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Commutative Rings, Dominating Sets and total graphs

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Abstract

This article wants to examine the commutative rings, firstly. For this purpose, the definitions, examples, ideals, modules, famous rings, and other related concepts of commutative rings are reviewed. Secondly, we also examine the dominating sets of commutative rings in this article. Let $G = (V, E)$ represent a graph. A set $S \subseteq V$ is considered a dominant set of G if every vertex in V is either included in S or is close to a vertex inside S. The dominance number $\gamma(G)$ of graph G is the smallest size of its dominating sets. We study and characterize the dominating sets of the zero-divisor graph $\Gamma(R)$ and ideal-based zero-divisor graph $\Gamma_1(R)$ of a commutative ring R. At last, we review some total graphs associated with commutative rings especially total graph of T (Γ (R)).

Keywords: Commutative rings, ideal rings, dominating set, graph

الحلقات التبادلية والمجموعات المهيمنة والرسوم البيانية الإجمالية

مهند عبدالله كاظم
قسم الرياضيات
مدرس في مديرية تربيه القادسية

الملخص

تهدف هذه المقالة إلى دراسة الحلقات التبادلية، أولاً. ولهذا الغرض، نستعرض التعريفات والأمثلة والمثل العليا والوحدات والحلقات الشهيرة والمفاهيم الأخرى ذات الصلة بالحلقات التبادلية. ثانيًا، ندرس أيضًا المجموعات المهيمنة للحلقات التبادلية في هذه المقالة. لنفترض أن G=(V,E) تمثل رسمًا بيانيًا. تُعتبر المجموعة S⊆V مجموعة مهيمنة لـ G إذا كان كل رأس في V متضمنًا في S أو قريبًا من رأس داخل S. رقم الهيمنة γ(G) للرسم البياني G هو أصغر حجم لمجموعاته المهيمنة. ندرس ونميز المجموعات المهيمنة للرسم البياني للمقسوم عليه صفر Γ(R) والرسم البياني للمقسوم عليه صفر القائم على المثالية 〖Γ〗_1 (R) للحلقة التبادلية R. وأخيرًا، نراجع بعض الرسوم البيانية الإجمالية المرتبطة بالحلقات التبادلية وخاصة الرسم البياني الإجمالي لـ T (Γ (R)).

الكلمات المفتاحية: الحلقات التبادلية، الحلقات المثالية، المجموعة المهيمنة، الرسم البياني

1. Introduction

A ring is a set R endowed with two binary operations that combine any two components of the ring to produce a third element. Addition and multiplication are indicated by " + " and "·", accordingly, as in " $a + b$". To constitute a ring, these two operations must adhere to many properties: the ring must be an abelian group with respect to addition and a monoid concerning multiplication, with multiplication distributing over addition; specifically, $.a \cdot (b + c) = (a \cdot$

$b) + (a \cdot c)$. The numbers 0 and 1 represent the identity elements for addition and multiplication, accordingly. Since multiplication is commutative, meaning $a \cdot b = b \cdot a$, afterwards the ring R is termed commutative. Unless otherwise specified, all rings in the rest of this essay will be commutative. One significant illustration, and maybe even the most important, is the ring of integers Z with addition and multiplication. This constitutes a commutative ring, as the multiplication of integers is a commutative operation. Z is often used as an abbreviation for the German term Zahlen, signifying "numbers." [1]. A field is a commutative ring in which $0 \neq 1$ and each non-zero member a possesses a the reverse b like $a \cdot b = 1$. Consequently, every field is a commutative ring by definition. Fields are made up of real, complex, and rational numbers.

Let R be a commutative ring; afterwards the collection of all polynomials in the variable X with coefficients in R is the polynomial ring, abbreviated $R[X]$. This is also applicable to different variables [2].

Since V is a topological space, such as a subset of R^n, the set of real- or complex-valued continuous functions on V constitutes a commutative ring. The same applies to differentiable or holomorphic functions once both terms are specified, like in the case of V being a complex manifold [3].

Ring divisibility encompasses a more complex notion than fields, whereby every nonzero element is multiplicatively invertible. If element an in ring R has a multiplicative inverse, then element an is a unit. The zero divisors, or elements a like that there is a non-zero element b of the ring like that $a\,b = 0$, are another specific kind of element.

