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Regular Attractor by Strict Lyapunov Function for Random Dynamical Systems

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ABSTRACT: The main objective of this article is to study regular random attractors in gradient random dynamical systems (RDS) based on the random strict Lyapunov function. This goal is achieved by defining a strict random Lyapunov function and then using it as a sufficient condition for the existence of a global attractor for gradient random dynamical systems. Secondly, a sufficient condition is found that ensures the existence of a regular attractor for the gradient RDS. Indeed, we show that the compact global attractor for an RDS with the finite set of random fixed points can be viewed as an unstable manifold. This global attractor includes complete trajectories that converge to the random fixed points. Also, it is shown that all components of the compact global attractor pertain to the whole trajectory if the gradient RDS has a finite number of random fixed points. Finally, it is shown that under certain conditions, if the gradient RDS in a Banach space has a random strict Lyapunov function, then it has a regular compact attractor.

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1. INTRODUCTION

The Lyapunov function has been used in several researches on the stability of (deterministic) dynamical systems. For example, see [1], [7],[8],[9], [12], [13] and [14]. However, in the following, a literature that dealt with the Lyapunov functions for the purpose of studying the stability of random dynamical systems and random sets, will be mentioned. Lyapunov's second method for random dynamical systems was created by L. Arnold and B. Schmalfuss [2]. They proved that that a necessary and sufficient conditions for the stability of a random set is the existence of the Lyapunov function. A plethora of literature reviewing the stability of random dynamical systems with Lyapunov functions can be found, including works by [3], [10], and [11].

The rest of the research is structured as follows: section 2 presents some essential concepts and facts related to the research topic. In Section 3 the concepts of random strict Lyapunov function for RDS and gradient RDS are introduced and studied. Section 4 presents and study the concept of regular random attractor for RDS.

Throughout, the triple $\Omega \equiv (\Omega, F, P)$ consider as a probability space and $X \equiv (X, \|\cdot\|)$ be a Banach space.

List of abbreviations

MDS	Metric dynamical system.
RDS	RFandom dynamical system.
TRV	Tempered random variable.
UA	Uniform attractor.
GA	Global attractor.
CGA	Compact global attractor.
RSLF	Random strict Lyapunov function.

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2. PRELIMINARIES

This section is devoted to mentioning some basic concepts in random dynamical systems.

Definition 2.1 [1,5] The measurable action $\theta: \mathbb{R} \times \Omega \longrightarrow \Omega$ that is verified $\mathbb{P}(\theta_t B) = \mathbb{P}(B)$ for every $B \in \mathcal{F}$ and $t \in \mathbb{R}$ is called **metric dynamical system** (shortly, MDS).

Definition 2.2 [1,5] A pair (θ, φ) where θ is a MDS and $\varphi: \mathbb{T} \times \Omega \times X \longrightarrow X$ is a function verify

$$\varphi(0, \omega) = id, \varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)\ , where\ \ t, s \in \mathbb{R}\ and\ \omega \in \Omega. \tag{1}$$

is called the **random dynamical system** (shortly, RDS). The function φ called co-cycle over θ.
If, in addition, the functions $\varphi(\cdot, \omega, \cdot): \mathbb{R} \times X \longrightarrow X$ is continuous for every $\omega \in \Omega$, then the co-cycle with the MDS θ called random dynamical system (RDS) over θ.

Definition 2.3 [1,5] Let (X, d) be a metric space.

(i) The multi-valued map $M: \Omega \longrightarrow 2^X/\{\emptyset\}$ is called a **random set** if for every $x \in X$, the function $\rho: \Omega \longrightarrow \mathbb{R}^+$,where $\rho(\omega) \coloneqq \mathrm{dist}_X(x, M(\omega))$ is measurable. If for each $\omega \in \Omega$ the set $M(\omega)$ is closed (compact) in X, then $M(w)$ is called a closed (resp. compact) random set.

(ii) A **tempered random variable** (TRV) is a random variable $\varepsilon: \Omega \longrightarrow \mathbb{R}$ satisfies $\lim_{t \longrightarrow +\infty} \frac{1}{|t|} \log|\varepsilon(\theta_t \omega)| = 0$.

Definition 2.4: Let $D(\omega)$ be a random set in a RDS (θ, φ) .

(i) The random trajectory [10] emanating from $D(\omega)$is defined by $\gamma_D^t(\omega) \coloneqq \bigcup_{\tau \geq t} \varphi(\tau, \theta_{-\tau}\omega) D(\theta_{-\tau}\omega)$.

(ii) A random curve and $\gamma(\omega) \equiv \{u_t: \Omega \longrightarrow X: t \in \mathbb{R}\}$in X is said to be a ***full trajectory*** if

$$\varphi(t, \theta_{-t}\omega) u(\tau, \theta_{-\tau}\omega) = u(t + \tau, \omega) \text{ for any } \tau \in \mathbb{R} \text{ and } t \geq 0.$$

Definition 2.5 [5] For a random set $D(\omega)$, we define the omega-limit set of $\gamma_D^t(\omega)$ starting from $D(\omega)$ by the random set.

$$\Gamma_D(\omega) = \left\{x \in X: \exists\ \{t_n\} \subset \mathbb{R}^+, \{y_n\} \subset D\left(\theta_{-t_n}\omega\right) \ni x = \lim_{n \longrightarrow +\infty} \varphi(t_n, \theta_{-t_n}\omega) y_n\right\}.$$

In the following, we will define some types of attractors of random sets similar to what was stated in [2], and [5].

Definition 2.6 [2,5]**:** Let $M(\omega)$ be a random compact set in a RDS (θ, φ) .

