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Invariant Quantity of Prime Knots and Links Using Seifert van-Kampen Theorem

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ABSTRACT: This paper is devoted to calculate the linking number (and the fundamental group) of prime knots. The result shows that the linking number of the knot of type kth prime with crossing number n has linking number also equal to n. Seiferet van-Kampen theorem is implemented to calculate the linking number of some types of links.

Keywords: Knot, Link, Linking number, Fundamental group, Pure cubical complex.

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1. INTRODUCTION

Linking number is one of the topological invariants which is the simplest topological relation between two closed curves (e.g. if these curves are unlinked then the linking number is zero), the linking numbers being known a Gauss linking number (in a double integral formula) [2]. It is used in study of knot and DNA topology (In molecular biology, the linking number becomes particularly important when studying circular DNA molecules, as it is related to the number of helical turns in the DNA, the number of supercoils, and how the DNA can be manipulated by enzymes like topoisomerases) [5]. In computer graphics and robotics is also used to understand the pathways of moving objects and in avoiding collisions [7], also the entanglement of polymer is important for understanding the molecular behaviour [3]. The concept appears in various physical systems as well, such as in fluid dynamics where links can represent vortex lines [4]. The linking number not only has rich mathematical significance but also plays a crucial role in practical applications across various fields [6].

The spaces themselves are extremely flexible and often difficult to study in detail. As an alternative, since algebraic topologists are only concerned in spaces up to homotopy equivalency, we can correlateispaces with more rigid algebraic objects that represent important properties of a space. As an example of an algebraic object, we consider the fundamental group of a space in this section.

Considering the preceding instance, connectivity is unable to distinguish between $\mathbb{R}^2$ and $\mathbb{R}^2-\{0\}$. But by contrasting their essential groupings, we can tell these two areas apart. Any loop in $\mathbb{R}^2$ that is based at a certain point can easily be stretched, squeezed, or otherwise changed to become any other loop based at that same point. A loop in $\mathbb{R}^2-\{0\}$ that surrounds $\{0\}$, but it "gets stuck" on the hole at $\{0\}$ and cannot be molded into any other loop. Therefore, by examining their fundamental groups, we can differentiate between $\mathbb{R}^2$ and $\mathbb{R}^2-\{0\}$.

A knot that is, in a sense, indecomposable is referred to as a prime knot or prime link in knot theory. It cannot be expressed as the knot sum of two non-trivial knots since it is a non-trivial knot. Composite knots or composite links are knots that are not prime. Determining whether a given knot is prime or not can be a challenging task.

Definition 1.1. A prime knot is a knot that cannot be decomposed, i.e. the knot cannot be constructed from the knot sum of two non-trivial knot

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Notation 1.2. the K^{th}.knot having n crossing given in the symbol n_k.

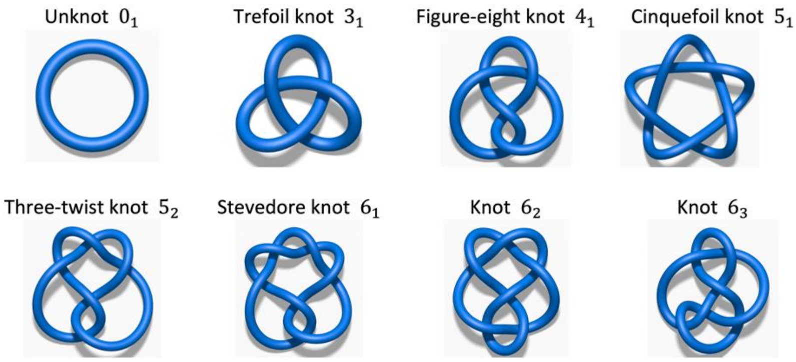


FIGURE 1. – Sample of Prime Knots.

2. BACKGROUND

Any quantity defined on the set of all knots that has the same value for any two knots that are equivalent is called a knot invariant. For instance, a knot invariant is a knot group.

A combinatorial quantity defined on knot diagrams is commonly referred to as a knot invariant. Therefore, two knot diagrams must represent separate knots if they differ with regard to some knot invariant. We cannot, however, conclude that two knot diagrams are the same even if they share values with regard to a [single] knot invariant, as is typically the case with topological invariants.

Other invariants

1. The linking number is a numerical invariant that characterizes how two closed curves in three dimensions space can be linked.
2. Finite type invariant: This sort of invariant, also known as Vassiliev or Vassiliev– Goussarov invariant, is used in Knot theory.
3. Stick number: The knot's comparable polygonal path with the fewest edges

Definition 2.1. A knot is an embedding of the unit circle into R^3 .

