Adomain Decomposition Method for Solving Integro-Differentional Equations of Fractional orders

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Abstract :

In this paper, we will study a modified Integro-Differential Equations Of Fractional Order of the form :

$$D^{\alpha} y = g(x) + I^{\beta}(K(s,y))$$

where the fractional order of differentiation and integration are different from each other, in which the Adomain decomposition method have been used to solve such type of problems for linear and non linear and cases, where such type of problems are so difficult to solve using other methods.

1. Introduction :

Fractional calculus are old as classical calculus and they have been successfully applied to many fields of applied mathematics, such as viscoelastic materials, signal processing, control quantum mechanics, metrology, finance, life sciences (see [8]and[6]).

The use of fractional order differential and integral operators in mathematical models has become increasingly wide spread in recent years (see [4] and [8]).

Several forms of fractional integro-differential equations have been proposed in standard models, and there has been significant interest in developing numerical schemes for their solution (see [6] and[7]).

However, much of the work published up to date has been concerned with linear single term equations or of these, equations of order less than unity or of these equations has a fractional derivatives only have been investigated(see[5][4]).

In this paper ,an improved integro-differential equations will be considered, in which fractional order of derivative and integration are considered of different orders, and Adomain decomposition considered to solve such equations.

2. Preliminaries :

It is important to recall that fractional calculus is so difficult to understand and because of this difficulty, will refer the reads for more details to[10],[4],[6],[8] we shall present in this section only some definitions of

fractional derivatives and integration.
The usual formulation of the fractional derivative, given in standard references such as [11], [10] may be given using different aspect.
The Riemann-Liouville definition of fractional derivative, of a function u which is given by:

$$D^q u(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_{t_0}^{t} (t-s)^{m-q-1} u(s)\,ds$$

Where m is appositive integer defined by $m - 1 < q \le m$, $q \in R^+, t_0 \in R^+$ and Γ is the gamma function.

The Grunwald definition of fractional derivatives is given by:

$$D^q u(t) = \lim_{N \to \infty} \left\{ \frac{\left(\frac{t}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} u\left(t - j\left(\frac{t}{N}\right)\right) \right\}$$

where $n \in \mathrm{N}$
The Caputo definition of fractional derivative is given by:

$$D^q u(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} u^{(m)}(s)\,ds$$

for $m - 1 < q \le m$, $m \in R^+$, $t > 0$.
Also, some common formulation for the fractional integral can be derived directly from a traditional expression of the repeated integration of a function, and several definitions of fractional integration may be given, such as:
the Riemann-Liouville definition of fractional integral for a function u is:

$$J^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) d(s), q > 0 \;\ldots\ldots\ldots\ldots*$$

where $J^0 u(t) = I$ is the Identity operator.

The Weyle definition of fractional integrals for a function u, where u(t) is a periodic function and its mean value for one period is zero, is given by:

$$J^\alpha u(t) = \frac{1}{\Gamma(q)} \int_{-\infty}^{\infty} (t-s)^{q-1} u(s) d(s)$$

It is remarkable that formula (*) is used as the definition of the integral without any conditions at the present time. Other types of fractional integrations are given in [2].

An important property concerning fractional derivatives and integration is the following[1]

$$J^q D^q u(t) = u(t) - \sum_{k=0}^{m-1} u^{(k)}(o^+)\frac{t^k}{k!}, t > 0 \text{.........}@$$

$$D^q J^q u(t) = u(t)$$

The properties of the operator J^q can be found for $q \geq 0, p > 0$, which are:

1. $J^q J^p u(t) = J^{q+p} u(t)$

2. $J^q J^p u(t) = J^p J^q u(t)$

3.Solution of fractional integro differential equations using Adomian Decomposition Method.

Adomian decomposition method (ADM for short) is one of the new methods that may be used to solve initial value problems in fractional integro-differential equations of various kinds that arising in the fields of medicine,engineering, physical and biological science. It is important to notice that a large amount of research workers had studied the application of ADM to a wide class of linear and nonlinear integro-differential equations of fractionalorder. In resent years, the decomposition method has emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential, integral, integro-differential, higher order ordinary differential equations and partial differential equations.