(i) If $\varphi(t, \omega) M(\omega) \subseteq M(\theta_t \omega)$ for all $t > 0$ and $\omega \in \Omega$, then $M(\omega)$ is called ***forward invariant***.

(ii) A random variable $x: \Omega \longrightarrow X$ is called **random fixed point** for (θ, φ) if $\varphi(t, \omega) x(\omega) = x(\theta_t \omega)$ for every $t \in \mathbb{R}$.

(iii) If there is a TRV δ such that

$$\lim_{t \to +\infty} \sup_{y \in S(\theta_{-t}\omega)} \inf_{x \in M(w)} \|\varphi(t, \theta_{-t}\omega) y - x\| = 0, w \in \Omega\ ,$$

where $S(\omega) \coloneqq S\big(M(\omega), \delta(\omega)\big) = \left\{x \in X: \inf_{z \in M(\omega)} \|x - z\| < \delta(\omega)\right\}$, then $M(\omega)$ is called an ***attractor*** of (θ, φ).

(iv) If there exist TRV $\delta(\varepsilon)$ and for each TRV $\varepsilon(\omega)$, there exists $\tau \coloneqq \tau(\varepsilon)$ such that

$$\sup_{y \in \gamma_x^\tau(\omega)} \inf_{x \in M(w)} \|\varphi(t, \theta_{-t}\omega) y - x\| < \varepsilon(\omega), \text{for } x \in S[M(\omega), \delta(\theta_{-t}\omega)],$$

then $M(\omega)$ is called a ***uniform attractor*** **(UA)** of (θ, φ).

(v) A bounded random set is called **global attractor** (**GA**) if it is attractor and UA**.**

Remark 2.7. Let $\mathfrak{A}(\omega)$ be a GA for the RDS $(\theta, \varphi)$and $\gamma(\omega) \equiv \{u_t: \Omega \longrightarrow X: t \in \mathbb{R}\}$ be a full trajectory.

(i) $\gamma(\omega) \subset \mathfrak{A}(\omega)$ if and only if γ is a bounded random set.

(ii) For every $x \in \mathfrak{A}(\omega)$ there exists a full trajectory γ such that $u(0, \omega) = x$ and $\gamma \subset \mathfrak{A}(\omega)$.

Definition 2.8. Let $\mathcal{N}$ be the set of random fixed points of a RDS (θ, φ):

$$\mathcal{N}: \{v \in X^\Omega: \varphi(t, \theta_{-t}\omega) v(\theta_{-t}\omega) = v(\omega) \text{ for all } t \geq 0\}$$

the ***unstable manifold*** $\mathcal{M}^u(\mathcal{N})$ starting from the set $\mathcal{N}$ is defined by a set of all $y \in X$ such that there exists a full trajectory $\gamma \equiv \{u_t: \Omega \longrightarrow X: t \in \mathbb{R}\}$ with

$$u_0(\omega) = y \text{ and } \lim_{t \longrightarrow -\infty} inf_{v \in \mathcal{N}} \|u_t(\omega) - v(\omega)\| = 0 \tag{2}$$

Remark 2.9 . It is clear that $\mathcal{M}^u(\mathcal{N})$ is a (strictly) invariant random set.

Proposition 2.10. *Let $\mathcal{N}$ be the set of random fixed points of a RDS (θ, φ) having a random* GA $\mathfrak{A}(\omega)$. *Then $\mathcal{M}^u(\mathcal{N}) \subset \mathfrak{A}(\omega)$.*

Proof. Let $y \in \mathcal{M}^u(\mathcal{N})$ and $\gamma \equiv \{u_t: \Omega \longrightarrow X: t \in \mathbb{R}\}$ be the trajectory having property (2). Then there exists $s \leq 0$ such that the set

$$\gamma_s(\omega) \equiv \{u_t: \Omega \longrightarrow X: -\infty < t \leq s\} \subset \{z: \inf_{v \in \mathcal{N}} \|z - v(\omega)\| \leq 1\}$$

Hence $\gamma_s(\omega)$ is bounded. Also $\gamma_s(\omega)$ is backward invariant, i.e., $\gamma_s(\omega) \subset \varphi(t, \theta_{-t}\omega)\gamma_s(\theta_{-t}\omega)$ for every $t > 0$. Consequently, $\gamma_s(\omega) \subset \mathfrak{A}(\omega)$. Since $y \in \varphi(-s, \theta_s\omega)\gamma_s(\omega)$, this indicates the desired deduction.

Definition 2.11 (Fréchet derivative)[4]. Let O be an open set in X. A mapping $F: O \longrightarrow Y$ is said to be ***Fréchet differentiable*** on O if for any $u \in O$ there exists a bounded linear operator $F'(u)$ from X into Y such that

$$\frac{\|F(v) - F(u) - F'(u)(v - u)\|_Y}{\|v - u\|_X} \longrightarrow 0 \text{ as } \|v - u\|_X \longrightarrow 0.$$

As in [6], we will introduce the following concepts of hyperbolic random fixed point and the index of instability .

Definition 2.12(*Hyperbolic random fixed point*). Suppose that the cocycle operator $\varphi(t, \omega)$ of an RDS (θ, φ) is of class C^1; that is, $\varphi(t, \omega)$ has a continuous Fréchet derivative with respect to $u \in X$ for each $t > 0$ and each $\omega \in \Omega$. A random fixed point z of RDS (θ, φ) is said to be ***hyperbolic*** if the Fréchet derivative $\varphi' \equiv D\varphi(1, \omega)z$ of $\varphi(t, \omega)z$ at the $t = 1$ is a linear operator in X with the spectrum $\sigma(\varphi')$ possessing the property

$$\sigma(\varphi') \cap \{w \in \mathbb{C}: |w| = 1\} = \emptyset.$$

The index ind (z) (of instability) of the random fixed point z is defined as a dimension of the spectral subspace of the operator φ' corresponding to the set $\sigma_+(\varphi') \equiv \{z \in \sigma(\varphi'): |z| > 1\}$.