Definition 2.2A relation ~ on a set X that possesses the following characteristics is called an equivalence relation on X:

1. If $a \sim a$ for each a in X (reflexivity).
2. If $a \sim b$, then $b \sim a$ (symmetry).
3. If $a \sim b$ and $b \sim c$, then $a \sim c$ (transitivity).

Definition 2.3. An equivalence relation on a set X and an element a of X determine an equivalence class, which is the subset. $E = \{b \mid b \sim a\}$ of X.

Path **homotopy** is an equivalence relation in the space that must be taken into account when establishing the fundamental group of that space.

Definition 2.4. Assume that X represent a space, and a , b be its points. A continuous map f: $[x, y] \rightarrow$ X of a closed interval into X where $f(x) = a \; and \; f(y) = b$ is called a path in X from a to b.

Definition 2.5. Consider the continuous mappings f and f ` from X to Y. If there is a continuous map F: $X \times [0, 1] \rightarrow$ Y where the map f is homotopic to f ‘

$$F(x,0) = f(x)\ and\ F(x,1) = f\,\grave{}\,(x).$$

F is called a **homotopy** between f and f `

Definition 1.5. Given the usual subspace topology, assume that f and f ` be continuous maps from the interval $I = [0,1]$ to X. Should these two paths share the same starting point (x_0) and ending point (x_1), and if there exists a continuous map F: I × I → X such that

$$F(s,0) = f(s)\ and\ F(s,1) = f\,\grave{}\,(s)$$
$$F(0,t) = X_0\ and\ F(1,t) = X_1.$$

F is termed a **path homotopy** among f and f `

An operation on path homotopy equivalence classes will now be defined.

Definition2.6. Assume that g and f be paths in X that go from $x_0$to $x_1$and x_1 to x_2, respectively. From x_0 to x_2, the path in X is defined as the product g ∗ f of g and f.

$$h(s) = \begin{cases} g(2s), & for\ s \in \left[0, \frac{1}{2}\right] \\ f(2s-1) & for\ s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

An operation is induced on the equivalency classes of pathways described by the equation by this product on paths

$$[g] * [f] = [g * f].$$

Since the operation ∗ is limited to two pathways where the beginning point of the second path is the endpoint of the first, the set of all path homotopy equivalence classes of paths in a space is not a group under the aforementioned operation. However, we will only discuss the operation ∗ if we consider pathways that begin and end at the same place.

Definition 2.7. Let X represent a space. A path in X that starts and finishes at the same location, X_0, is termed a loop in X based at X_0.

Definition 2.8. The set of paths homotopy classes of loops based at $X_o$is the fundamental group of a space X with respect to the base point X_o, represented by π_1(X, X_o),

We want to demonstrate that the fundamental group is, in fact, a topological invariant since we introduced it as a tool for differentiating across spaces. As an example of how to demonstrate this.

The point X_0 in X is mapped to y_0 in Y by the map h: (X, X_o) → (Y, y_o). Then, h ∘ f is a loop in Y based at y_0 for any loop f in X based at X_0. We may define a homomorphism among the basic groups of X and Y produced by the continuous map h thanks to this link among loops in X and loops in Y.

Definition 2.9. Let h: $(X, X_0\ ) \to (Y, y_0)$ be an ongoing map. We can then obtain a map.

$$h* : \pi_1(\text{X}, X_0) \to \pi_1(\text{Y},\ y_0\ ),$$

referred to as the homomorphism h induces with respect to the base point X_0 , which is defined by

$$h*\ ([f]) = [h \circ f].$$

Definition2.10. If there is a path in X that connects each pair of points in X, then X is path connected.

Definition 2.11. If a space X is path-connected and π_1(X, X_0) is the trivial group for all $X_0 \in$ X, then the space is simply connected.

Example 2.12. R^n is only linked. Since we can obtain a straight line homotopy F: I × I → R^n among any two loops f and g in R^n located at a point X_o specified by, we may conclude that R^n is clearly path-connected and that its basic group is simple.

$$F(x,t) = (1 - t)f(x) + tg(x)$$

3. LINKING NUMBER USING FUNDAMENTAL GROUP

Theorem 3.1. Assume that k, w: (X, X_0) → π_1(Y, y_0). If f and g are homotopic and the image of X_0 of X is fixed at y_0 during the homotopy, then the induced homomorphisms k_* and w_* are equal.

Proof. We have a homotopy K: X × I → Y from k to w where K(X_0, s) = y_0 for each s. Then, if p is a loop in X based at X_0, there is a homotopy P : I × I → Y clear by P = K ∘ (p × id) from k ∘ p to w ∘ p. P is a path homotopy because p is a loop based at X_0 and K maps X_0 × I to y_0. So, by the definition of the induced homomorphism, k_* = w_*.

