

Optical Perspective of the Fourier Transformers

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Abstract

The present work is a theoretical and experimental study of a biconvex lens as an optical processor. The MATLAB software concerning the Fourier optics is used as the basic numerical tool for these work. In addition to providing functions for calculating Fraunhofer diffraction. The FFT command enables the calculation of the diffraction pattern of an arbitrary aperture. Relatively simple MATLAB® scripts are constructed to calculate the diffraction patterns of arbitrary graphics such as geometry shapes, pictures of faces, letters. This paper also describes a few demonstrations that can be used to reinforce what is covered on the work. The demonstrations are based on a simple (2F set-up) system. Diffractive imaging relies on many other technological innovations, including: a CCD camera, digital camera HD (12 Mega Pixels) with a lens ($f=7.23\text{mm}$). The experimental results agree with the theoretical calculations.

الخلاصة

تم في هذا البحث دراسة نظرية وعملية باستخدام عدسة محدبة الوجهين كمعالج إشارة بصرية. في هذا العمل تم استخدام برمجيات **MATLAB** كأداة لإجراء الحسابات النظرية الخاصة بحسابات دوال فتحات الحيود وأنماطه. بالإضافة إلى توفير الدوال لحساب حيود فرانوفر، تمكنا **FFT** من حساب أنماط الحيود لفتحة اعتباطية. بنص بسيط مكتوب بلغة **MATLAB®** لحساب أنماط الحيود لإشكال اعتباطية مثل أشكال هندسية، صور وجوه، أحرف. هذا البحث يصف أيضا عدة إثباتات يمكن استخدامها لدعم هذا العمل. هذه الإثباتات تستند على أساس استخدام منظومة **2F**. صور الحيود تعتمد على عدة طرق تكنولوجية، من ضمنها: كاميرا **CCD**، كاميرا رقمية **HD (12 ميكا بكسل)** بعدسة بعدها البؤري ($f=7.23\text{mm}$). والنتائج العملية متطابقة مع النتائج النظرية.

1- Introduction

Diffraction theory is often taught as a purely mathematical treatment or used to analyze very simplistic apertures such as slits and holes. The effect is a general characteristic of wave phenomena occurring whenever a portion of a wavefront, be it sound, a matter wave, or light, is obstructed in some way. The various segments of the Wavefront that propagate beyond the obstacle interfere, causing the particular energy-density distribution referred to as the diffraction pattern. There is no significant physical distinction between interference and diffraction (Eugene Hecht, 2002). As shown in figure (1).

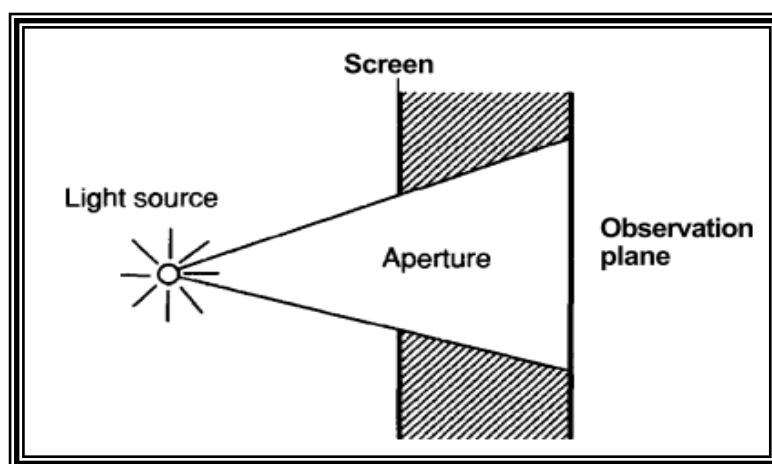


Fig . 1 : Arrangement used for observing diffraction of light (Joseph W. Goodman , 1996).