STRICT RANDOM LYAPUNOV FUNCTION

In this section the strict Lyapunov function (in the random case) for a RDS (θ, φ) is defined. The gradient RDS (see [6]) is defined by using strict Lyapunov function. Then we establish the relation between the existences of strict Lyapunov function and the existences of the compact global attractor (CGA)

Definition 3.1. [2] Let $Y(\omega) \subset X$ be a forward invariant random set of an RDS (θ, φ) . A function $\mathcal{L}: \Omega \times Y(\omega) \longrightarrow \mathbb{R}^+$ defined on $Y(\omega)$ is called a ***random Lyapunov's function*** for (θ, φ) if

(i) $\mathcal{L}(\omega, \cdot): Y(\omega) \longrightarrow \mathbb{R}^+$ is continuous for every ω ,
(ii) $\mathcal{L}(\cdot, y): \Omega \longrightarrow \mathbb{R}^+$ is measurable for every y
(iii) $\mathcal{L}(\theta_t\omega, \varphi(t, \omega)y) \leq \mathcal{L}(\omega, y)$ for any $y \in Y(\omega)$.

Definition 3.2 The random Lyapunov function $\mathcal{L}: \Omega \times Y(\omega) \longrightarrow \mathbb{R}^+$ is called ***strict*** on $\Omega \times Y(\omega)$ if $\mathcal{L}(\theta_t\omega, \varphi(t, \omega)y) = \mathcal{L}(\omega, y)$ for *all* $t > 0$ and for some $y \in Y$ implies that y is a fixed point of (θ, φ).

From now on, we will symbolize the "random strict Lyapunov's function" by the abbreviation "RSLF".

Definition 3.3 [3] If there exists RSLF for (θ, φ) on X, then $(\theta, \varphi)$is called ***gradient***. This random Lyapunov's function is usually called ***global***.

Example 3.4. Let θ be a MDS. Consider the random ordinary differential equation

$$\dot{x}(t) = -\nabla F(\theta_t\omega, x(t)), t > 0$$

where $F: \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a C^2 function with $F(x) \longrightarrow +\infty$ as $|x| \longrightarrow \infty$. This random ordinary differential equation generates a RDS (θ, φ) which has a RSLF $\mathcal{L}: \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $\mathcal{L}(\omega, x) \coloneqq F(\omega, x)$.

Theorem 3.5. *Let a RDS* (θ,φ) *have a* CGA $\mathfrak{A}(\omega)$. *Suppose that there is a* RSLF *on* $\mathfrak{A}(\omega)$. *Then* $\mathfrak{A}(\omega) = \mathcal{M}^u(\mathcal{N})$, *where* $\mathcal{M}^u(\mathcal{N})$ *denotes the unstable manifold starting from the set* $\mathcal{N}$ *of random fixed points. Furthermore, the GA* $\mathfrak{A}(\omega)$ *involves of full trajectories* $\gamma(\omega) \equiv \{u_t : \Omega \longrightarrow X\}$ *such that*

$$\lim_{t\to-\infty} inf_{v\in\mathcal{N}} \|u_t(\omega) - v(\omega)\| = 0 \ and \ \lim_{t\to+\infty} inf_{v\in\mathcal{N}} \|u_t(\omega) - v(\omega)\| = 0 \quad (3)$$

Proof. By Proposition 2.10 we have $\mathcal{M}^u(\mathcal{N}) \subset \mathfrak{A}(\omega)$. Therefore, we must demonstrate that $\mathfrak{A}(\omega) \subset \mathcal{M}^u(\mathcal{N})$. Let $y \in \mathfrak{A}(\omega)$. By Remark 2.7 (ii) there is a full trajectory $\gamma(\omega) \equiv \{u_t : \Omega \longrightarrow X\}$ passing through y, $u_t(\omega) = y$. Since $\gamma(\omega) \subset \mathfrak{A}(\omega)$, the set $\gamma(\omega)$ is compact random set. Thus the α −limit set

$$\Gamma_\gamma^-(\omega) = \cap_{\tau<0} \overline{\cup\{u_t(\omega) : t \le \tau\}}$$

is a nonvoid compact random set. The random set $\Gamma_\gamma^-(\omega)$ is invariant: $\varphi(t,\theta_{-t}\omega)\Gamma_\gamma^-(\theta_{-t}\omega) = \Gamma_\gamma^-(\omega)$. Let's demonstrate that the RSLF $\mathcal{L}: \Omega \times X \longrightarrow \mathbb{R}^+$ is a constant on $\alpha_\gamma(\omega)$. Indeed, if $u \in \Gamma_\gamma^-(\omega)$, then there is a sequence $\{t_n\}$,$t_n \longrightarrow -\infty$ and $u_{t_n}(\omega) \longrightarrow u$ as $n \longrightarrow -\infty$ (see Definition 2.5). Consequently, $\mathcal{L}(\omega,u) = \lim_{n\to\infty} \mathcal{L}(\omega, u_{t_n}(\omega))$. By the monotonicity of $\mathcal{L}$ along trajectories, we have $\mathcal{L}(\omega,u) = \sup_{\tau<0} \mathcal{L}(\theta_\tau\omega, u_\tau(\omega))$.Consequently, the above limit does not depend on $\{u_n\}$ and $\mathcal{L}(\omega,u)$ is a constant on $\Gamma_\gamma^-(\omega)$. Since $\Gamma_\gamma^-(\omega)$ is invariant, then

$$\mathcal{L}(\theta_t\omega, \varphi(t,\omega)u(\omega)) = \mathcal{L}(\omega, u(\omega)) \text{ for all } t > 0 \text{ and } u \in \Gamma_\gamma^-.$$

