e^{\lambda_n t} \\ X_2^{(1)} e^{\lambda_1 t} & \ldots\ldots\ldots & X_2^{(n)} e^{\lambda_n t} \\ X_n^{(1)} e^{\lambda_1 t} & \ldots\ldots\ldots & X_n^{(n)} e^{\lambda_n t} \end{vmatrix}$$

$$= e^{\lambda_1 t + \lambda_2 t + \ldots\ldots + \lambda_n t} \begin{vmatrix} X_1^{(1)} & \ldots\ldots\ldots X_1^{(n)} \\ X_2^{(1)} & \ldots\ldots\ldots X_2^{(n)} \\ \vdots & \\ X_n^{(1)} & \ldots\ldots\ldots X_n^{(n)} \end{vmatrix}$$

on the right, the exponential function is never zero, and the determinant is not zero (| A | ≠ 0) because its columns are the linearly independent eigen vectors that form a basis.

Theorem (1): (General Solution)[4]

If the constant matrix A in the system (1) has a linearly indepen set of n eigen vectors, then corresponding solution $Y^{(1)}$, $Y^{(2)}$, ……, $Y^{(n)}$ in (4) form a basis of solutions of (1), and the correspondingh general solution is

$$Y = c_1 x^{(1)} e^{\lambda_1 t} + c_2 x^{(2)} e^{\lambda_2 t} + \ldots\ldots\ldots + c_n x^{(n)} e^{\lambda_n t}$$

……..(5)

Q1: How to plot solution, phase plane:

We shall now concentrate on homogeneous linear system(1) with coefficients consisting of two equations:

$$Y^{/} = AY \quad \text{in components,} \quad \begin{aligned} Y_1' &= a_{11}\, y_1 + a_{12}\, y_2 \\ Y_2' &= a_{21}\, y_1 + a_{22}\, y_2 \end{aligned} \qquad \text{........(6)}$$

$$Y(t) = \begin{bmatrix} y_1\,(t) \\ y_2\,(2) \end{bmatrix}$$

- The $Y_1\ y_2$ – plane is called the phase plane of (1)
- If we fill the phase plane with trajectories of (6), we obtain the so – called phase portrait of (6)

<u>Example (1)[2]</u>: Trajectories in the phase plane (phase portrait) find and plot solutions of the system

$$Y' = AY = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} Y \ , \quad thus \quad \begin{aligned} Y_1' &= -3\,y_1 + y_2 \\ Y_2' &= y_1 - 3\,y_2 \end{aligned} \qquad \text{.......(7)}$$

Solution: by substituting $Y = X\,e^{\lambda t}$ and $Y' = \lambda\, X\, e^{\lambda t}$ and dropping the exponential function we get $A\,X = \lambda\, X$

The characteristic equation is

$$\det(A - \lambda\, I) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} \quad \lambda^2 + 6\,\lambda + 8 = 0$$

this gives the eigen ralues (λ_1 = -2) and (λ_2 = -4)

eigenvectors are then obtained from

$$\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (-3-\lambda)\, x_1 + x_2 = 0 \Rightarrow$$

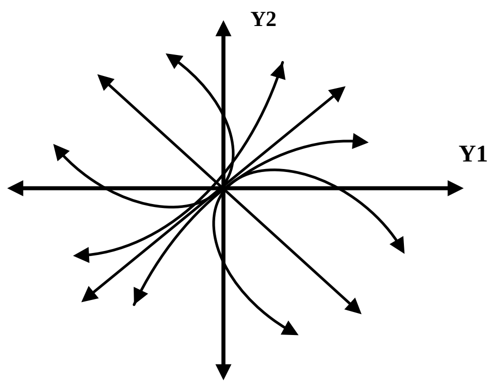
if $\lambda_1 = -2 \Rightarrow (-3+2)\,x_1 + x_2 = 0 \Rightarrow -x_1 + x_2 = 0 \Rightarrow [x_2 = x_1]$

we can take $x^{(1)} = [1 \quad 1\,]^T$ $\qquad y^{(1)} = x^{(1)}\, e^{-2t}$, $\quad y^{(2)} = x^{(2)}\, e^{-4t}$

for $\lambda_2 = -4 \Rightarrow x_1 + (-3-\lambda)\,x_2 = 0 \Rightarrow \quad x_1 + (-3+4)\ x_2 = 0 \Rightarrow \ (\ x_1 = -x_2\ )$

eigenvector is $x^{(2)} = [\,1 \quad -1]^T$. This gives the general solution

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\, y^{(1)} + c_2\, y^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$