The goal of this paper is to describe how the scientific analysis tool MATLAB® can be used to perform complex mathematical calculations with Fraunhofer diffraction domains and experimental implementation . Fraunhofer diffraction deals with the limiting cases where the light approaching the diffracting object is parallel and monochromatic, and where the image plane is at a distance large compared to the size of the diffracting object (Okan K. Ersoy , 2007) . Fraunhofer diffraction is far field approximation, where the observe pattern in the focal plane of a lens . The far field diffraction, or Fraunhofer diffraction of the Kirchhoff–Fresnel integral have the same mathematical appearance . The only difference is that in far field approximation the diffraction pattern is observed on a faraway screen, whereas in Fraunhofer diffraction the observation screen is placed at the focal plane of a lens and that may be closer to the aperture (K.D. Moller et al. , 2003). Linear transforms , especially Fourier and Laplace transforms, are widely used in solving problems in science and engineering. The Fourier transform is used in linear systems analysis, antenna studies, optics, random process modeling, probability theory, quantum physics, and boundary-value problems (Brigham , E.Oren ,1988) and has been very successfully applied to restoration of astronomical data (Brault,J.W.andWhite,O.R , 1971) .

2- Theory

Fraunhofer diffraction is the theory of transmission of light through apertures under the assumption that the incident wave is multiplied by the aperture function . Fraunhofer diffraction is far field approximation , where the observed pattern is located at the focal plane of a lens (Okan K. Ersoy , 2007) , which usually called Fourier plane . So , we will use the fraunhofer approximation to determine the propagation of light in the free space beyond the aperture . As shown in figure (2) , in this figure which showing position of Fraunhofer (far field) region (Keigo Iizuka , 2008) .

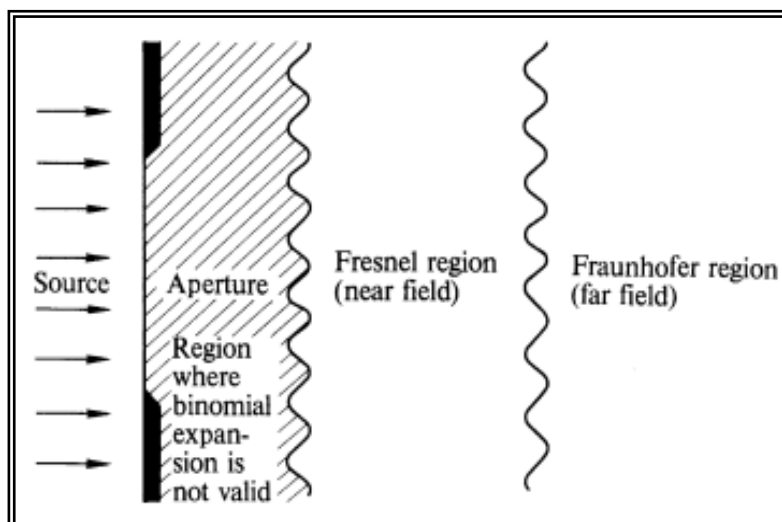


Fig . 2 : Diagram showing the relative positions of the Fresnel (near field) and Fraunhofer (far field) regions (Keigo Iizuka , 2008) .

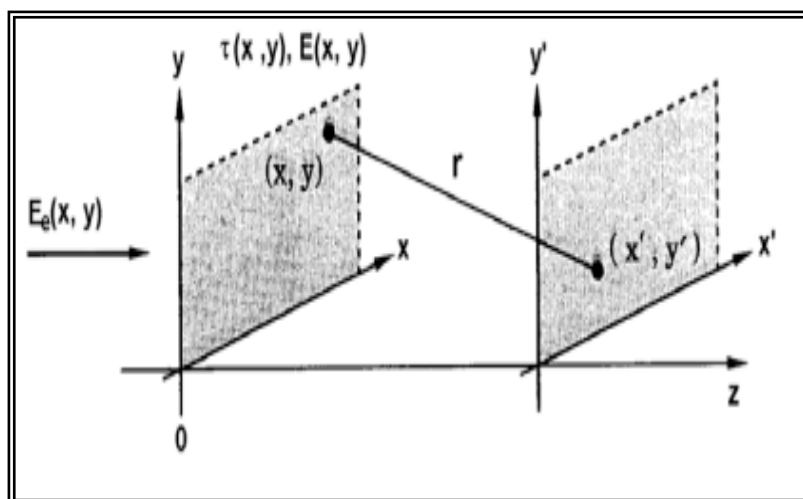


Fig . 3 : A plane wave $E_e(x, y)$ is diffracted in the plane with $\tau(x, y)$ for $z=0$ (David Voelz , 2011).