Thus $\Gamma_\gamma^-(\omega) \subset \mathcal{N}$. Consequently,

$$\lim_{t\to\infty} \inf_{v\in\Gamma_\gamma^-(\omega)} \|u_t(\omega) - v(\omega)\| = 0. \quad (4)$$

If (4) is false, there is a sequence $\{t_n\}$, $t_n \longrightarrow -\infty$ so that

$$\inf_{v\in\Gamma_\gamma^-(\omega)} \|u_{t_n}(\omega) - v(\omega)\| \ge \delta > 0 \ for\ all\ n = 1,2,\ldots \quad (5)$$

Since $\bar{\gamma}$ is compact, there exist $z \in X$ and a subsequence $\{t_{n_m}\}$ so that $u(t_{n_m},\omega) \longrightarrow z$ as $m \longrightarrow \infty$. Furthermore, by Definition 2.5, $z \in \Gamma_\gamma^-(\omega)$. Since this goes against the property in (5), (4) is true. As $\Gamma_\gamma^-(\omega) \subset \mathcal{N}$, equation (4) indicates the first relation in (3) and henceforth $y \in \mathcal{M}^u(\mathcal{N})$ and $\mathfrak{A}(\omega) = \mathcal{M}^u(\mathcal{N})$. Using the same concept as earlier, we demonstrate the second relation in (3). We take into account the ω-limit $\Gamma_\gamma^+(\omega) = \cap_{t>0} \overline{\cup\{u_\tau(\omega) : \tau \ge t\}}$ which is a nonvoid compact strictly invariant random set. Since $\mathcal{L}$ is monotone and $\Gamma_\gamma^+(\omega)$ is invariant , then $\mathcal{L}(\omega,x)$ is a constant on $\Gamma_\gamma^+(\omega)$ and so, $\mathcal{L}(\theta_t\omega, \varphi(t,\omega)u(\omega)) = \mathcal{L}(\omega,u(\omega))$ for all $t > 0$ and $u \in \Gamma_\gamma^+(\omega)$. Then $\Gamma_\gamma^+(\omega) \subset \mathcal{N}$. As previously stated, using the contradiction argument,

$$inf_{v\in\mathcal{N}} \|u_t(\omega) - v(\omega)\| \le \lim_{t\to+\infty} inf_{v\in\Gamma_\gamma^+(\omega)} \|u_t(\omega) - v(\omega)\| = 0. \quad (6)$$

The proof of Theorem 3.5 is thus finished.
Theorem 3.6. *If a gradient RDS* (θ,φ) *has a CGA* $\mathfrak{A}(\omega)$, *Then*

$$\lim_{t\to+\infty} inf_{y\in\mathcal{N}(\omega)} \|\varphi(t,\theta_{-t}\omega)x - y\| = 0 \text{ for any } x \in X; \quad (7)$$

Proof. Consider $\Gamma_x^+(\omega)$, $x \in X$ and use the same reasoning as at the conclusion of Theorem 3.5's proof.
Corollary 3.7. *If a gradient RDS* (θ,φ) *has a CGA* $\mathfrak{A}(\omega)$ *and* $\mathcal{N}$ *is a finite, then*

(i) The GA $\mathfrak{A}(\omega)$ *contains of full trajectories* $\gamma = \{u_t : \Omega \longrightarrow X : t \in \mathbb{R}\}$ *relating couples of random fixed points: every* $u \in \mathfrak{A}(\omega)$ *belongs to some full trajectory* $\gamma \subset \mathfrak{A}(\omega)$ *and for every* $\gamma \subset \mathfrak{A}(\omega)$ *there is a couples* $\{z, z^*\} \subset \mathcal{N}$ *so that*

$$\lim_{t\to-\infty} u_t(\omega) = z \text{ a.s. and } \lim_{t\to+\infty} u_t(\omega) = z^*.$$

(ii) *For every* $v \in X$ *there is a fixed point z with* $\lim_{t\to+\infty} \varphi(t,\theta_{-t}\omega)x = z$ almost surly.

Proof The proof follows from Theorems 3.5 and 3.6.

3. REGULAR RANDOM ATTRACTOR

This section is dedicated to the study of the regular attractor, which is considered an important tool in studying the asymptotic behavior of RDSs. As is the case in the deterministic scenario mentioned in [6],we present the following formulation to characterize other aspects of global attractors for gradient systems. Let a RDS (θ,φ) act in a Banach space X and $\mathcal{N}$ be the set of its equilibrium points. It is supposed that the set $\mathcal{N}$ is finite, $\mathcal{N} = \{z_1, \ldots, z_N\}$ and it has an attractor $\mathfrak{A}(\omega)$. Moreover, it is supposed that Lyapunov's function Φ is defined on $\Omega \times \mathfrak{A}(\omega)$ and the indexation of the points $z_j \in \mathcal{N}$ such that

$$\Phi(\omega,z_1) \le \Phi(\omega,z_2) \le \cdots \le \Phi(\omega,z_n)\,. \quad (8)$$

Let

$$M_k = \bigcup_{j=1}^{k} \mathcal{M}_{z_j}^u(\omega)\,, M_0 = \emptyset. \quad (9)$$

In the following we will introduce the concept of regular random attractor for the RDSs.
Definition 4.1. An random attractor $\mathfrak{A}(\omega)$ is called a regular if

$$\mathcal{M}^u_{z_j}(\omega) \cap \mathcal{M}^u_{z_j}(\omega) = \emptyset;\ when\ i \neq j, \mathfrak{A} = M_N \tag{10}$$

and for $j = 1,2,\ldots,N$ the following conditions hold:
(i) $M_k(\omega)$ is a invariant random compact set.