The convergence of this method have been investigated by Cherruault and cooperators[9]. In this method, the solution is considered as the sum of an infinite series, rapidly converging to an accurate solution. The advantage of ADM` is that it provides a direct scheme for solving the problem without any need for linearization or discretization. Essentially, the method provides a systematic computational procedure for equations of physical significance.

The analysis of the method for solving fractional integro different ional equations may be give for linear and non linear cases, as follows:

3.1 <u>Linear case</u>

Consider the linear fractional integro-differention in operator form:

$$D^\alpha y = g(x) + J^\beta K(y), \alpha, \beta \in R^+$$

or

$$D^{\alpha} y = g(x) + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} y(s) d(s) \ldots\ldots\ldots \otimes$$

Where g is a given function and $y_0(x_0) = y_0$ is the given initial condition.

Taking J^{α} to the both sides of eq. ($\otimes$), give and using property (@),one can get:

$$J^{\alpha} D^{\alpha} y = y = J^{\alpha} g + J^{\alpha} \left[\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} y(s) d(s) \right]$$

And the initial condition is then transformed to:

$$y_0(x) = y(x_0) + J^{\alpha} g$$

and hence the solutions may be obtained iteratively as:

$$y_1 = J^{\alpha} \left[\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} y_0(s) d(s) \right]$$

$$y_2 = J^{\alpha} \left[\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} y_1(s) d(s) \right]$$

$$y_3 = J^{\alpha} \left[\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} y_2(s) d(s) \right]$$

$$y_{n+1}(x) = J^{\alpha} \left[\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} y_n(s) d(s) \right]$$

Hence a sequence of approximate solutions will be obtained $\{y_i\}$,i=1,2,… wich will converge to the exact solution [10].

As an illustration, we will consider two examples, the first one for linear case and the other one for nonlinear-case.

Example (1)

Consider the fractional integro-differential equation:

$$D^{0.5} y(x) = g(x) + J^{1.5} y(x)\ ,\ y(0) = 0, x \in [0,1] \ldots\ldots\ldots\ldots \#$$

Where $g(x) = \frac{6}{\Gamma(3.5)} x^{2.5} - \frac{6}{\Gamma(4.5)} x^{3.5}$

According to the Adomain decomposition method, the approximate solution is given by:

$$y(x)=\sum_{k=0}^{m-1}\frac{\partial^k y(0)}{\partial x^k}\frac{x^k}{k!}+\frac{6}{\Gamma(3.5)}J^{1.5}x^{2.5}-\frac{6}{\Gamma(4.5)}J^{1.5}x^{3.5}+$$

$$J^{1.5}\left[\frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{0.5}y(s)d(s)\right]$$

and therefore:

$$y_0(x)=\sum_{k=0}^{0}\frac{\partial^k}{\partial x^k}\frac{x^k}{k!}+\frac{6}{\Gamma(3.5)}J^{1.5}x^{2.5}-\frac{6}{\Gamma(4.5)}J^{1.5}x^{3.5}$$

$$=\frac{1}{4}x^4-\frac{1}{20}x^5$$

$$J^{1.5}x^{2.5}=\frac{\Gamma(2.5+1)}{\Gamma(2.5+1+1.5)}=\frac{\Gamma(3.5)}{\Gamma(5)}$$

Where

$$J^{1.5}x^{3.5}=\frac{\Gamma(3.5+1)}{\Gamma(3.5+1+1.5)}=\frac{\Gamma(4.5)}{\Gamma(6)}$$

And also

$$y_1(x)=J^{1.5}\left[\frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{0.5}(0.25s^4-0.05s^5)ds\right]$$

$$=0.25\ \frac{\Gamma(5)}{\Gamma(8)}x^7-0.05\ \frac{\Gamma(6)}{\Gamma(9)}x^8$$

$$=0.25\ \frac{4!}{7!}x^7-0.05\ \frac{5!}{8!}x^8$$

$$=\frac{0.25}{3!}x^7-\frac{0.05}{3!}x^8$$

Similarly

$$y_2(x)=J^{1.5}\left[\frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{0.5}(0.0011904761\quad 905\ s10^{7}-0.0001488095\quad 238\ s^{8})ds\right]$$

$$=\frac{1}{720}0.0011904761\quad 905\ x^{10}-\frac{1}{990}0.0001488095\quad 238\ x^{11}$$

$$\therefore\ y_2(x) = (0.0000165343\ 915)\,x^{10} - (0.0000001503127)x^{11}$$

And in a same manner, we my find:

$$y_3(x) = 0.0000000096354x^{13} - 0.0000000000688x^{14}$$

$$y_4(x) = 0.000000000029x^{16}$$

Which will yields to $y_5 = 0$ and hence $y_6 = y_7 = \ldots = 0$

Therefore, the solution using Adomain decomposition method is given by:

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \ldots$$

$$= (0.25x^4 - 0.05x^5) + (0.0011904761905x^7 - 0.0001488095238x^8)$$

$$+ (0.0000165343915x^{10} - 0.0000001503127x^{11}) + (0.0000000096354x^{13}$$

$$- 0.0000000000688x^{14}) + 0.000000000029x^{16}$$

The residue error obtained by substituting y(x) in eq(#) for is different values Of $x \in [0,1]$.

Table (1) The residue error of example(1)

x	Redis of error(LHS-RHS)^2
0	0
0.1	0.0000298690826
0.2	0.0008740488229
0.3	0.0060538120929
0.4	0.0232050248014
0.5	0.0642395273417
0.6	0.0371909207961
0.7	0.2816175282741
0.8	0.4932804497599
0.9	0.7957206811371
1	1.2016070285921

3.2 Nonlinear case:

Moreover, the method may be used to the nonlinear integro-differential Equation of fractional order:

$$D^{\alpha}u(t)=f(t)+J^{\beta}N(u(t)),\ u(0)=0$$

Or equivalently :

$$D^{\alpha}u(t)=f(t)+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}N(u(s))ds, u(0)=u_0 \text{………**}$$

Where $0<\alpha<1$,$0<\beta<1$, $f(t)$ is assumed to be bounded for all $t\in I=[0,T]$

And $\left|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds\right|\le M$, for all $0\le s\le t\le T$

Where M is finite constant.

The nonlinear term N(u) is Lipschitzian with :

$$|N(u)-N(z)|\le L|u-z|$$

And may be decomposed in the form :

$$N(u(t))=\sum_{n=0}^{\infty}A_n(t) \text{…………***}$$

where A_n are the Adomian polynomials, given by :

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]\Big|_{\lambda=0}$$

Then N(u) will be a function of $\lambda, u_0, u_1, \dots$.

Now ,substituting eq.(***) in eq. (**), yields to:

$$D^{\alpha} u(t) = f(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(\sum_{n=0}^{\infty} A_n(s)\right) ds \text{........****}$$

Operating with J^{β} on both sides of equation (****) give:

$$u(t) = \sum_{k=0}^{n-1} u^k(0) \frac{t^k}{k!} + J^{\beta} f(t) + J^{\beta}\left(\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(\sum_{n=0}^{\infty} A_n(s)\right) ds\right)$$

The components $u_0, u_1, \dots$are determined recursively by:

$$u_0 = \sum_{k=0}^{n-1} u^k(0) \frac{t^k}{k!} + J^{\beta} f(t)$$

$$u_{k+1} = J^{\beta}\left(\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A_k(s) ds\right)$$

The Adomain's polynomials can be generated from Taylor expansion of $N(u(t))$ about the first component u_0,which means that :

$$N(u) = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} N^{(n)}(u_0)$$

In [3],Adomain's polynomials are arranged to have the form :

$$A_0 = f(u_0)$$

$$A_1 = u_1 f^{(1)}(u_0)$$

$$A_2 = u_2 f^{(1)}(u_0) + \frac{u_1^2}{2!} f^{(2)}(u_0)$$

$$A_3 = u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + \frac{u_1^3}{3!} f^{(3)}(u_0)$$

<u>Theorem(1)</u> :

The series solution $u(t) = \sum_{n=0}^{\infty} u_n(t)$ of equation

$D^{\alpha} u(t) = f(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s) ds,\ u(0) = u_0\ , 0 < \beta < 1$ using ADM

Converges whenever 0 < k < 1 ,where $k = \frac{LMT^{\beta}}{\Gamma(\beta)}$ and $|u_1| < \infty$

<u>Proof</u> :

Let s_n and s_m be an arbitrary partial sums with n $n \geq m$,and to prove that $\{s_n\}$ is a Cauchy sequence in Bnach space B.