In fig (3) where the observed plane wave which is diffracted in one plane . For this wave in the (x, y) plane directly behind the plane ($z=0$) with the following transmission distribution $\tau(x, y)$ (David Voelz , 2011) :-

$$E(x, y) = \tau(x, y) E_e(x, y) \quad (1)$$

Where $E_e(x, y)$: electric field distribution of the incident wave. The further expansion can be described by the assumption that a spherical wave emanates from each point $(x, y, 0)$ behind the diffracting structure (Huygens' principle). This lead to Kirchhoff's diffraction integral (David Voelz , 2011) :-

$$E(x', y', z) = \frac{1}{i\lambda} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(x, y) \frac{e^{ikr}}{r} \cos(\vec{n}, \vec{r}) dx dy \quad (2)$$

with $\lambda = \text{spherical wave length}$.

$\vec{n} = \text{normal vector of the } (x, y) \text{ plane}$.

$$k = \text{wave number} = \frac{2\pi}{\lambda}.$$

Equation (2) corresponds to a accumulation of spherical waves, where the factor $\left(\frac{1}{i\lambda}\right)$ is a phase and amplitude factor and $\cos(\vec{n}, \vec{r})$ a directional factor which results from the Maxwell field equations. The Fresnel approximation (observations in a remote radiation field) considers only rays which occupy a small angle to the optical axis (z -axis). In this case, the directional factor can be neglected and the $\left(\frac{1}{r}\right)$ dependence becomes : $\left(\frac{1}{r} = \frac{1}{z}\right)$. In the exponential function, this cannot be performed as easily since even small changes in (r) result in large phase changes. This results in the *Fresnel approximation* of the diffraction integral.

$$E(x', y', z) = \frac{e^{ikz}}{i\lambda} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(x, y) \exp\left\{\frac{ik}{2z} ((x' - x)^2 + (y' - y)^2)\right\} dx dy \quad (3)$$

For long distances from the diffracting plane with concurrent finite expansion of the diffracting structure, will obtain the *Fraunhofer approximation* as given by eq. (4).

$$E(x', y', z) = C(x', y', z) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(x, y) \exp\left\{2\pi i \left(\frac{x'}{\lambda z} x + \frac{y'}{\lambda z} y\right)\right\} dx dy \quad (4)$$

$$\text{with : } C(x', y', z) = \frac{e^{ikz}}{i\lambda z} \exp\left\{\frac{i\pi}{\lambda z} (x'^2 + y'^2)\right\}$$

Where $C(x', y', z)$ is the phase factor and $E(x', y', z)$ is the electric field distribution in the plane (x', y') for $(z = \text{const})$ (David Voelz, 2011).

3- Theoretical Results

When the distance away from the grating is large or a lens is used to focus the diffraction pattern to the image plane then the diffraction pattern becomes a Fourier transform as given by (Joseph W. Goodman, 1996).

$$E(x, y, z) = C \mathfrak{T} \{ E \} \Big|_{\substack{v_x = \frac{x}{\lambda z} \\ v_y = \frac{y}{\lambda z}}} \quad (5)$$

Where : $E(x, y, z)$: is the electric field distribution .

C : is the phase factor .

v_x and v_y are spatial frequencies.

λ = wave length .