(ii) $M_k(\omega)$ is a invariant random set.

(iii) $M_k(\omega)$ is a stable [2] with respect to (θ,φ).

(iv) and $\partial\mathcal{M}^u_{z_i}(\omega) \subset M_{i-1}(\omega)$, where $\partial\mathcal{M}^u_{z_i}(\omega) \equiv \overline{\mathcal{M}^u_{z_i}(\omega)}/\mathcal{M}^u_{z_i}(\omega)$.

(v) $\partial\mathcal{M}^u_{z_i}(\omega)$ is an invariant random set with respect to (θ,φ).

(vi) For every compact random set $K(\omega) \subset \mathcal{M}^u_{z_i}(\omega)/\{z_i\}$ we have

$$\lim_{t\longrightarrow+\infty} \max\left\{\inf_{y\in M_{i-1}(\omega)}\|\varphi(t,\theta_{-t}\omega)k - y\|: k \in K(\theta_{-t}\omega)\right\} = 0.$$

(vii) Every set $\mathcal{M}^u_{z_i}(\omega)$ is a C^1 −manifold of dimension d_i, this manifold is diffeomorphic to $\mathbb{R}^{d_i}$, and the embedding $\mathcal{M}^u_{z_i}(\omega) \subset X$ is of class C^1 in a neighborhood of $v \in \mathcal{M}^u_{z_i}(\omega)$.

Theorem 4.2 *Let (θ,φ) be a gradient RDS in a Banach space X with a RSLF $\Phi(\omega,u)$ has the following properties.*

(i) *It admits a CGA $\mathfrak{A}(\omega)$.*

(ii) *$\varphi(t,\omega) \in C^{1+\alpha}$ for some $\alpha > 0$ and there exists a vicinity $\mathcal{O} \supset \mathfrak{A}$ such that*
$$\|D\varphi(t,\theta_{-t}\omega)u - D\varphi(t,\theta_{-t}\omega)v\| \leq C_T\|u-v\|^\alpha, u,v \in \mathcal{O}, t \in [0,T].$$

(iii) *$(t,u) \longmapsto \varphi(t,\theta_{-t}\omega)u$ is continuous on $\mathbb{R}^+ \times \mathfrak{A}(\omega)$.*

(iv) *The operators $\varphi(t,\omega)$ are injective on $\mathfrak{A}(\omega)$ for every $t > 0$ and $\varphi(t,\omega)^{-1}$ are continuous on $\mathfrak{A}(\omega)$ for* every *ω.*

(v) *The Fréchet derivatives $D\varphi(t,\omega)u$ of $\varphi(t,\omega)u$ at every point $u \in \mathfrak{A}(\omega)$ have zero kernel.*

(vi) *The set $\mathcal{N} = \{z_1,\ldots,z_n\}$ of random fixed points is finite and every point $z_i \in \mathcal{N}$ is hyperbolic.*

Let the indexation of fixed points be such that
$$\Phi(\omega,z_1) \leq \Phi(\omega,z_2) \leq \cdots \leq \Phi(\omega,z_n)$$
and $M_k(\omega) = \bigcup_{j=1}^k \mathcal{M}^u_{z_j}(\omega)$, $M_0 = \emptyset$;, where $\mathcal{M}^u_{z_j}(\omega)$ is the unstable manifold starting from z_j.
Suppose that the function $t \longmapsto \Phi(\theta_t\omega,\varphi(t,\omega))$ is strictly decreasing for $u \notin \mathcal{N}$.
Then $\mathfrak{A}(\omega) = M_n(\omega)$ and regular random attractor. Moreover, $d_i = ind(z_i)$.