A valuable idea known as a deformation retraction arises from the homotopy invariance of the fundamental group. It allows us to "deform" an unfamiliar space to make it resemble a more familiar one.

Definition 3.2. Let A be one of X's subspaces. If the identity map of X is homotopic to a map that conveys each of X into A so that every point of A stays constant during the homotopy, then A is a deformation retract of X. A deformation retraction of X onto A is the homotopy K: X × I → X, such that K(x, 0) = x and K(x, 1) ∈ A, for a each x ∈ X, and K(a, t) = a, for each a ∈ A.

It is advantageous that deformation retracts have the same fundamental groups as the spaces into which they could be "deformed" due to the homotopy invariance of fundamental groups.

Corollary 3.3. Assume that B be a deformation retract of X, and X_o ∈ B. Then the inclusion map L: (B, X_o) → (X, X_o) induces an isomorphism of fundamental groups π_1(B, X_o) → π_1(X, X_o),

3.1. CW COMPLEX

A CW Complex is a topological space constructed by joining the n − dimensional disc border onto h − dimensional spheres with h < n. The following is an inductive definition of the structures:

Definition3.4. A CW complex X is constructed inductively starting from X^o which is a collection of points. X^{n+1} is constructed by a collection of attaching maps of the form $j_\alpha : S^n \to X^n$ and producing the quotient space of the disjoint union $X^n \sqcup_\alpha D_\alpha^{n+1}$ such that x ∈ ∂D_α^n is identified with j_α (x).

Definition 3.5. An open disc $D^n \setminus \partial D^n$ that has been attached to X^{n-1} is termed an n − cell. $X^n \subset$ X is termed an n-skeleton.

We can begin by creating basic CW complexes from scratch in order to improve our mental understanding of CW complexes.

Example 3.6. (S^2 as a CW complex). Although there are many ways to express S^2 as a CW complex, the simplest construction begins with a single point X^o = {a}. We do not need any l-cells, so instead, we go straight to building our 2-cells. Note that since we have no 1-cells, $X^o = X^1$. Now, we use a map k : $\partial D^2 \to X^1$ where for all d ∈ ∂D^2 , k : d → a. This completes our construction of S^2 . In fact, extrapolating from this instance, we can see that S^n can be represented as a single-pointed, single-n-cell CW complex. The process of building a torus is another comparatively easy illustration.

Example3.7. (Torus as a CW complex). We can begin with just one vertex (x) to construct a torus once more. Next, two D^1 (which is simply a line segment) are attached to the point x using the maps $j_1, j_2 : \partial D^1 \to X^0$. This has the appearance of two rings sharing a point. The rings will be referred to as S_1 and S_2. The loop corresponding to the attaching map is $S_1 \cdot S_2 \cdot S_1^{-1} \cdot S_2^{-1}$. We then apply one final attaching map, $j_3 : \partial D^2 \to X^1$. This brings our torus construction to a close.

The picture of the loop to which the disc is being attached does not always change by one rotation around the boundary of the attaching disk. It is important to remember this while considering CW complexes. For instance, we can connect D^2 onto S^1 to create a CW complex X, where one revolution around ∂D^2 equals two revolutions around S^1. Our topological space is very different from just placing D^2 onto S^1 (which we'll term Y) since a loop around the edge of Y belongs to the same homotopy class as the constant loop in Y, whereas a similar loop in X does not share the same equivalence class as the constant loop. In reality, adding a disc to loop l in some dimension k creates a continuous space for mapping loop l to the constant loop, combining [l] and [c] in π_K (X). in the process of building a CW complex. RP^2, or the real projective plane, is another name for X.

3.2. Pure Cubical Complex

The subspaces of the tessellated space $\mathbb{R}_c^n$ that are covered in this article have structures that are inherited from the lattice produced by n orthonormal basis vectors. Combinatorially, a facet of $\mathbb{R}_c^n$ is equivalent to an n-cube. The next section develops the theory for cubical complexes, abstract cubical complexes, and cubical complexes based on these tessellations.
Because pure cubical complexes only deal with cubes of the same dimension, the associated techniques and representation are made simpler. Cubical complexes are required because homology computations and other related tasks require the distinction between edges, vertices, and other cubes. However, dealing with cubes of several dimensions complicates representation and methods compared to pure cubical complexes. Abstract cubical complexes may work better in higher-dimensional or sparser data sets.