If R has no non-zero zero divisors, it is termed an integral domain (or domain). An element a that satisfies $a^n = 0$ for a certain positive integer n is termed nilpotent. A ring is considered localised once some elements are rendered invertible, or when multiplicative inverses are included into the ring. Specifically, while S is a multiplicatively closed subset of R (i.e., if $s, t \in S$, then st), afterwards the localisation of R at S, or the ring of fractions with denominators in S, is often termed $S^{-1}R$ comprises symbols

$$\frac{r}{s}\ with\ r \in R, s \in S$$

subject to specific rules that resemble the cancellation seen in logical numbers. In this language, Q represents the localization of Z at all nonzero integers. This approach is applicable to any integral domain R rather than Z. The localization of $(R \setminus \{0\})^{-1}R$ is a field, referred to as the quotient field of R [4].

2. Famous commutative rings

Noetherian rings

A ring is termed Noetherian, in tribute to Emmy Noether, when each ascending sequence of ideals

$$0 \subseteq I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \ldots$$

becomes stationary, meaning it remains equal beyond a certain index n. Any ideal can also be formed by a finite number of elements, or equivalently, finitely developed sections of finitely constructed modules. Being Noetherian is a crucial finiteness criterion, and it is upheld by a variety of procedures that are commonly used in geometry.

When R is Noetherian, afterwards the polynomial ring $R[X_1, X_2, \ldots, X_n]$ is likewise Noetherian (according to Hilbert's basis theorem), as are any localisation $S^{-1}R$ and each quotient ring R / I.

Every non-Noetherian ring R is the union of its Noetherian subrings. This principle, termed Noetherian approximation, facilitates the generalisation of specific theorems to non-Noetherian rings [10].

Artinian rings

A ring is termed Artinian (after Emil Artin) if each descending chain of ideals terminates.

$$R \supseteq I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq I_{n+1}$$

ultimately becomes stagnant. Noetherian rings are far broader in scope than Artinian rings, even though the two requirements appear to be symmetric.

For instance, Z is Noetherian as each ideal may be created by a single element; nevertheless, it is not Artinian, because the chain

$$Z \supsetneq 2Z \supsetneq 4Z \supsetneq 8Z \ldots$$

indicates. In actuality, every Artinian ring is Noetherian according to the Hopkins–Levitzki theorem. To be more exact, the Noetherian rings with a Krull dimension of zero are Artinian rings [10].

3. Spectrum of a commutative ring

Z is a unique factorisation domain, as previously stated. In the nineteenth century, algebraists found that this does not hold true for more general rings. For instance, in $Z[\sqrt{-5}]$, there are two authentically separate representations of 6 as a product:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

One way around this issue is to use prime ideals instead of prime components. If the product ab of any two ring components, a and b, is in p,, then at least one of the components a or b must already be in p. This is known as a prime ideal. A prime ideal is a proper (that is, rigorously included in R) ideal p (By definition, the opposite outcome applies to any ideal). Consequently, a prime element generates a prime ideal that is primary in an equivalent manner. In rings like Z [$\sqrt{-5}$], prime ideals are not always main. This restricts the use of prime components in ring theory. A fundamental principle of algebraic number theory is the notion that in any Dedekind ring (including Z [$\sqrt{-5}$] and, more broadly, the ring of integers in a number field), every ideal (including the one formed by 6) consistently decomposes into a product of prime ideals.

Every maximum ideal is a prime ideal, or succinctly, is prime. Furthermore, an ideal I is considered prime if and only if the factor ring R / I constitutes an integral domain. Demonstrating that an ideal is prime, or equivalently that a ring is devoid of zero-divisors, may be somewhat challenging. Another approach to articulate this is to state as the complement R \ p is multiplicatively closed. The localization $(R\ p)^{-1}R$ is significant sufficient to have its own notation: R_p. This ring has a single maximal ideal, namely pRp. These rings are referred to be local.