Therefore :

$$\|s_n - s_m\| = \max_{t \in I} |s_n - s_m|$$

$$= \max_{t \in I} \left| \sum_{i=m+1}^{n} u_i(t) \right|$$

$$= \max_{t \in I} \left| \sum_{i=m+1}^{n} J^{\beta} \left(\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \overline{A}_{i-1}(s) ds \right) \right|$$

$$= \max_{t \in I} \left| J^{\beta} \left(\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sum_{i=m+1}^{n} \overline{A}_{i-1}(s) \right) \right|$$

From $\overline{A}_n = N(s_n) - \sum^{n-1} \overline{A}_i$ we have : $\sum_{i=m}^{n-1} \overline{A}_i = N(s_{n-1}) - N(s_{m-1})$

$$\|s_n - s_m\| = \max_{t\in I}\left| J^{\beta}\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[N(s_{n-1} - N(s_{m-1})]ds\right)\right|$$

$$\le \max_{t\in I} J^{\beta}\left|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds\right|N(s_{n-1}) - N(s_{m-1})|$$

$$\le \frac{LMT^{\beta}}{\Gamma(\beta)}\|s_{n-1} - s_{m-1}\|$$

Hence :

$$\|s_n - s_m\| \le k\|s_{n-1} - s_{m-1}\|$$

Let $n = m+1$, then :

$$\|s_{m+1} - s_m\| \le k\|s_m - s_{m-1}\|$$

$$\le k^2 \|s_{m-1} - s_{m-2}\|$$

$$\vdots$$

$$\le k^m \|s_1 - s_0\|$$

And from the triangle inequality:

$$\|s_n - s_m\| \le \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \ldots + \|s_n - s_{n-1}\|$$

$$\le (k^m + k^{m+1} + \ldots + k^{n-1})\|s_1 - s_0\|$$

$$\le k^m \left(\frac{1-k^{n-m}}{1-k}\right) \|u_1(t)\|$$

Since $0<k<1$,so $(1-k^{n-m})<1$, and then :

$$\|s_n - s_m\| \le \frac{k^m}{1-k} \max_{t\in I}|u_1(t)| \ldots\ldots\ldots\ldots *****$$

But $|u_1(t)|<\infty$

Since f(t) is bounded, so as $m\to\infty$,then $\|s_n - s_m\|\to 0$

So $\{s_n\}$ is a Cauchy sequence in B, and therefore the series $\sum_{i=0}^{\infty} u_i(t)$ converges.

<u>Theorem(2)</u> :

The maximum absolute error of the series solution $u(t)=\sum_{n=0}^{\infty} u_n(t)$ to equation

$D^{\alpha}u(t)=f(t)+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)ds$ is estimated to be:

$$\max_{t\in I}\left|u(t)-\sum_{i=0}^{\infty} u_i(t)\right| \le \frac{k^m}{1-k} \max_{t\in I}|u_1(t)|$$

<u>Proof</u>:

From equation(*****) in theorem (1), we have :

$$\|s_n - s_m\| \le \frac{k^m}{1-k} \max_{t\in I}|u_1(t)|$$

As $n\to\infty$, then $s_n \to u(t)$, so we have :

$$\|u(t) - s_m\| \le \frac{k^m}{1-k} \max_{t\in I}|u_1(t)|$$

And the maximum absolute truncation error in the interval I is estimated to be :

$$\max_{t\in I}\left|u(t)-\sum_{i=0}^{\infty} u_i(t)\right| \le \frac{k^m}{1-k} \max_{t\in I}|u_1(t)|.$$

<u>Example</u> (2)

Consider the nonlinear fractional integro-differential equation:

$$D^{0.25}y(x)=\frac{1}{\Gamma(1.25)}x^{0.25}-\frac{2}{\Gamma(3.75)}x^{2.75}+J^{0.75}y^{2}(x),\ x\in[0,1]$$

$$y(0)=0$$

Therefore, according to the Adomain decomposition method:

$$y_0(x)=\sum_{k=0}^{0}y^{k}(0)\frac{x^k}{k!}+J^{0.75}h(x)=0+J^{0.75}\left[\frac{1}{\Gamma(1.25)}x^{0.25}-\frac{2}{\Gamma(3.75)}x^{2.75}\right]$$