3-1 Fourier Transform using Matlab for rect-function

The one-dimensional (1-D) rectangular function is given by : -

$$\text{rect}\left(\frac{x}{a}\right) = \begin{cases} 1, & |x| < a/2 \\ 0, & \text{otherwise} \end{cases} \quad (6)$$

Where (a) is the domain of the function (or the width of the aperture) .The two-dimensional of the rectangular function is given by : -

$$\text{rect}(x/a, y/b) = \text{rect}(x/b) \text{rect}(y/b) \quad (7)$$

The Fourier transform of the (2-D) rectangular function is given by (Ting Chung Poon , 2007) : -

$$\begin{aligned} \mathfrak{T}_{xy} \{ f(x, y) \} &= \mathfrak{T}_{xy} \{ \text{rect}(x/a, y/b) \} \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \text{rect}(x/a, y/b) \exp[-2\pi i (v_x x + v_y y)] dx dy \quad (8) \end{aligned}$$

The aperture function , $\{ \text{rect}(x, y) \}$ and its Fourier transform is shown in fig. (4).

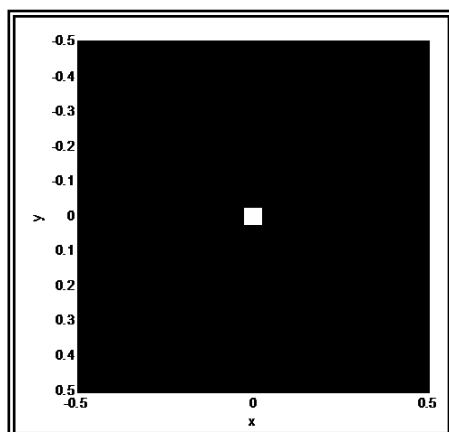


Fig . 4 : a - rect- function:
gray-scale plot .

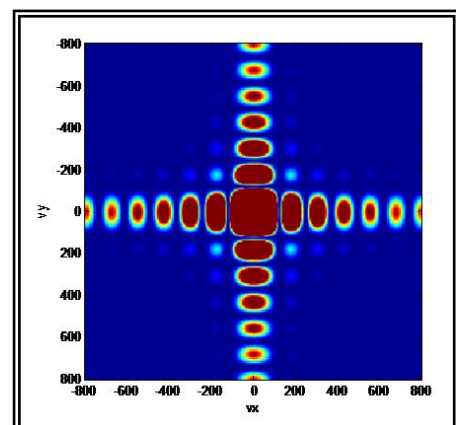


Fig . 4 : b - Square-absolute value of Fourier
transform of the rect- function: gray-scale plot .

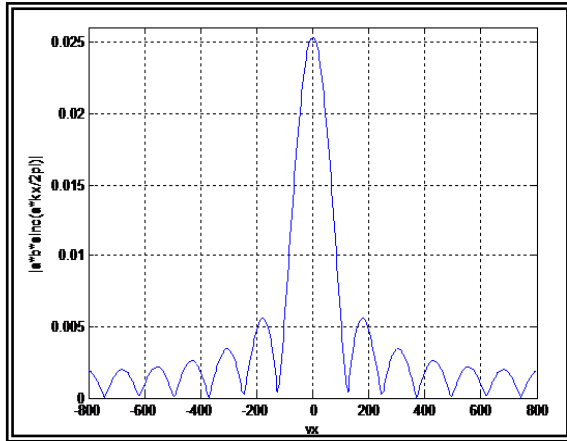


Fig . 4 : c- Square-absolute value of Fourier transform of a rect- function: cross-section plot.

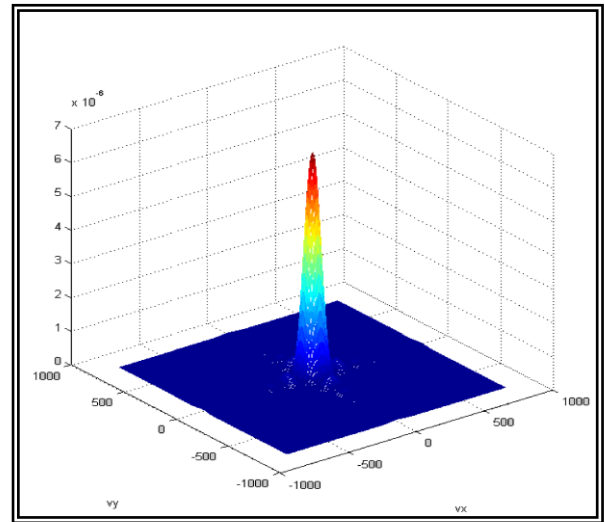


Fig . 4 : d- The power distribution of a rect- function by 3-D .