The spectrum of a ring R, referred to as Spec R, is the collection of all prime ideals of R. It has a topology, namely the Zariski topology, that embodies the algebraic characteristics of R: a collection of open subsets is provided by

$$D(f) = \{p \in Spec\ \ R, f \notin p\}$$

where f represents any component in a ring. Interpreting f as a function that yields the value f mod p (i.e., the representation of f in the residue field R/p), this subset is the locus where f is non-vanishing. The spectrum clarifies the understanding that localisation and factor rings are complimentary: the natural mappings $R \rightarrow R_f$ and $R \rightarrow R / fR$ correspond, when the spectra of the corresponding rings are equipped with their Zariski topology, to complementary open and closed immersions, accordingly. Even for fundamental rings, like that seen for R = Z on the right, the Zariski topology significantly diverges from that on the set of real numbers [12].

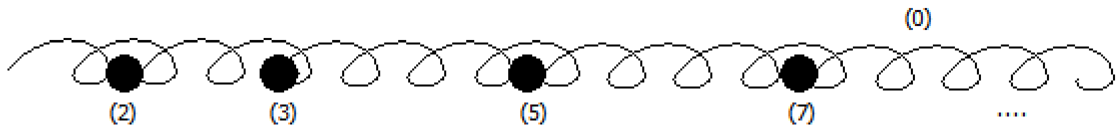

Figure 1. There is a point for the zero ideal in Spec (Z). The entire space is the closure of this point. The ideals (p), for which p is a prime number, correspond to the remaining points. We've closed these points.

The spectrum comprises the collection of maximum ideals, sometimes referred to as $mSpec\ (R)$. For an algebraically closed field k, mSpec ($k[T_1, ..., T_n] / (f_1, ..., f_m)$) corresponds bijectively to the set

$$\{x = (x_1, \ldots, x_n) \in k^n$$

The geometric features of solution sets of polynomials are reflected in maximal ideals, which provides a preliminary incentive to explore commutative rings. It is helpful to take into account non-maximal ideals while analyzing the geometric characteristics of a ring for a number of reasons. For instance, the irreducible components of Spec R correspond to the minimal prime ideals, or those that do not strictly contain smaller ones. There are only finitely many irreducible components in Spec R for a Noetherian ring R.

This is a geometric reformulation of the concept of fundamental decomposition, which asserts that every ideal may be expressed as a product of a finite number of original ideals. This fact is the farthest generalization of the decomposition of Dedekind rings into prime ideals [13].

4. Dominating sets of some graphs of commutative rings

Let R denote a commutative ring with unity. The zero-divisor graph of R, represented as $\Gamma(R)$, is an undirected network where the vertices consist of the nonzero zero-divisors of R. Two unique vertices x and y are contiguous if and only if the product $xy = 0$. Consequently, $\Gamma(R)$ constitutes an empty graph if and only if R qualifies as an integral domain. The idea of a zero-divisor graph of a commutative ring was first presented by Beck in [22], although the focus of this work was primarily on colorings of zero-divisor of rings. The aforementioned concept first emerged in the research of Anderson and Livingston [20], which encompasses numerous essential findings related to $\Gamma(R)$. This concept, in contrast to the previous work of Anderson and Nasser [16] and Beck [22], does not consider zero as a vertex of $\Gamma(R)$.

Numerous writers have examined the zero-divisor graph of a commutative ring in great detail [14–20]. The zero-divisor graph has also been generalized to a $\mathrm{k}-$ zero divisor hypergraph by Eslahchi and Rahimi in [24], and extended to various algebraic structures in DeMeyer et al. [23], Redmond [28]. Furthermore, an implicit study of the idea of a dominating set in a zero-divisor graph was conducted in [26].

To ensure comprehensiveness, we include many definitions and concepts used throughout this section of the text to maintain its self-sufficiency. We define $\delta(G)$ as the minimal degree of the vertices in G. In a nontrivial linked graph G, the distance $d(u, v)$ between vertices u and v is defined as the length of the shortest route connecting u and v inside G.

The girth of a graph G, which contains a cycle, is defined as the minimum length of its cycles and is represented as gr(G). If G is acyclic, we define its girth as infinite. A complete r-partite graph is defined as a graph whose vertex set can be divided into r subsets, ensuring that no edge connects vertices within the same subset, but every vertices in one subset is connected to every vertex in the other subsets. A complete bipartite graph, also known as a 2-partite graph, with partite set sizes m and n is represented as $K_{m,n}$. A complete bipartite graph of the type $K_{1,n}$ is formed when its centres are connected. A graph in which every pair of different vertices is connected by an edge is referred to as a full graph K_n with n vertices. A clique is defined as a complete subgraph of a graph G. The clique number, $\omega(G)$, is the maximum integer $n \geq 1$ such that $K_n \subseteq G$, and $\omega(G)$ is infinite if $K_n \subseteq G$ for every $n \geq 1$ [29].