Note that any v ∈ L_c^n generates a Dirichlet-Voronoi cell
$D(v) = \{x \in \mathbb{R}^n : ||v - x|| \leq ||w - x||$ for any $w \in L_c^n\}$, and that the n-dimensional cubical lattice L_c^n is an additive subgroup created by n orthonormal vectors in $\mathbb{R}^n$.

The Euclidean space from which A tessellated space with cells $D(v)$ as its facets is inherited by $\mathbb{R}^n$. The symbol for this tessellated space is $\mathbb{R}_c^n$.

Definition 3.9. A pure cubical space X is a tessellated subspace of $\mathbb{R}_c^n$. The dimensions of X is a list of lengths, $[d_1, d_2, \ldots, d_n]$, where d_i is the maximal difference among the i^{th} coordinates of any two points in X. By applying a translation if necessary we can let X is a union of translations of facets $D(v)$ where any v has only positive integer coordinates with respect to the given basis.

Definition 3.10. A pure B-complex $H = (A, B, T)$ consists of

- a binary array A = (a_λ) of dimension d,
- a basis $T = \{t_1, t_2, \ldots, t_d\}$ for $\mathbb{R}^d$,
- a finite set B ⊆ $\mathbb{Z}^d$. We call B the ball.

Definition 3.11. Recall our Definition 3.2 of a pure B − complex. An n-dimensional pure cubical complex K is a pure B − complex with orthonormal basis vectors, $(t_1, t_2, \ldots, t_n)$.
Let Γ be the set of all $k = (k_1, k_2, \ldots, k_n)$ where $k_i \in \{-1, 0, 1\}$. The neighbourhood N_k (K_λ) of an entry K_λ in an n-dimensional pure cubical complex K consists of all entries $K_{\lambda+K}$ for κ ∈ Γ. The neighbourhood N_X (X_λ) of a facet X_λ in an n-dimensional pure cubical space consists of the union of aech facets $X_{\lambda+K}$ in $\mathbb{R}_c^n$. Previously described algorithms for tessellated spaces can be applied to pure cubical complexes.

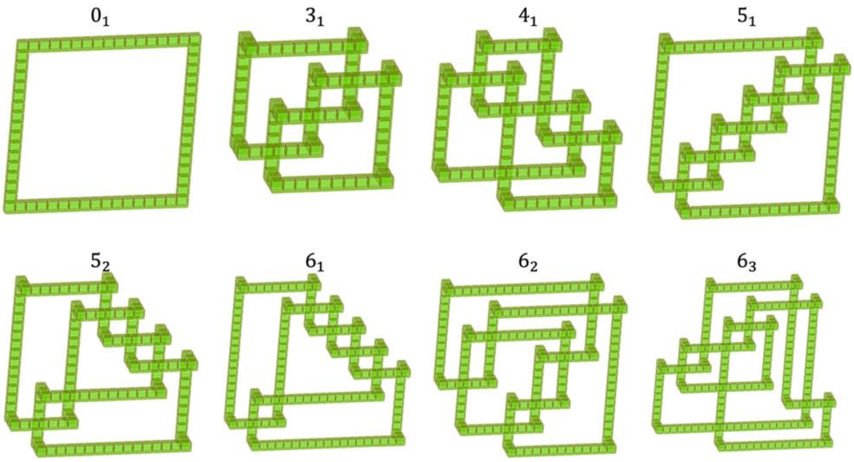


FIGURE 2. – ***Pure cubical complex the prime knots.***

Definition 3.12. A cubical complex K of dimension d is a binary array with index λ ranging over the set
$$\Lambda = \{1, 2, \ldots, 2_{n1} + 1\} \times \{1, 2, \ldots, 2_{n2} + 1\} \times \ldots \times \{1, 2, \ldots, 2_{nd} + 1\} \subset N^d$$

That arises from a CW-subspace of a pure cubical complex
The dimensions of K are the integer vector $(2_{n1}+1, 2_{n2}+1, \ldots, 2_{nd}+1)$. There is one axiom: if some entry $K_\lambda = 1$ then the entry $K_{\lambda^{`}} = 1$, such that $K_{\lambda^{`}} \in \Lambda$ is obtained by adding ±1 to an even entry in the index λ.
We term ξ(λ) to be the number of even entries in λ.

Definition 3.13. A cubical space $X \subset \mathbb{R}_c^n$ is a CW-subspace. The cell X_λ of X is a ξ(λ)-cube of the d-cubes $X_{\lambda+k}$ present in X in $\mathbb{R}_c^n$ such that $k \in [-1, 0, 1]^n$ such that ξ(k + λ) = d. By $[-1, 0, 1]^n$ we mean all possible lists of length n with entries chosen from [−1, O, 1].