Proof. We shall consider for brevity the case when
$$\Phi(\omega,z_1) \leq \Phi(\omega,z_2) \leq \cdots \leq \Phi(\omega,z_n) \tag{11}$$
Obviously, this condition is slightly more restrictive than (1). Note that
$$\mathcal{M}^u_{z_i}(\omega) \cap \mathcal{M}^u_{z_j}(\omega) = \emptyset\ ;\ when\ i \neq j.$$
Indeed, if $u \in \mathcal{M}^u_{z_i}(\omega) \cap \mathcal{M}^u_{z_j}(\omega)$, there is t such that $u = \varphi(t,\theta_{-t}\omega)u_1$ and $\varphi(t,\theta_{-t}\omega)u_2$ where u_1 is in a small neighbourhood of z_i and u_2 is in a small neighbourhood of z_j. Since $\varphi(t,\theta_{-t}\omega)$ is injective , $u_1 = u_2$. For $z_i \neq z_j$. we obtain the contradiction.
Let $\xi_j = \Phi(\omega,z_j)$. Let the numbers $\xi_+ > \xi_j$. and $\xi_- < \xi_j$. be so close to ξ_j that the segment $[\xi_-,\xi_+]$ does not contain values $\xi_j = \Phi(\omega,z_j)$ of the function Φ when $i \neq j$. Consider the sets
$$X_+(\omega) = \{u \in U(\omega): \Phi(\omega,u) \leq \xi_+\},$$
$$X_-(\omega) = \{u \in U(\omega): \Phi(\omega,u) \leq \xi_-\}.$$
The sets $X_+(\omega)$ and $X_-(\omega)$ are invariant and compact, therefore the restrictions of $\varphi(t,\theta_{-t}\omega)$ to them have attractors $U_-(\omega)$ and $U_+(\omega)$ respectively , $U_-(\omega) \subset U_+(\omega)$. It follows that
$$U_+(\omega) = M_j(\omega), U_-(\omega) = M_{j-1}(\omega), U_+(\omega) = U_-(\omega) \cup \mathcal{M}^u_{z_j}(\omega). \tag{12}$$
Using induction in j, we deduce (10) from (12). The conditions (i)-(iii) of Definition 4.1 follow from corresponding properties of $U_+(\omega)$ and from (11). The set $\mathcal{M}^u_{z_j}(\omega)$ does not intersect with $U_-(\omega)$.
Indeed, $U_-(\omega) = M_{j-1}(\omega)$ by (7) and $\mathcal{M}^u_{z_i}(\omega) \cap \mathcal{M}^u_{z_j}(\omega) = \emptyset$ when $i < j$, consequently $\mathcal{M}^u_{z_i}(\omega) \cap M_{j-1}(\omega) = \emptyset$.
If $\mathcal{O}$ is an open neighbourhood of $U_-(\omega)$, then by (7) $U_+(\omega)/\mathcal{O} = \mathcal{M}^u_{z_i}(\omega)/\mathcal{O}$. Since $U_+(\omega)$ is compact, then $U_+(\omega)\setminus \mathcal{O}$ is compact, $\mathcal{M}^u_{z_i}(\omega)/\mathcal{O}$ is compact and $[\mathcal{M}^u_{z_i}(\omega)/\mathcal{O}] = \mathcal{M}^u_{z_i}(\omega)/\mathcal{O}$. Since $\mathcal{M}^u_{z_i}(\omega)$ is a subset of the compact $U_+(\omega)$, then
$$\begin{aligned}[\mathcal{M}^u_{z_i}(\omega)/\mathcal{O}] &= [\left(\mathcal{M}^u_{z_i}(\omega)/\mathcal{O}\right) \cup \left(\mathcal{M}^u_{z_i}(\omega) \cap \mathcal{O}\right)] \\ &= [\left(\mathcal{M}^u_{z_i}(\omega)/\mathcal{O}\right) \cup \left(\mathcal{M}^u_{z_i}(\omega)/\mathcal{O}\right)]\end{aligned}$$

$$\subset [\mathcal{M}^u_{z_i}(\omega)/\mathcal{O}] \cup [\mathcal{O}].$$

Therefore, since $\mathcal{M}^u_{z_i}(\omega)/\mathcal{O}$ is closed,

$$\partial\mathcal{M}^u_{z_i}(\omega) = [\mathcal{M}^u_{z_i}(\omega)]/\mathcal{M}^u_{z_i}(\omega) \subset [\mathcal{M}^u_{z_i}(\omega)/\mathcal{O}] \cup [\mathcal{O}]/\mathcal{M}^u_{z_i}(\omega) \subset [\mathcal{O}]. \quad (13)$$

Hence $\partial\mathcal{M}^u_{z_i}(\omega) \subset [\mathcal{O}]$ for any neighbourhood $\mathcal{O}$ of the set $U_-(\omega)$ and, consequently,

$$\partial\mathcal{M}^u_{z_i}(\omega) \subset [U_-(\omega)] = U_-(\omega),$$

which yields (iv) of Definition 4.1. Now we verify that (vi) holds. If $K(\omega)$ is a compact and $K(\omega) \subset M_j(\omega)/\{z_j\}$, then

$$sup_{x\in K(\omega)}\, \Phi(\omega, x) < \Phi(\omega, z_j). \quad (14)$$

Indeed, $\Phi(\omega, x) \leq \Phi(\omega, z_{j-1}) < \Phi(\omega, z_j)$ on $M_{j-1}(\omega)$. On $\mathcal{M}^u_{z_i}(\omega)/\{z_j\} \equiv \mathcal{M}^u(\omega)/\{z_j\}$, since for given $u \in \mathcal{M}^u(\omega)$ there exists a negative semi trajectory $u_t(\omega)$, $t \geq 0$, $u_0(\omega) = 0$ and by the definition of a Lyapunov function and thanks to its continuity $\Phi(\omega, u(t)) \uparrow \Phi(\omega, z_j)$ as $t \longrightarrow -\infty$ and $\Phi(\omega, u_t(\omega)) < \Phi(\omega, z_j)$. Therefore $\Phi < \Phi(\omega, z_j)$ on $K(\omega)$ and, thanks to compactness of $K(\omega)$ and continuity of Φ , (14) is valid. The inequality (14) implies that

$$K(\omega) = \{u \in U: \Phi(\omega, u_t(\omega)) \leq \Phi(\omega, z_j) - \varepsilon\} = X_-(\omega),$$

when $\varepsilon > 0$ sufficiently small, and (vi) holds since $M_{j-1}(\omega)$. is an attractor of the semigroup $\{\varphi(t,\omega)\}$ restricted to X_-. Now we show that $\partial M^u(\omega)$ is strictly invariant, i.e. (v) holds . To do it we prove that $[M^u(\omega)]$ is strictly invariant. If $y \in [M^u(\omega)]$, then $y = \lim\ x_i, x_i \in M^u(\omega)$. Since $M^u(\omega)$ is strictly invariant, then $x_i = \varphi(t, \theta_{-t}\omega)y_i$, $y_i \in M^u(\omega)$. It follows from precompactness of $M^u(\omega)$that $y_{i_n} \longrightarrow y_0$, $y_0 \in [M^u(\omega)]$ for some subsequence $\{i_n\}$. $\varphi(t, \theta_{-t}\omega)y_0 = y$ since S is continuous. Since y is arbitrary point of $[\mathcal{M}^u_{z_j}(\omega)]$, this implies the inclusion

$$[\mathcal{M}^u_{z_j}(\omega)] \subset \varphi(t, \theta_{-t}\omega)[\mathcal{M}^u_{z_j}(\theta_{-t}\omega)].$$