Let $G = (V, E)$ represent a graph. A set $S \subseteq V$ is considered a dominant set if every vertex in V is either included in S or is close to a vertex inside S. The dominance number $\gamma(G)$ of graph G is the smallest cardinality of any dominating set of G. A dominating set S is classified as a clique dominating set if the generated subgraph $< S >$ forms a clique. The clique domination number $\gamma_{cl}(G)$ is the smallest size of a clique dominating set of G [25].

This section aims to examine the characteristics of dominant sets in zero-divisor graphs and idea-based zero-divisor graphs. In this section, all rings were commutative with identity $1 \neq 0$. For a subset A of a ring R, we define $A^* = A \setminus\{0\}$. We shall represent the rings of integers modulo n, the integers, and a finite field with q members as Z_n, Z, and IF_q, accordingly.

Numerous categories of graphs exhibit well-defined dominating sets and dominance numbers. For example, we provide some of them in the following instances:

(a) $\gamma(K_n) = 1$ for a complete graph K_n.

(b) Let G be a complete $r - partite$ graph $(r \geq 2)$ with partite sets $V_1, V_2, \dots V_r$.

If $|V_i| \geq 2$ for $1 \leq i \leq r$, afterwards $\gamma(G) = 2$, as one vertex of V_1 and one vertex of V_2 dominate G. If $|V_i| = 1$ for some i, afterwards $\gamma(G) = 1$.

(c) $\gamma\left(K_{1,n}\right) = 1$ for a star graph $K_{1,n}$.

(d) The dominance number of a bistar graph is 2, since the set including the two centres of the graph constitutes a dominating set.

(e) Let C_n, P_n be a cycle and a path with n vertices, accordingly. Afterwards $\gamma(C_n) = \left\lceil\frac{n}{3}\right\rceil = \gamma(P_n)$.

(f) Let F_1 F_2 be infinite fields with $|F_1| = m \geq 3$ and $|F_2| = n \geq 3$. The graph $\Gamma(F_1 \times F_2)$ is bipartite[18], and hence $\gamma\left(\Gamma(F_1 \times F_2)\right) = 2$.

(g) $\Gamma(F_1 \times Z_4)) = 2$.

Now we study the domination number of a zero-divisor graph.

Example 1. Let R be a ring and $r \geq 3$. Reference [14] demonstrates that in a full $r - partite$ graph $\Gamma(R)$, at most one part may contain more than one vertex. Consequently, given the aforementioned case (b), it is evident that $\gamma(\Gamma(R)) = 1$. Furthermore, in [14], it is observed that for each prime number p and any positive integer n, there exists a finite ring R such that its zero-divisor graph $\Gamma(R)$ is a complete $p^n - partite$ network. For instance, if IF$_{p^n}$ is a finite field with p^n elements, then $R =$ IF$_{p^n}[x, y]/(xy, y^2 - x)$ represents the requisite ring.

Example 2. Obviously, a graph G has domination number equal to 1 means that G has a vertex adjacent to every other vertex od G. In [20], it is shown that for any commutative ring *R*, there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R = Z_2 \times$ A where A is an integral domain, or *Z(R)* is an annihilator ideal (and hence is prime).

Remark 3. Let *R* be a commutative ring with 1, and let Z(R) be the set of zero-divisors in *R*. If Z(R) constitutes an ideal of *R*, then no subset of Z(R) can create R, since this would necessitate the inclusion of 1 in Z(R), resulting in a contradiction. Thus, if a dominant set of $\Gamma(R)$ creates *R*, therefore Z(R) cannot constitute an ideal of *R*.