Lemma 3.14. [11] Let Γ be a pure (d - 1)-dimensional simplicial complex. ("Pure" means that all maximal faces have d elements.) Then

$$\sum_{F\in\Gamma}(-1)^{d-\#F}\ h(1\mathrm{k}_\Gamma F, x) = -(x-1)^d \widetilde{\mathcal{X}}\ (\Gamma)$$

where $\widetilde{\mathcal{X}}\ (\Gamma) = -1 + f_0 + f_1 + \cdots$, the reduced Euler characteristic of Γ.

Theorem 3.15. [11] Let Δ be a pure (d - l)-dimensional simplicial complex, and let Δ′ be a simplicial subdivision of Δ. Then

$$h(\Delta', x) = \sum_{F\in\Delta}(\Delta'_F, x)\ h(1\mathrm{k}_\Delta F, x)$$

Theorem 3.16. Assume that K be a pure cubical complex. For all cubical subdivision K' of K we have (4.1)

$$h^{(sc)}(K', x) = \sum_{F\in\mathcal{F}(K)} \ell_F\ (K'_F, x) h(1\mathrm{k}_k(F), x)$$

Proof. Denoting by $R(K', x)$ the right-hand said of (4.1) and setting $P = \mathcal{F}(K)$, we compute that

$$R(K', x) = \sum_{G\in P} \ell_G\ (K'_G, x) h(1\mathrm{k}_k(G), x)$$

$$= \sum_{G\in P}\left[\sum_{F\le_p G}(-1)^{\dim(G)-\dim(F)}\ h^{(sc)}(K'_F, x)\right] h(1\mathrm{k}_k(G), x)$$

$$= \sum_{F\in P} h^{(sc)}(K'_F, x)\left[\sum_{F\le_p G}(-1)^{\dim(G)-\dim(F)}\ h(1\mathrm{k}_k(G), x)\right]$$

$$= -\sum_{F\in P}[\sum_{E\in K'_F\backslash\{\emptyset\}}(2x)^{\dim(E)}\ (1-x)^{\dim(K)-\dim(E)}\ ]\ \widetilde{\mathcal{X}}\ (1\mathrm{k}_k(F))$$

$$= -\sum_{E\in K'\backslash\{\emptyset\}}(2x)^{\dim(E)}\ (1-x)^{\dim(K)-\dim(E)} \sum_{\sigma(E)\le_p F}\widetilde{\mathcal{X}}\ \big(1\mathrm{k}_k(F)\big).$$

The defining equation yields the fourth of the preceding equalities.

$$h^{(sc)}(K'_F, x) = \sum_{E\in K'_F\backslash\{\emptyset\}}(2x)^{\dim(E)}\ (1-x)^{\dim(F)-\dim(E)}$$

and the equality

$$\sum_{F\le_p G}(-1)^{\dim(G)-\dim(F)}\, h(1\mathrm{k}_k(G), x) = -(1-x)^{\dim(K)-\dim(F)}\widetilde{\mathcal{X}}\ \big(1\mathrm{k}_k(F)\big),$$

For the given F ∈ P. The latter is comparable to

$$\sum_{F\le_p G}(-1)^{\dim(K)-\dim(G)}\, h(1\mathrm{k}_k(G), x) = -(1-x)^{\dim(K)-\dim(F)}\widetilde{\mathcal{X}}\ \big(1\mathrm{k}_k(F)\big),$$

Then, using [17, Lemma3.1] to $1\mathrm{k}_k(F)$, a pure simplicial complex of dimension $\dim(K)-\dim(F)-1$, follows suit. Similar to the demonstration of Theorem 3.2 in [17, p. 813], we discover that

$$\sum_{\sigma(E)\leq_p F} \widetilde{\mathcal{X}}\left(1\mathrm{k}_k(F)\right) = -1$$

and so

$$R(K',x) = \sum_{E\in K'_F\setminus\{\emptyset\}} (2x)^{\dim(E)}\ (1-x)^{\dim(F)-\dim(E)} = h^{(sc)}(K',x)$$

as stated in the theorem's wording.

Proposition 3.17. For all cubical subdivision Γ of a cube C of dimension d ≥ 1 we have

$$x\,\ell_C(\Gamma,x) = (x+1)\,L_C(\Gamma,x).$$

Consequently, we have
$L_0 = L_{d+1} = 0$ and $L_{i+1} = \ell_i - \ell_{i-1} + \cdots + (-1)^i\,\ell_0$ for $0 \leq i \leq d-1$, where $L_C(\Gamma) = (L_0, L_1, \ldots, L_{d+1})$ and $\ell_C(\Gamma) = (\ell_0, \ell_1, \ldots, \ell_d)$.