Now we prove the inverse inclusion. Let $y = \lim x_i$, $x_i \in M^u(\omega)$. By continuity of $\varphi(t, \theta_{-t}\omega)$ we have $\varphi(t, \theta_{-t}\omega)\, y = \lim\ \varphi(t, \theta_{-t}\omega)\, x_i$. Since $\varphi(t, \theta_{-t}\omega)\, x_i \in M^u(\omega)$, then $\varphi(t, \theta_{-t}\omega)\, y \in M^u(\omega)$ which yields $\varphi(t, \theta_{-t}\omega)[\mathcal{M}^u_{z_j}(\theta_{-t}\omega)] \subset [\mathcal{M}^u_{z_j}(\omega)]$. This inclusion and the inverse inclusion already proved imply the equality $\varphi(t, \theta_{-t}\omega)\left[\mathcal{M}^u_{z_j}(\theta_{-t}\omega)\right] = [\mathcal{M}^u_{z_j}(\omega)]$. Since $\varphi(t, \theta_{-t}\omega)[M^u(\omega)] = [M^u(\omega)]$, then

$$\varphi(t, \theta_{-t}\omega)([M^u(\omega)]/M^u(\omega)) = [M^u(\omega)]/M^u(\omega)$$

and (v) of Definition 4.1 is verified. Now we proceed to prove that (vii) of Definition 4.1 holds. First we notice that for any sufficiently small neighbourhood W_2 of the point $z_j = z$ there exists such a neighbourhood $W_1 \subset W_2$ of the point z that

$$\mathcal{M}^u_z(\omega) \cap W_1 = \mathcal{M}^u_{z,W_1}(\omega) \cap W_1 \quad (15)$$

In fact, it shall be demonstrated below that

$$\Phi(\omega, u(t)) \leq \Phi(\omega, z) - \varepsilon \text{ on } \mathcal{M}^u_z(\omega)/W_2,$$

where $\varepsilon > 0$. Since Φ is continuous on $U(\omega)$,

$$|\Phi(\omega, u_t(\omega)) - \Phi(\omega, z)| < \varepsilon/2 \text{ when } u_t(\omega) \in \mathcal{M}^u_z(\omega) \cap W_1(\omega),$$

if W_1 is sufficiently small. Let $u \in \mathcal{M}^u_z(\omega) \cap W_1(\omega)$ and $u_t(\omega)$ be a negative semitrajectory, $u_0(\omega) = 0$, $u_t(\omega) \longrightarrow z$ as $t \longrightarrow -\infty$. Obviously,

$$\Phi(\omega, z) > \Phi(\omega, u_t(\omega)) > \Phi(\omega, u_0(\omega)) \quad \text{when } t \leq 0.$$

Therefore $\{u_t(\omega): t \in \mathbb{R}^-\} \cap \{\mathcal{M}^u_z(\omega)/W_2\} = \emptyset$ and, consequently, $u_t(\omega) \in \mathcal{M}^u_z(\omega) \cap W_2(\omega)$ for every $t \leq 0$. Therefore $u_0(\omega) \in \mathcal{M}^u_{z,W_2}(\omega)$ and $\mathcal{M}^u_z(\omega) \cap W_1(\omega) \subset \mathcal{M}^u_{z,W_2}(\omega)$. This inclusion together with the evident inclusion $\mathcal{M}^u_{z,W_2}(\omega) \subset \mathcal{M}^u_z(\omega)$ imply (10) . Propositions 3.1 and 3.2 imply that for sufficiently small W_3

$$\mathcal{M}^u_{z,W_2}(\omega) \cap W_3(\omega) \subset \mathcal{M}^+_z(\omega) \cap W_3(\omega), \quad (16)$$

where $\mathcal{M}^+_z(\omega)$ is the manifold constructed in Theorem 3.1. From (15) and (16) we obtain

$$\mathcal{M}^u_z(\omega) \cap W_3(\omega) = \mathcal{M}^+_z(\omega) \cap W_3(\omega) \quad (17)$$

if the neighbourhood W_3 of the point z is sufficiently small. We take $W_3 = \{\, u_t(\omega) : \|u_t(\omega) - z\| < \delta \,\}$ and conclude by Theorem 3.1 that $\mathcal{M}^u_z(\omega) \cap W_3(\omega)$ is a smooth manifold of class $C^{1+\alpha}$. The next formula is valid:

$$\mathcal{M}^u_z(\omega) = \bigcup_{k=0}^{\infty} \{\varphi(k, \theta_{-k}\omega)(\mathcal{M}^u_z(\omega) \cap W_3(\omega))\} \quad (18)$$