$Y^{(1)}$ (+)c_1 $\{^1{}_1\}$ e^{-zt}

Y^2 (t) = c_2 $(^1{}_{-1}\}e^{-4t}$

Fig (1): Trajectories of the system (8) (Improper node)

2- Critical Points of the System (6)[1]:

The point y=0 in fig(1) seems to be acommon point of all trajectories, and we want to explore the reason for this remarkable observation. The answer will follow by calculus.

Indeed, from (6) we obtain

$$\frac{dy_2}{dy_1} = \frac{dy_2\,/\,dt}{dy_1\,/\,dt} = \frac{y_2'}{y_1'} = \frac{a_{21}\,y_1 + a_{22}\,y_2}{a_{11}\,y_1 + a_{12}\,y_2} \qquad \text{……..(9)}$$

This associates with every point P: (y_1 , y_2) aunique tangent direction $\frac{dy_2}{dy_1}$ of the trajectory passing through P, except for the point P= P_o ! (0, 0), where the right side of (9) becomes 0/0. This point P_o, at which $\frac{dy_2}{dy_1}$ becomes undertermined is called a critical point of (6).

3- Five Types of Critical Points[4]:

There are five types of critical points depending on the geometrical shape of the trajectories near them. They are called improper nodes, proper nodes, saddle points, centers, and spiral points.

Example (2)[3]: (continued) improper node (fig -1-y)

An improper node is a critical point (P_o) at which all the trajectories, except for two of them, have the same limiting direction of the tangent. The two excptional trajectories also have a limiting direction of the tangent at P_o which, however, is different.

The system (8) has an improper node at (0), as its portrait fig(1) shows. The common limiting direction at (0) is that of the eigenvector $x^{(1)} = [\ 1 \quad 1\ ]^T$ because e^{-4t} goes to zero faster then e^{-2t} as (t) increases. The exceptional limiting tangent direction is that of $x^{(2)} = [\ 1 \quad 1\ ]^T$.

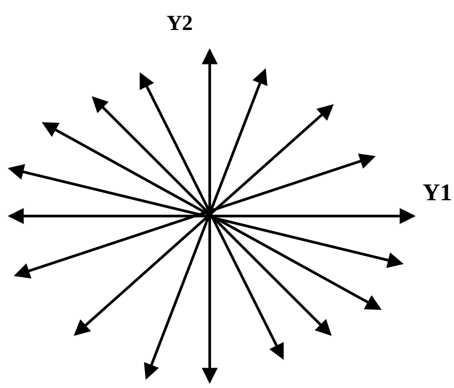


Fig (2): Trajectories of the system (10) (proper node)

<u>Example (3)[1]</u>: Proper node (fig -2-)

A proper node is a critical point P_o at which every trajectory having (d) as its limiting direction. The system

$$\left[Y' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Y \right. \; , \quad thus \quad \begin{matrix} Y_1' = y_1 \\ Y_2' = y_2 \end{matrix}$$

……..(10)

has a proper node at the origin (fig -2-) because a general solution is

$$Y = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \quad \text{or} \quad \boxed{y_1 = c_1 e^t}$$