Proof. Since $\tilde{x}(\Gamma_F) = 0$ for $F \in f(C)$, using

$$((x+1)h^{(c)}(\mathrm{K},\mathrm{x}) = 2^d + xh^{(sc)}(\mathrm{K},\mathrm{x}) + (-2)^d\,\tilde{x}(K)x^{d+2}\text{, to } \Gamma_F \text{ we obtain}$$

$$(x+1)\,h^{(c)}(\Gamma_F,\mathrm{x}) = 2^{\dim(F)} + xh^{(sc)}(\Gamma_F,\mathrm{x}).$$

The outcome is obtained by adding $(\mathrm{L}_C(\Gamma,x) = \sum_{F\in F(C)}(-1)^{d-\dim(F)}\ h^{(c)}(\Gamma_F,\mathrm{x}))$ by x + 1 and by the previous equality.

Theorem 3.18. Assume that K be a pure cubical complex. For all cubical subdivision K′ of K we possess

$$h^{(c)}(\mathrm{K}',\mathrm{x}) = h^{(c)}(\mathrm{K},\mathrm{x}) + \sum_{F\in K:\dim(F)\geq 1} L_F(K`_F,\mathrm{x})\ \ \mathrm{h}(\mathrm{l}k_k(\mathrm{F}),\mathrm{x}).$$

Proof. We utilize Proposition (6.3), equation, and multiply (4.1) by x.

$$xh^{(sc)}(\mathrm{K},\mathrm{x}) = \sum_{v\in vert(K)} h(lK_K(v),x)\,,$$

and the fact that $\ell_F(K`_F,\mathrm{x}) = 1$ for all face with zero-dimensional F ∈ K to so

$$x\,h^{(sc)}(K',x) = (x+1)\sum_{F\in K:\dim(F)\geq 1} L_F(K`_F,\mathrm{x})\ \ \mathrm{h}(\mathrm{l}k_k(\mathrm{F}),\mathrm{x}). + \sum_{v\in vert(K)} h(lK_K(v),x)\,,$$

$$= (x+1)\sum_{F\in K:\dim(F)\geq 1} L_F(K`_F,\mathrm{x})\ \ \mathrm{h}(\mathrm{l}k_k(\mathrm{F}),\mathrm{x}) + x\,h^{(sc)}(K,x)$$

Applying $((x+1)h^{(c)}(\mathrm{K},\mathrm{x}) = 2^d + xh^{(sc)}(\mathrm{K},\mathrm{x}) + (-2)^d\,\tilde{x}(K)x^{d+2}$, to K and K′ and noting that $\tilde{x}(K) = \tilde{x}(K')$ we obtain

$$(x+1)\,h^{(c)}(K',x) - xh^{(sc)}(K',x) = (x+1)\,h^{(c)}(K,x) - xh^{(sc)}(K,x).$$

After summing the two equalities from before and dividing by x + 1, the outcome is obtained.

3.3. Seifert van-Kampen Theorem

Theorem 3.19. Assume that X be a topological space and let $U, V \subset X$ be open subsets where $U \cap V$ is nonempty and path-connected. Assume that $x \in U \cap V$ be a basepoint. Then

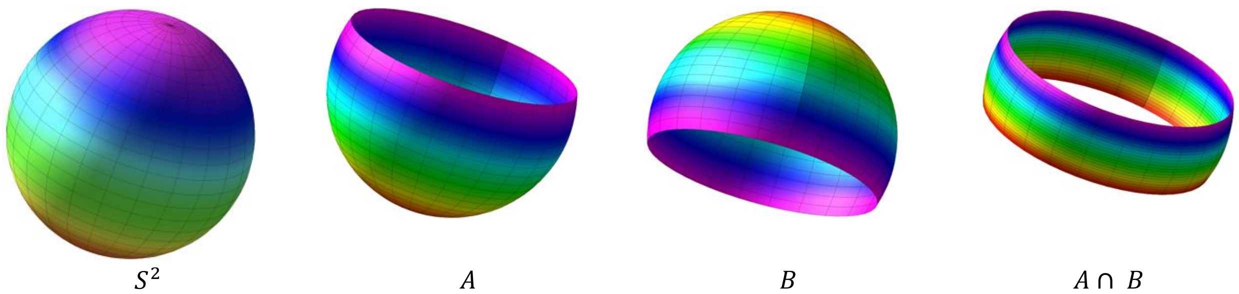
$$\pi_1(X,x) = \pi_1(U,x) * \pi_1(V,x)$$

Here, $A * c\ B$ denotes the amalgamated product. Let you have groups A, B, C and homomorphisms $f: C \to A$ and $g: C \to A$. In our case, $A = \pi_1(U,x)$, $B = \pi_1(V,x)$ *and*

$C = \pi_1(U \cap V, x)$, the map f is the pushforward map i_* where $i: U \cap V \to U$ is the inclusion, and gg is the pushforward j_* where $j: U \cap V \to V$ is the insertion.