Fig . 4 : Rect- function and its Fourier transform .

3-2 The Triangle Function

The *triangle function* is a shape like a triangle, as its name indicates. Figure (5: a) illustrates the wave form. The function can be obtained by the convolution of $\Pi(x)$ with itself as shown (Keigo Iizuka , 2008) .

$$\Lambda(x) = \Pi(x) * \Pi(x) \quad (9)$$

Using the fact : -

$$\mathfrak{F}\{\Pi(x) * \Pi(x)\} = \mathfrak{F}\{\Pi(x)\} \mathfrak{F}\{\Pi(x)\} \quad (10)$$

the Fourier transform of eq. (9) is found to be in (Fig. 5: b)

$$\mathfrak{F}\{\Lambda(x)\} = \sin^2 c^2 f \quad (11)$$

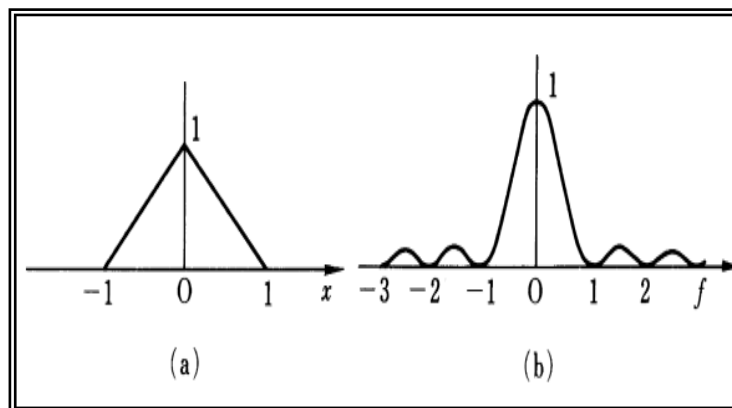


Fig . 5: (a) Triangle function $\Lambda(x)$ and (b) its Fourier transform $\sin^2 c^2 f$ (Eugene Hecht , 2002) .

3-2-1 Fourier Transform of Bitmap Images

When the (2-D) function or image is given with a bitmap file, we can use the m-file given in algorithm (1) to find its Fourier transform. Figure (6 : a) is the bitmap image used when the image file of the size is (256×256) . It is easily generated with Microsoft® Paint. Figure (6:b) is the diffraction pattern (or the Fourier transform) of the tri-function (Ting Chung Poon , 2007) .

Algorithm 1: *fft2D bitmap_image.m : m-file for 2-D Fourier transform of bitmap image.*

```
-----
%fft2Dbitmap_image.m
%Simulation of Fourier transformation of bitmap images
clear
I=imread('triangle.bmp','bmp'); %Input bitmap image
I=I(:,:,1);
figure(1) %displaying input
colormap(gray(255));
image(I)
axis off
FI=fft2(I);
FI=fftshift(FI);
max1=max(FI);
max2=max(max1);
scale=1.0/max2;
FI=FI.*scale;
figure(2) %Gray scale image of the absolute value of transform
colormap(gray(255));
image(10*(abs(256*FI)));
axis off
-----
```

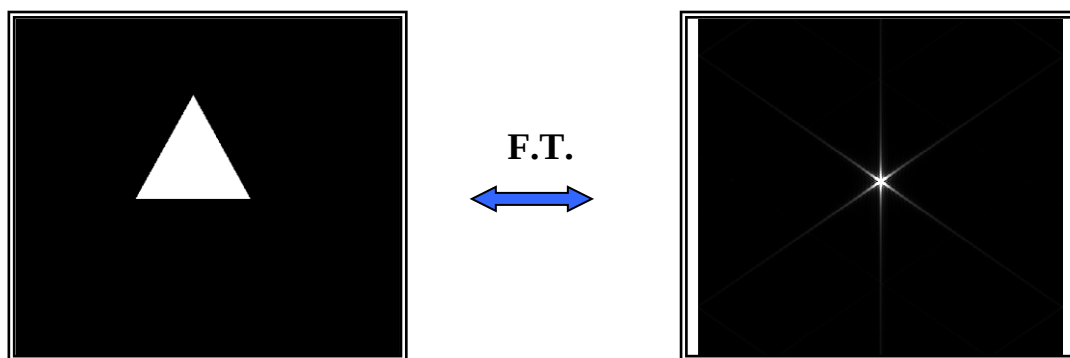


Fig . 6 : a- Bitmap image of a triangular aperture function.