$$\boxed{y_2 = c_2 e^t}$$

$$\text{or} \quad \boxed{c_1 y_2 = c_2 y_1}$$

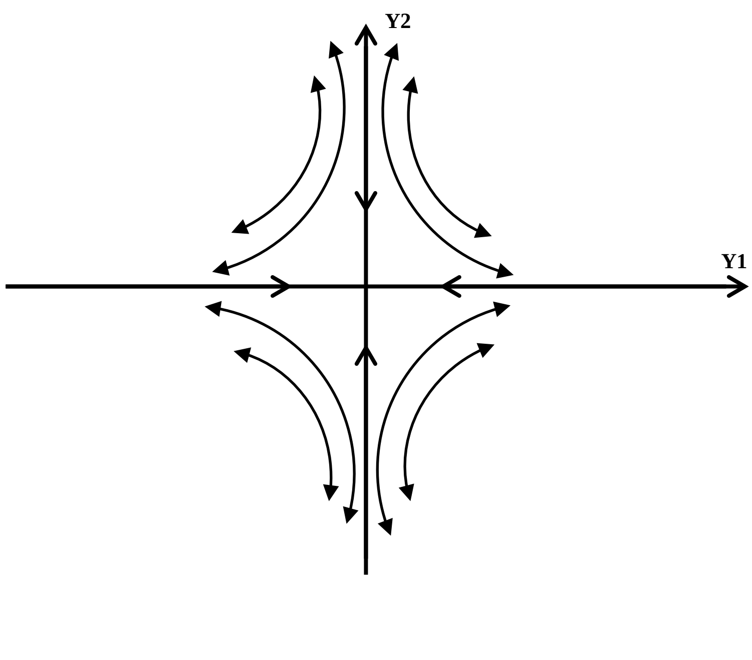


Fig (3): Trajectories of the system (11) (saddle point)

<u>Example (4)[3]</u>: Saddle node (fig -3-)

A saddle point is a critical point (P_o) at which there are two incoming trajectories, two outgoing trajectories, and all the orther trajectories in a neighborhood of (P_o) by pass P_o.

The system

$$\left[Y' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} y \ , \quad thus \quad \begin{matrix} Y_1' = y_1 \\ Y_2' = y_2 \end{matrix} \right.$$

……..(11)

has a saddle point at the origin because a general solution is:

$$Y = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \qquad \text{or} \qquad \begin{matrix} y_1 = c_1 e^t \\ y_2 = c_2 e^{-t} \end{matrix} \qquad \text{or} \qquad y_1 y_2 = const$$

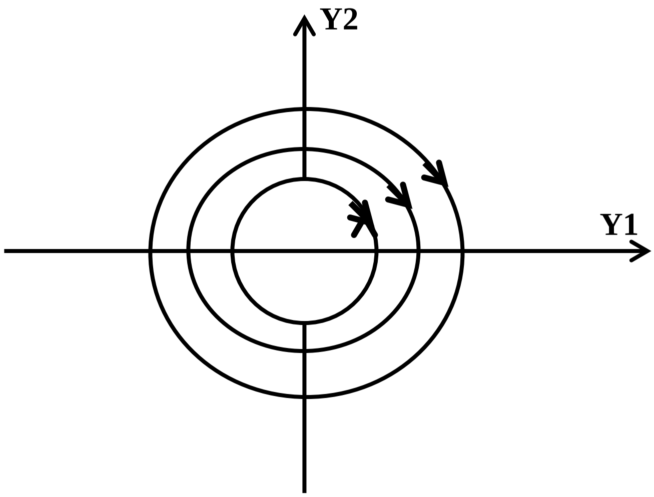


Fig (4): Trajectories of the system (12) (center)

<u>Example (5)[2]</u>: Center (fig -4-)

A center is a critical point that is enclosed by infinitely many closed trajectories.

The system

$$\left[Y' = \begin{bmatrix} 1 & 1 \\ -4 & 0 \end{bmatrix} y \right. , \quad thus \quad \begin{matrix} Y_1' = y_2 \\ y_2' = -4y_1 \end{matrix}$$

has a center theorigin, as we now show. The characteristic equation is $\lambda_2 + 4 = 0 \Rightarrow 1t$ has the eigenvalues (2i) and (-2i) and eigen vectors [1 2i]^T and [1 -2i]^T, respectively

$$\Rightarrow (12*)\ Y = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it} , \quad thus \quad \begin{matrix} Y_1 = c_1 e^{2it} + c_2 e^{-2it} \\ y_2 = 2i c_1 e^{2it} - 2i\ c_2 e^{-2it} \end{matrix}$$

<u>Example (6)[1]</u>: Spiral Point (fig -5-)

A spiral point is a critical point (P_o) about which the trajectories spiral, approaching P_o as $t \to \infty$ (or tracing these spiral in the opposite sense, a way from P_o).