Given A, B, C, f, g, you can define the amalgamated product $A *_c B$ = ⟨generators of A, generators of B | relations of A, relations of B, amalgamated relations⟩.
The amalgamated relations come from elements $c \in C$: each $c \in C$ gives a relation $f(c) = g(c)$.

Example 3.20. Consider the sphere S^2. Let $S^2 = A \cup B$ where A and B are the two hemispheres (north and south) with non-empty overlapping as shown in the following figure.



The overlapping region $A \cap B$ is annular space as showing in figure above (on the right). It is obvious $\pi_1(A) = \pi_1(B) = 1$ and the fundamental group of $A \cap B$ which is equivalence to a circle then $\pi_1(A \cap B) = Z$. Now the amalgamated product has the generators of both $\pi_1(A)$ and $\pi_1(B)$ which are trivial. So, there is no effect on the relators and this gives that $\pi_1(S^2) = \{1\}$.

3.2.1 Pure Cubical Complex of Links

Any link can be presented as a pure cubical complex as an array of dimension 5×A, where the binary matrix A of dimension $n \times n$. Each entry represents a cube or empty in the space depends on the value.

Example 3.21. The prime knot of type the 5th prime with 2 crossing is presented in the following Figure:

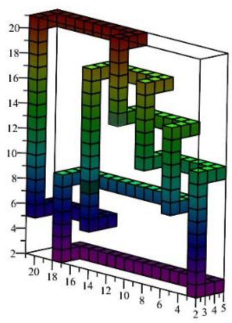


FIGURE 3. – ***Pure cubical complex of the 5th prime knot with crossing 5.***

This knot has crossing number equal to 2.
The following is the GAP session used to calculate the linking number using the GAP program which is an open access program can be used to do calculation in Algebra and GAP is the abbreviation of Group Algorithm Programming.

```
gap> K:=PureCubicalKnot(5,2);
```

```
prime knot 2 with 5 crossings
gap> P:=PurePermutahedralComplex(K!.binaryArray);
Pure permutahedral complex of dimension 3.
gap> C:=ZigZagContractedPureComplex(P);
Pure permutahedral complex of dimension 3.
gap> Y:=PermutahedralComplexToRegularCWComplex(C);;
gap> i:=BoundaryPairOfPureRegularCWComplex(Y);
Map of regular CW-complexes
gap> CriticalCellsOfRegularCWComplex(Source(i));
[ [ 2, 1 ], [ 1, 58 ], [ 1, 154 ], [ 0, 180 ] ]
gap> phi:=FundamentalGroup(i,180);
[ f1, f2 ] -> [ f1, <identity ...> ]
gap> RelatorsOfFpGroup(Source(phi));
[ f2*f1^-1*f2^-1*f1 ]
```

First, we must create a CW-complex for the Knot and then calculate the fundamental group of the complement with a base point equal to 180 as shown in the step 6 which is the 0-cell with number equal to 180.

From the result in the last two steps, we can see that the fundamental group is generated by two generators f_1 and f_2 with relator $f_2 * f_1^{-1} * f_2^{-1} * f_1$ which mean that the given knot has linking number equal to 2. Consequently, we can calculate the linking number of any link with two components by calculating the fundamental group of its complement using the Seifert van − Kampen theorem which state that the fundament group of any space with more than one component is equal to

$$\pi_1(A \cup B) = \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$

Example 3.22. Consider the link of two string that given in the following figure with two colors (K_1 in red and K_2 in blue).

$$K_1(t) = (2 + \cos t) \cos t \ \hat{\imath} + (2 + \sin t) \cos t \ \hat{\jmath} + \sin 5t \ \hat{k}$$
$$K_2(t) = (2 + \cos(5t)) \cos(2t + \pi) \ \hat{\imath} + (2 + \sin(5t)) \cos(2t + \pi) \ \hat{\jmath} - \sin 2t \ \hat{k}$$