$$=x-\frac{2}{\Gamma(4.5)}x^{3.5}$$

$$y_{n+1}=J^{0.75}\left[\frac{1}{\Gamma(0.75)}\int_0^x(x-s)^{-0.25}A_n(s)d(s)\right],n=0,1,\ldots \qquad [1]$$

are the Adomain polynomials for the nonlinear term $N(y)=y^2$

$$y_1(x)=J^{0.75}\left[\frac{1}{\Gamma(0.75)}\int_0^x(x-s)^{-0.25}(s-\frac{2}{\Gamma(4.5)}s^{3.5})ds\right]$$

$$=-0.025x^{6}+0.17194x^{3.5}+1.24908x^{8.5}$$

$$y_2(x)=J^{0.75}\left[\frac{1}{\Gamma(0.75)}\int_0^x(x-s)^{-0.25}\left[\begin{array}{l}2(s-\frac{2}{\Gamma(4.5)}s^{3.5})(-0.025s^{6}+0.17194s^{3.5}+\\ 0.00124908s^{8.5})ds\end{array}\right]\right]$$

$$y_2(x)=-1.73157\quad 10^{-4}x^{11}+2.5\ 10^{-2}x^{6}-3.8569\quad 10^{-4}x^{8.5}-$$

$$8.90992\quad 10^{-6}x^{13.5}$$

Similarity, in a same manner:

Hence

$$y_3 = 0.000017437731066x^{13.5} + 0.003360804774983x^{8.5}$$
$$-0.000510114271856x^{11} - 0.000009329943918x^{16}$$
$$+0.000000427607336x^{17.5}$$

Hence $A_2(s) = 2y_0y_2 + y_1{}^2$, and

Similarly, we can find :

$$A_3(s) = 2y_1y_2 + 2y_0y_3$$

$$= 0.000232455\ \ 498641\,s^{14.5} - 0.0000200249\ \ 18736\ s^{17}$$
$$-0.0035583668\ \ 69914\ s^{12} + 0.0153186095\ \ 49966\ s^{9.5}$$
$$+0.0000032207\ \ 22882\ s^{19.5} + 0.0000008552\ \ 14672\ s^{18.5}$$
$$-0.0000001470\ \ 19012\ s^{21} - 0.0000000222\ \ 58405\ s^{22}$$

$$y_4(x) = 0.00000372085841x^{16} - 0.000025685672669x^{18.5}$$
$$-0.000073795991817x^{13.5} + 0.000434998360226x^{11}$$
$$+0.000003408728347x^{21} + 0.000009747626049x^{20}$$
$$-0.000000140074676x^{22.5} - 0.000000019853404x^{23.5}$$

And so on .

Hence, the approximate solution using ADM up to for terms is given by:

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x)$$

$$\Rightarrow y(x) = x + 0.004224186774983x^{8.5} - 0.000065268180751x^{13.5}$$

$$-0.00024827291163x^{11} - 0.000005609085508x^{16}$$

$$+0.000000427607336x^{17.5} - 0.000025685672669x^{18.5}$$

$$+0.000003408728347x^{21} + 0.000009747626049x^{20}$$

$$-0.000000140074676x^{22.5} - 0.000000019853404$$

Table (2) the residue error of example (2)

x	Redis of error(LHS-RHS)^2
0	0
0.1	0.1822631978817
0.2	0.1700654307148
0.3	0.1409613537276
0.4	0.1089803768896
0.5	0.0788145079365
0.6	0.0524963201967
0.7	0.0310080415341
0.8	0.0148602631693
0.9	0.0043982669861
1	0.0000737934553

the residue error for different values of $x \in [0,1]$ are given in table (2).

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طريقة تحليل (Adomain) لحل المعادلات التفاضلية التكاملية من الرتب الكسرية

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الخلاصة :

في هذا البحث، سنقوم بدراسة نوع محور من المعادلات التفاضلية التكاملية ذات الرتب الكسرية بالشكل :

$$D^{\alpha} y = g(x) + I^{\beta}(K(s,y))$$

حيث إن رتبة التفاضل والتكامل الكسريين تكون مختلفة ،حيث تم استخدام طريقة تحليل أدومن لحل هكذا نوع من المشاكل ولكلتا الحالتين الخطية وغير الخطية ، حيث ان هذا النوع

من المشاكل تكون من الصعب ايجاد حل لها بالطرق الاخرى.