By Condition (iv) of Theorem 4.2 $\varphi(k, \theta_{-k}\omega)$ maps homemorphically $\mathcal{M}^u_z(\omega)$ onto $\mathcal{M}^u_z(\omega)$. Therefore, by (18) a small neighbourhood $\aleph$ of a point $u \in \mathcal{M}^u_z(\omega)$ coincides with the image of the neighbourhood $\aleph_1 = \varphi(k, \theta_{-k}\omega)^{-1}(\aleph) \subset \mathcal{M}^u_z(\omega) \cap W_3$. By condition (v) of Theorem 4.2 $\varphi(k, \theta_{-k}\omega)$ is a diffeomorphism of $\aleph$ and $\aleph_1$ of class $C^{1+\alpha}$. Indeed, let $u_0 + E^0_+$ be a hyperplane tangent to $\mathcal{M}^u_z(\omega)$ at the point $u_0 \in W_3$, let $E^1_+ = \varphi'(k, \theta_{-k}\omega)E^0_+$, and $u_1 = \varphi(k, \theta_{-k}\omega)u_0$. By condition (v) the operator $\varphi'(k, \theta_{-k}\omega)$ restricted to the finite-dimensional space E^0_+ is a linear isomorphism between linear spaces E^0_+ and E^1_+. Denote by π_0 and π_1 projections onto $E^0_+ + u_0$ and E^1_+ respectively. Since $\mathcal{M}^u(\omega)$ is tangent to $u_0 + E^0_+$, then π_0 is a diffeomorphism of $\mathcal{M}^u(\omega)$ and $u_0 + E^0_+$ in a neighbourhood of the point u_0 . Let π_0^{-1} be the inverse mapping and consider the mapping $G: E^0_+ \longrightarrow E^1_+$,

$$G(v) = \pi(\varphi(k, \theta_{-k}\omega)\pi_0^{-1}(u_0 + v) - \varphi(k, \theta_{-k}\omega)u_0).$$

This mapping is of class $C^{1+\alpha}$ and its differential at the point $v = 0$ coincides with $\varphi'(k, \theta_{-k}\omega)$ restricted to E_{+}^{0} and therefore it is invertible. Consequently, G is a diffeomorphism in a neighbourhood of zero. One can easily see that the mapping

$$h: u \longrightarrow E_{+}^{1} \longrightarrow \varphi(k, \theta_{-k}\omega)\mathcal{M}^{u}(\theta_{-k}\omega)$$

which is defined in a neighbourhood of zero by the formula

$$h(u_1 + w) = \varphi(k, \theta_{-k}\omega)\pi_0^{-1}(G^{-1}(w + u_0)), w \in E_{+}^{1},$$

determines $\mathcal{M}_z^u(\omega)$in a neighbourhood of the point u_1 as the graph of the function

$$h_1 = (I - \pi_1)(h + u_1).$$

This function of class $C^{1+\alpha}$ is defined on the hyperplane $u_1 + E_{+}^{1}$ and takes values in $(I - \pi_1)E$. Thus, $\mathcal{M}_{z_j}^u(\omega)$ is a $C^{1+\alpha}$ −manifold of dimension n_j; $n_j = \dim E_{+}(z_j) = \operatorname{ind} z_j$ and this manifold is embedded into E , the embedding being of class $C^{1+\alpha}$ in a neighbourhood of any point $u \in \mathcal{M}_{z_j}^u(\omega)$.

We now show that $\mathcal{M}^u(\omega) = \mathcal{M}_z^u(\omega)$, $z = z_j$, is diffeomorfic to $\mathbb{R}^n$, $n = \operatorname{ind} z$. If a manifold M is such that

$$M = \cup_{j=0}^{\infty} M_j,\ M_j \subset M_{j+1} \text{ for every } j \in \mathbb{Z}^{+},$$

and any M_j is diffeomorphic to $\mathbb{R}^n$, then M is diffeomorphic to $\mathbb{R}^n$. Let

$$M_j(\omega) = \mathcal{M}^u(\omega) \cap \{u: \|u - z\| < \rho\}. \tag{19}$$

Let $M_j(\omega) = \varphi\big(j, \theta_{-j}\omega\big)M_0(\theta_{-j}\omega)$ .By (4.11) $\mathcal{M}^u(\omega) = \cup\, M_j(\omega)$. By (18), since ρ is arbitrarily small in (5), $M_0\,(\omega) = M_{+}\,(\omega)$ where $M_{+}\,(\omega)$ is defined in (3.3). By Theorem 3.1 $M_{+}(\omega) \subset \varphi\big(j, \theta_{-j}\omega\big)M_{+}(\theta_{-j}\omega)$, hence

$$\varphi\big(j, \theta_{-j}\omega\big)M_{+}(\theta_{-j}\omega) \subset \varphi\big(j+1, \theta_{-(j+1)}\omega\big)M_{+}(\theta_{-(j+1)}\omega) \text{ and,}$$

consequently, $M_j(\omega) \subset M_{j+1}(\omega)$. $M_j(\omega)$is diffeomorphic to $\mathbb{R}^n$. Indeed, $M_0(\omega) = M_{+}(\omega)$ is diffeomorphic to the ball $\{\|u_{+}\| < \rho\}$ in $\mathbb{R}^n$ by (5), and M_0 is diffeomorphic to $\mathbb{R}^n$ as this ball is diffeomorphic to $\mathbb{R}^n$. As it was already proved, $\varphi\big(j, \theta_{-j}\omega\big)$ maps diffeomorphically M_0 onto $\varphi\big(j, \theta_{-j}\omega\big)M_0\big(\theta_{-j}\omega\big) = M_j(\omega)$; hence $M_j(\omega)$ is diffeomorphic to $\mathbb{R}^n$. Thus all the conditions on $M_j(\omega)$are verified and $M_z(\omega) = \cup\, M_j(\omega)$ is diffeomorphic to $\mathbb{R}^n$ ■

5. CONCLUSIONS

Through this article, some conclusions have been reached:

(1) The compact global attractor for an RDS can be considered as an *unstable manifold starting from the set of random fixed points*, and this global attractor *involves full trajectories* that converge to the random fixed points.

(2) If the *gradient RDS has finite numbers of random fixed points, then every element of the compact global attractor belongs to the full trajectory*.

(3) The gradient RDS *in a Banach space has a regular compact attractor if it has a random strict Lyapunov function* and satisfies the conditions of Theorem 4.2.

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