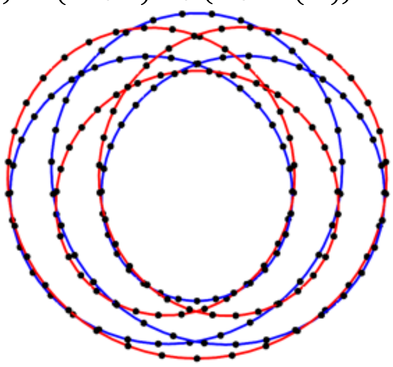


FIGURE 4. – Link of two strings $\mathbf{K_1}$ and $\mathbf{K_2}$.

```
LinkingNumberOfLinkOf2ndOrder:=function(K1,K2)
local K,P,C,Y,i,CC,a0,phi,rels,ab;
K:=PureCubicalKnot(K1);
K:=PureComplexComplement(K2);
P:=PurePermutahedralComplex(K!.binaryArray);
C:=ZigZagContractedPureComplex(P);
Y:=PermutahedralComplexToRegularCWComplex(C);
i:=BoundaryPairOfPureRegularCWComplex(Y);
CC:=CriticalCellsOfRegularCWComplex(Source(i));
a0:=CC[Length(CC)][2];
phi:=FundamentalGroup(i,a0);
ab:=NqEpimorphismNilpotentQuotient(Target(phi),1);;
return Size(Image(ab)/Image(ab,Image(phi)));
end;
```

Applying the above code to get the linking number of the link in Figure 4 to get the linking number:

```
> LinkingNumberOfLinkOf2ndOrder:=function(K1,K2)
5
```

We can also use Seifert van-Kampen theorem to find the linking number as follows:
First, we divide the data into two parts A and B with intersection non-empty $A \cap B$, then apply the van-Kampen Theorem in order to calculate the fundamental groups for each part and the common area, i.e.

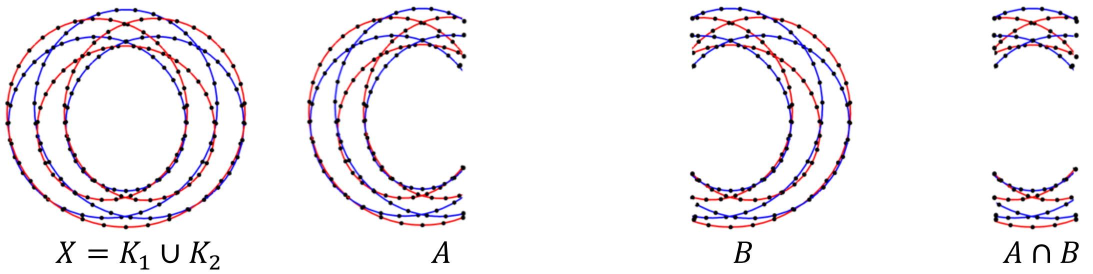
$$\pi_1\,(A), \pi_1\,(B), \pi_1(A \cap B)$$

Second, we do the product in the following manner

$$\pi_1(A \cup B) = \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$

Example 3.23: Consider the link given in the Example 3.22, we will apply the GAP function SeieferVanKampen(A,B) the inputs are the three data sets which represent the two pieces of the whole space.

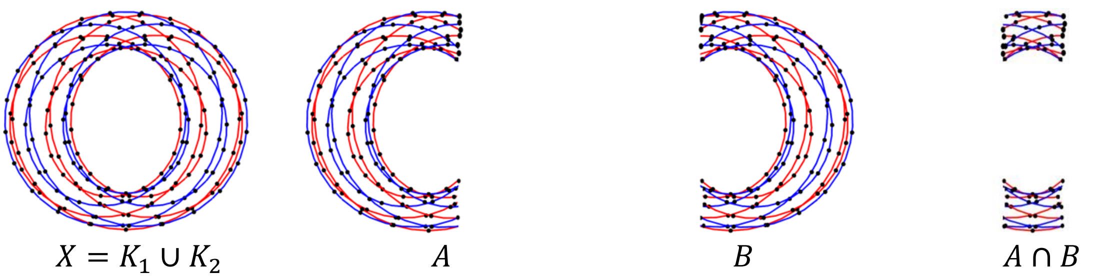
```
> SeieferVanKampen(A,B)
5
```



Example 3.23: Consider the link given in the following figure, we will apply the GAP function SeieferVanKampen(A,B) the inputs are the three data sets which represent the two pieces of the whole space and their intersection C.

```
> SeiferVanKampen(A,B)
5
```

Figure 5 : Link of two strings K_1 and K_2.



4. CONCLUSIONS

The Sefiert van-Kampen theorem is applicable and can be implemented for calculation the fundamental group and can be used in order to find the linking number of any link by converting the link into pure cubical space and then calculate the fundamental group of its complement, the results show that linking number of two component link have the same result in both the traditional method or the Sefiert van-Kampen theorem.

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