The system

$$\left[Y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} y \right. , \quad thus \quad \begin{matrix} Y_1' = -y_1 + y_2 \\ Y_2' = -y_1 - y_2 \end{matrix}$$

…….(13)

has a spiral point at the origin, as we show. The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$. It gives the eigenvalues (-1+i) and (-1-i).

corresponding eigenvectors are obtained from ($-1-\lambda$) $x_1 + x_2 = 0$ for λ=-1+I this becomes ($-i\ x_1 + x_2 = 0$) and we can take [1 i]^T as a eigenvector corresponding (-1 – i) is [1 -i]^T. this gives the complex general solution.

$$Y = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

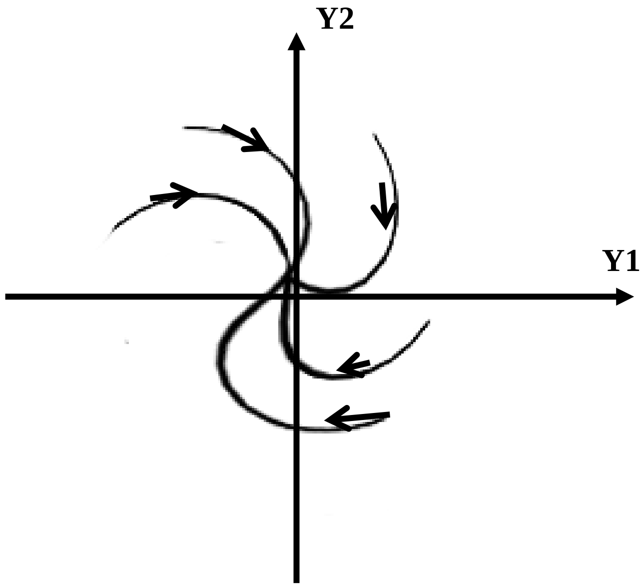


Fig (5): Trajectories of the system (13) (spiral point)

The next step would be the transformation of this complex to a real general solution by the Euler formula. But, as in the last example, we just wanted to see what eigenralues to expect in the case of a spiral point. Accordinaly, we start again from the beginning and instead of that rather lengthy systematic calculation. We use a shortcut. We multiply the first equation in (13) by y_1, the second by y_2, and add, obtaining

$$y_1 y_1' + y_2 y_2' = -(y_1^2 + y_2^2)$$

introducing polar coordinates r , t , where $r^2 = y_1^2 + y_2^2$, we see that this equation becomes $\frac{1}{2}(r^2)' = -r^2$

now $(r^2)' = 2rr'$ by differentiation

$$\frac{1}{2}(r^2)' = rr' = -r^2$$

$$r' = -r$$

$$Lnr = -t + c'$$

by taking exponentials we have $r = c\, e^{-t}$

for each real c this is a spiral, as claimed

4- No Basis of Eigenvectors Available[3]:

When could this happen and what could we do? Well, it cannot happen if the matrix (A) is symmetric ($a_{jk} = a_{kj}$) or skew – symmetric ($a_{jk} = -a_{kj}$, thus $a_{jj} = 0$), and it also does not happen in many other cases (in examples 5 and 6).

This is the case for any n, not just for n=2. if it happens, what can we do?

Suppose that an n×n matrix (A) has a double eigenvalue μ [that is, the product representation of det(A- λ I) has a factor ($\lambda - \mu$)^2] with only one eigenvector (and its multiples corresponding to it, instead of two linearly independent eigenvectors, so that we first get only one solution $Y^{(1)} = x e^{\mu t}$. In this case we can obtain a second independent solution by substituting.

$$Y^{(2)} = x t e^{ut} + u e^{ut}$$
………(14)

$$\Rightarrow Y'^{(2)} = x e^{\mu t} + \mu x t e^{\mu t} + \mu u e^{\mu t} = A y^{(2)} = A x t e^{\mu t} + A u e^{\mu t}$$

since $\mu x = A x$

the term $\mu x t e^{\mu t}$ and $A x t e^{\mu t}$ cancel, and division by $e^{\mu t}$ gives

$$x t \mu u = A u, \qquad \textit{thus} \quad (A - \mu I) u = x$$
………(15)

although $\det(A - \mu I) = 0$, this can always be solved for μ , as can be shown