Proposition 4. [29]Suppose for a fixed integer $n \geq 2$, that $R = R_1 \times R_2 \times \ldots \times R_n$, where R_i is an integral domain for each $i = 1,2, \ldots, n$. Then:

(a) $\gamma(\Gamma(R)) = n = \omega(\Gamma(R))$ if $n \geq 3$

(b) $\gamma(\Gamma(R)) = 2 = \omega(\Gamma(R))$ if $n = 2$ and $\min\{|R_1|. |R_2|\} \geq 3$ and

(c) $\gamma(\Gamma(R)) = 1 < \omega(\Gamma(R))$=2 if $n = 2$ and $\min\{|R_1|. |R_2|\} = 2$.

Corollary 5. For each positive integer k, there is a commutative ring R such that its zero-divisor graph has a minimal clique dominating set of size k, and hence, its domination number is k. Furthermore, for $k \geq 2$,, $\gamma(\Gamma(R)) = \omega(\Gamma(R))$=$k$.

Lemma 6. Let R be a commutative local Artinian ring, which may specifically be a finite ring, with a maximum ideal M. Afterwards $\gamma(\Gamma(R)) = 1$.

Theorem 7. Let R be as commutative Artinian ring (in particular, R could be a finite commutative ring). Suppose that that $R = R_1 \times R_2 \times \ldots \times R_n$, where R_i is a local ring with maximal ideal M_i for every $i = 1,2, \ldots, n$. After that:

(a) $\gamma(\Gamma(R)) = n$ if $n \geq 3$

(b) $\gamma(\Gamma(R)) = 2 =$ and $\min\{|R_1|. |R_2|\} \geq 3$ or $|R_1| = 2$ and the maximal ideal of R_2 is nonzero.

(c) $\gamma(\Gamma(R)) = 1$ and the maximal ideal of R_2 is nonzero

(d) $\gamma(\Gamma(R)) = 1$ if $n = 2$, $|R_1| = Z_2$ and R_2 is a field.

Remark 8. In Theorem 7 and Proposition 4, the minimal dominant set S constitutes a clique in $\Gamma(R)$. Thus, S constitutes a linked and total dominating set if $|S| \geq 2$, as the notion of a total dominating set is applicable only when S has a minimum of two items.

Furthermore, because every Artinian ring R (namely, R may be a finite commutative ring) always has a minimal clique dominating set, we may deduce the domination number of $\Gamma(R)$. That is, $\gamma(\Gamma(R)) \leq \omega(\Gamma(R))$, and it is not difficult to show that this inequality is sharp by using proposition 4, theorem 7, and the fact that $\omega(\Gamma(R)) = n(n \geq 3)$ for $R \cong R_1 \times R_2 \times \ldots \times R_n$, where either R_i is an integral domain or R_i is a local ring for each $1 \leq i \leq n$.

Remark 9. If a ring R has a prime ideal P of cardinality 2, hence $\gamma(\Gamma(R)) = 1$, since each edge possesses at least one vertex in P. For any prime ideal P in a ring R, the set $S = P\backslash\{0\}$ constitutes a dominant set of $\Gamma(R)$, since each edge has at least one vertex in S. Moreover, if P is a prime ideal of a finite ring R with the smallest cardinality among all prime ideals of R, then $\gamma(\Gamma(R)) < |P| - 1$. Consequently, from this and Theorem 7, we deduce that $\gamma(\Gamma(R)) \leq |p| - 1$ for each finite reduced ring $R = R_1 \times R_2 \times \ldots \times R_n$ (assuming $n \geq 3$), where each R_i is a field and P is a prime ideal of R with the smallest cardinality among all prime ideals of R."

Theorem 10. [29] Let R be a finite reduced commutative ring that is not a field. Thus, either $\gamma(\Gamma(R)) = 1$ or $\gamma(\Gamma(R)) = \omega(\Gamma(R))$. If $\gamma(\Gamma(R)) \neq 1$, then it equals the count of minimum prime ideals of R. Furthermore, if R has $k \geq 3$ minimum prime ideals, then $\gamma(\Gamma(R)) = k$.

5. The domination number of an ideal-based zero-divisor graph

We want to investigate the correlation between the dominance numbers of $\Gamma_1(R)$ and $\Gamma(R/I)$.

Let R be a commutative ring with nonzero identity, and let I be an ideal of R. The ideal-based zero-divisor graph of R, denoted by $\Gamma_1(R)$, is the graph whose vertices are the set $\{x \in R\backslash I | xy \in 1 \text{ for some } y \in\backslash I\}$.

In the case $I = 0, \Gamma_0(R)$ is denoted by $\Gamma(R)$. Moreover, $\Gamma_1(R)$ is empty if and only if I is prime.

Lemma 11. [28] Let I be an ideal of a ring R, and let x, y be in $R\backslash I$. Then:

(a) If $x + 1$ is adjacent to $y + 1$ in $\Gamma(R\backslash I)$, then x is adjacent to y in $\Gamma_1(R)$

(b) If x is adjacent to y in $\Gamma_1(R)$ and $x+1 \neq y+1$, then $x+1$ is adjacent to $y+1$ in $\Gamma_1(R)$

(c) If x is adjacent to y in $\Gamma_1(R)$ and $x+1 = y+1$, then $x^2, y^2 \in I$.

Theorem 12. Let S be a nonempty subset of $R\backslash I$. If S is a dominating set of $\Gamma(R\backslash I)$, then $S+1 = \{S+1|s \in S\}$ is a dominating set of $\Gamma(R\backslash I)$,

6. Graphs associated to a commutative ring

The examination of graphs linked to algebraic structures originated in 1878 when Arthur Cayley presented the Cayley Graph for finite groups [41]. In 1988, Beck defined the zero-divisor graph of a commutative ring R. The vertices of Beck's network consist of all elements of R, with two vertices being nearby if their product is a zero divisor of R. Anderson and Livingston [40] refined the notion of a zero divisor network by limiting the vertices to the non-zero zero divisors of the ring R. This graph is represented by Γ(R). The reader may consult references [36] and [42] for a survey of zero divisor graphs. The interaction between ring theory and graph theory has prompted the creation and exploration of several graphs related to algebraic structures. Two decades subsequent to Beck's graphs, Andersen and Badawi [37] presented a novel graph linked to a commutative ring R with a non-zero unity. This graph is referred to as the entire graph of R.

Definition 13. [37] Let R be a commuatative ring with a non-zero unity, let Z(R) be the set of all zero divisors in R. The total graph of R is the simple graph with vertex set R and two distinct vertices x and y are adjacent if their sum x + y ∈ Z(R). This graph is denoted by $T\ (\Gamma(R))$.

Several induced subgraphs of $T\ (\Gamma(R))$. are also studied in the literature. The graphs Re($\Gamma(R)$).,Z($\Gamma(R)$)., $Z_0(\Gamma(R))$ and $T_0((\Gamma(R))$ are defined to be the induced subgraphs of T $(\Gamma(R))$ with vertex sets Reg(R), Z(R), Z(R) \ {0} and $R \setminus \{0\}$, respectively, where Reg(R) is the set of all regular elements of R and Z(R) is the set of all zero divisors of R.

The definition of T $(\Gamma(R))$ brings back to our minds the infamous Cayley graph Cay(R,Z(R)*), where Z(R)* = $Z(R) \setminus \{0\}$.

Definition 14. Let R be a commutative ring with a non-zero unity, let Z(R) be the set of all zero divisors in R. The Cayley graph, Cay(R,Z(R)*), is the simple graph with vertex set R andtwo distinct vertices x and y are adjacent if x - y ∈ Z(R).

Theorem 15. [43] Let R be a finite commutative ring. Then the two graphs T (Γ (R)) and Cay(R,Z(R)*) are isomorphic if and only if at least one of the following conditions is true:

(i) R = R1 × R2 × . . . × Rk, k ≥ 1, where each R_i is a local ring of an even order;

(ii) R = R1 × R2 × . . . × Rk, k ≥ 2, where each R_i is a local ring and $\min\{|Ri/Z(Ri)|, i = 1, 2, \dots, k\} = 2$.

Definition 16. [42] Let R be a commutative ring with a non-zero identity, and let U(R) be the set of all unit elements in R. The unit graph of R is a simple graph with vertex set R, where two different vertices x and y are nearby if their total sum x + y ∈ U(R). The graph is represented as G(R).

Definition 17. [39] Let R be a commutative ring with identity, let M be an R-module, let $T(M) = \{m \in M : rm = 0 \; for\ some\ r \in R^*\}$ be the set of its torsion elements. The total graph of a module, T Γ(M), is defined to be the graph with vertex set M and two distinct vertices $m1, m2 \in M$ are adjacent if $m1 + m2 \in T(M)$.