<u>Example (7)[2]</u>: No basis of eigenvectors available. Degenerate node finde a general solution of $Y' = A y = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} Y$

Solution: The matrix (A) is not skew – symmetric

$\Rightarrow$ characteristic equation is

$$\det(A-\lambda I)=\begin{vmatrix}4-\lambda & 1\\ -1 & 2-\lambda\end{vmatrix}=\lambda^2-6\lambda+9=(\lambda-3)^2=0$$

it has a double root ($\lambda = 3$). Eigenvectors are obtained from $(4-\lambda)x_1+x_2=0$ thus from $x_1+x_2=0$

$$\Rightarrow x^{(1)}=[1 \quad -1]^T$$

$$\Rightarrow (A-3I)u=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}u=\begin{bmatrix}1\\ -1\end{bmatrix}, \quad thus \quad \begin{matrix}u_1+u_2=1\\ -u_1-u_2=-1\end{matrix}$$

and we can take simply (u = [0 1 $]^T$). This gives the answer (fig -6-)

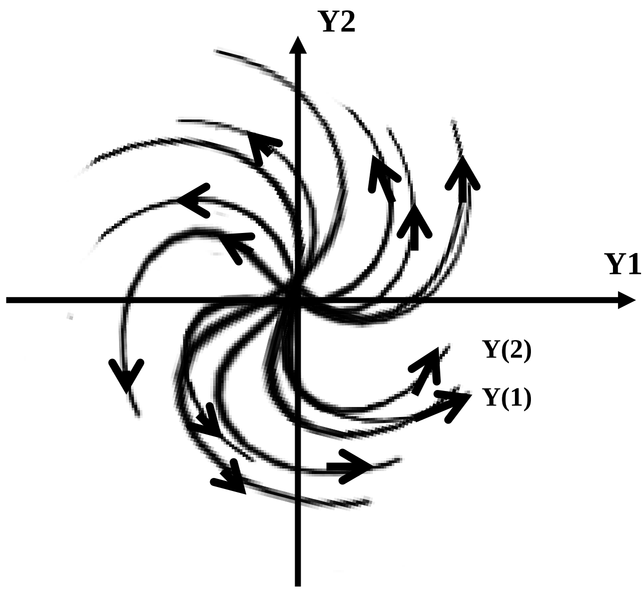


Fig (6): Degenerate node in Ex(7)

$$Y = c_1\, y^{(1)} + c_2\, y^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}$$

This critical point at the origin is often called a degenerate node (or sometimes an improper node, although it differs from that in example (1)). $c_1 Y^{(1)}$ gives the heavy stright line, with $c_1 > 0$ corresponding to the lower part and $c_1 < 0$ corresponding to the upper part. $Y^{(2)}$ gives the right part of the heavy curve from 0 through the second, first, and finally fourth quadrants. $Y^{(2)}$ gives the other part of that curve.

Now suppose that (1) consysts of three or more equations and that A has a triple eigenvalue (μ) with only a single linearly independent eigenvector corresponding to it.

Then we get a second solution (14) with a vector satisfying (15), as just discaussed, and a third of the form

$$y^{(3)} = \frac{1}{2} x t^2\, e^{ut} + u t e^{ut} + v e^{ut}$$

………(16)

with (u) satisfying (15) and (v) determined from

$$(A - \mu I) v = u \qquad ……..(17)$$

which can always be solved

We finally mention that if A has a triple eigenvalue (μ) and two linearly independent eigenvectors $x^{(1)}, x^{(2)}$ corresponding to it, then three linearly independent solution are

$$Y^{(1)} = x^{(1)} e^{ut}\ ,\ y^{(2)} = x^{(2)} e^{ut}\ , Y^{(3)} = x t e^{ut} + u e^{ut} \qquad ………(18)$$

where x is a linear combination of $x^{(1)}$ and $x^{(2)}$

$(A - \mu I) \mu = x$ is solvable for (u) ……...(19)

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