Observe that if M = R, then T (M) = Z(R) and hence, the resultant graph will be T (Γ (R)).

Definition 18. [46] The graph GS(R) is defined to be the simple graph with all elements of R as vertices, and two distinct vertices x and y of R are adjacent if and only if x + y ∈ S.

Definition 19. [31] The complete graph of a commutative ring R concerning a suitable ideal I consists of vertices representing all components of R, where two different vertices x and y ∈ R are near if x + y ∈ S(I). This graph is represented by $T(\Gamma_I(R))$.

6.1 The Total Graph T (Γ (R))

Two prominent graphs serve as foundational components for the entire graph T (Γ (R)). A full graph with n vertices is a graph where every pair of unique vertices is connected by an edge. This graph is represented as K_n. A complete bipartite graph is a graph in which the vertices can be divided into two subsets, V_1 and V_2, such that two unique vertices are near if and only if one vertex is in V_1 and the other is in V_2. If $|V_1| = m$ and $|V_2| = n$, the graph is represented as Km, n.

The examination of T (Γ (R)) categorically divides into two scenarios based on the status of Z(R) as an ideal of R. If R is a commutative ring and Z(R) is an ideal of R. Consequently, Z(Γ (R)) constitutes a full (induced) subgraph of T (Γ (R)). Andersen and Badawi [37] demonstrated that Reg(Γ (R)) constitutes the union of complete graphs if 2 ∈ Z(R); otherwise, Reg (Γ (R)) comprises the union of complete bipartite graphs. In summary, they achieved an accurate characterization of T (Γ (R)) when Z(R) is an ideal of R, as stated in the subsequent theorem.

Theorem 20. Let R be a commutative ring such that Z(R) is an ideal of R. Then

(i) Reg(Γ (R)) is complete if and only if either R/Z(R) $\cong Z_2$ or R $\cong Z_3$.

(ii) Reg(Γ (R)) is connected if and only if either R/Z(R) $\cong Z_2$ or R/Z(R) $\cong Z_3$.

(iii) Reg(Γ (R)) is totally disconnected if and only if R is an integral domain with char(R) = 2.

Theorem 21. [37] Let R be a commutative ring such that Z(R) is an ideal of R. Then the

following statements are equivalent.

(i) Reg(Γ (R)) is connected.

(ii) Either x + y ∈ Z(R) or x - y ∈ Z(R) for every x, y ∈ Reg(R).

(iii) Either x + y ∈ Z(R) or x + 2y ∈ Z(R) for every x, y ∈ Reg(R). In particular, either $2x \in Z(R)$ or $3x \in Z(R)$ (but not both) for every x ∈ Reg(R).

(iv) Either R/Z(R) ≅ Z2 or R/Z(R) ≅ Z3

The scenario in which Z(R) is not an ideal of R is much more complex. Z(G(R)) is linked and intersects with Reg(G(R)) [37].

Theorem 22. [37] Let R be a commutative ring such that Z(R) is not an ideal of R. Then Reg(Γ (R)) is connected implies that T (Γ (R)) is connected.

Theorem 23. [40] Let R be a commutative ring for which Z(R) is not an ideal of R. Subsequently, T (Γ (R[x])) is connected if and only if T (Γ (R)) is linked.

We say that a ring R is generated by a subset T of R, denoted by $(T) = R$ if R = $(t_1, ..., t_n)$ for some $t_1, ..., t_n$ ∈ T.

Theorem 24. [37] Let R be a commutative ring for which Z(R) is not an ideal of R. Consequently T (Γ (R)) is linked if and only if (Z(R)) = R. If R is a finite ring and Z(R) is not an ideal, then T (Γ (R)) is linked.

Theorem 25. [47] Let R be a commutative ring, then for any vertex $u \in V(T(\Gamma(R)))$,

$$deg(u) = \begin{cases} |Z(R)| - 1, if\ 2 \in Z(R) or\ u \in Z(R) \\ \qquad\qquad |Z(R)| \end{cases}$$

A regular graph is defined as a graph in which each vertex has a same number of adjacent vertices, meaning each vertex has an identical degree. Theorem 25 demonstrates that T (Γ (R)) is regular if and only if 2 ∈ Z(R).

Theorem 26. [38] Let R be a finite ring, after that

(i) Reg(Γ (R)) is a regular graph.

(ii) $Z(\Gamma(R))$ is regular graph if and only if R is a local ring.

6.2 The Total Graph of a module T $\Gamma(M)$

Theorem 27. [39] Let R be a commutative ring with identity, let M be an R-module. After that

(i) $T\Gamma(M)$ is complete if and only if $T(M) = M$.

(ii) $T\Gamma(M)$ is totally disconnected if and only if R has characteristic 2 and M is torsion-free.

(iii) If $T(M)$ is a proper submodule of M, then $T\Gamma(M)$ is disconnected.

6.3 The Total Graph $\Gamma_S(R)$

Theorem 28. [46] The graph $\Gamma_S(R)$ is complete if and only if $S = R$ or $char(R) = 2$ and $S = R \setminus \{0\}$.

Theorem 29. [46] Let R be a finite ring such that $R \neq Z3$. Also, suppose that S is a saturated multiplicatively closed subset of R. Then $\Gamma_S(R)$ is a forest if and only if it is a complete matching.

6.4 The Total Graph $T\Gamma_I(R)$

The next theorem illustrate the relation between $T\Gamma_I(R)$ and $T(\Gamma(R/I))$.

Theorem 30. [31] Let R be a commutative ring with the proper ideal I, and let x, $y \in R$. Then

(i) If $x + I$ and $y + I$ are (distinct) adjacent vertices in $T(\Gamma(R/I))$, then x is adjacent to y in $T(\Gamma_I(R))$.

(ii) If x and y are (distinct) adjacent vertices in $T(\Gamma_I(R))$ and $x + I \neq y + I$ then $x + I$ is adjacent to $y + I$ in $T(\Gamma(R/I))$.

(iii) If x is adjacent to y in $T(\Gamma_I(R))$ and $x + I = y + I$, then $2x, 2y \in S(I)$ and all distinct elements of $x + I$ are adjacent in $T(\Gamma_I(R))$.

Corollary 31. [31] Let R be a commutative ring with the proper ideal I. Then $T(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $T(\Gamma(R/I))$.

6.5 The Generalized Total Graph $GT_H(R)$

Theorem 32. [39] Let H denote a prime ideal inside a commutative ring R. Consequently, $GT_H(H)$ constitutes a full (induced) subgraph of $GT_H(R)$ and $GT_H(H)$ is disjoint from $GT_H(R \setminus H)$. Specifically, $GT_H(H)$ is connected, but $GT_H(R)$ is never linked.

Theorem 33. [39] Let H be a prime ideal of a commutative ring R

(i) $G_H(R \setminus H)$ is complete if and only if either $R/H \cong Z_2$ or $R \cong Z_3$.

(ii) $GT_H(R \setminus H)$ is connected if and only if either $R/H \cong Z_2$ or $R/H \cong Z_3$

(iii) $GT_H(R \setminus H)$ is totally disconnected if and only if $H = \{0\}$ and char(R) = 2.

Theorem 34. [39] Let H be a prime ideal of a commutative ring R. Then the following statements ae equivalent.

(i) $GT_H(R \setminus H)$ is connected.

(ii) Either $x + y \in H$ or $x - y \in H$ for every $x, y \in R \setminus H$.

(iii) Either $x + y \in H$ or $x + 2y \in H$ for every $x, y \in R \setminus H$. In particular, either $2x \in H$ or

$3x \in H$ (but not both) for every $x \in R \setminus H$.

(iv) Either $R/H \cong Z_2$ or $R/H \cong Z_3$

Theorem 35. [36] Let R be a commutative ring and H be a multiplicative-prime subset of R.

Then $GT_H(R)$ is connected if and only if (H) = R.

Theorem 36. [10] Let R be a commutative ring and H be a multiplicative-prime subset of R

that is not an ideal of R.

(i) $GT_H(H)$ is connected.

(ii) Some vertex of $GT_H(H)$ is adjacent to a vertex of $GT_H(R\setminus H)$. In particular, the subgraphs

$GT_H(H)$ and $GT_H(R \setminus H)$ of $GT_H(R)$ are not disjoint.

(iii) If $GT_H(R \setminus H)$ is connected, then $GT_H(R)$ is connected.

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