Fig . 6 : b- The diffraction pattern (or the Fourier transform) of the tri-function .

Fig . 6 : Bitmap image and its transform generated using the m-file in Matlab software .

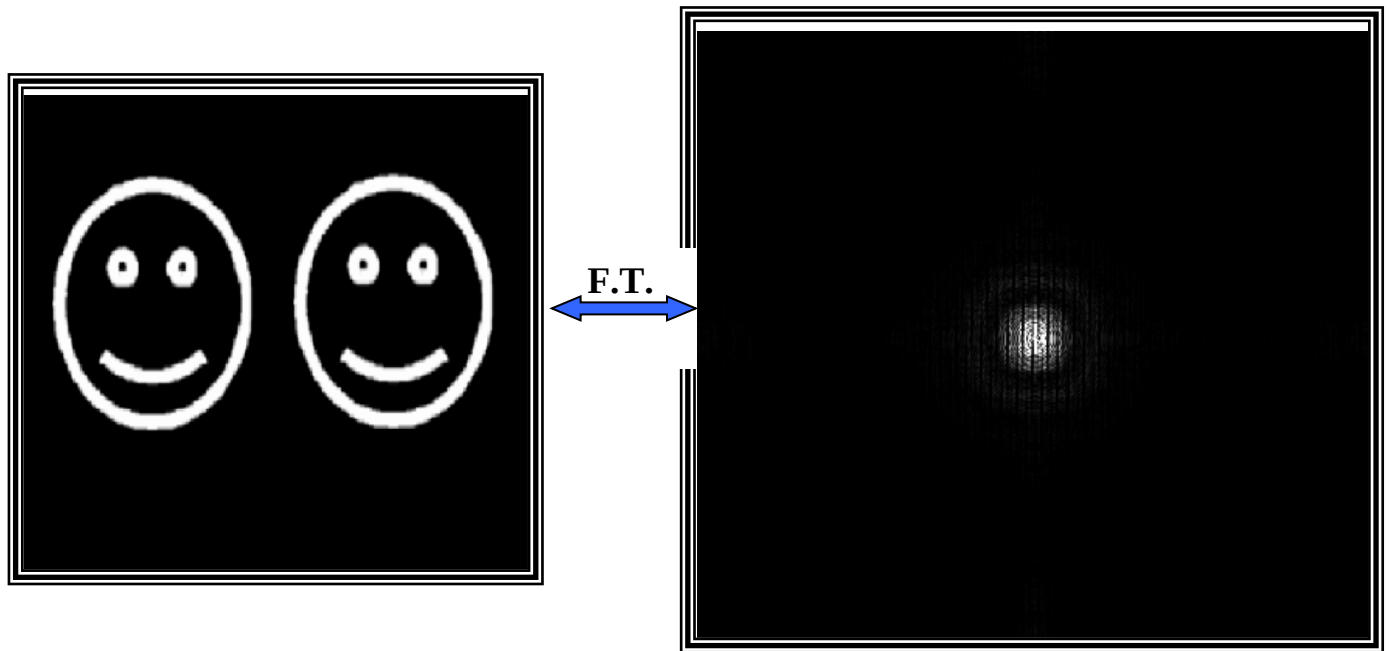


Fig . 7: a- Bitmap image of a face.

Fig . 7: b- Displaying the absolute value of the transform of a .

Fig . 7 : Bitmap image and its transform generated using the m-file in Matlab software .

3-3 Element Size

The extent of the diffraction pattern is complementary to the size of the single diffracting elements . Figure (8) shows this reciprocal behavior using elliptical apertures of different size (Cetin, A.E. , et al. , 2004) .

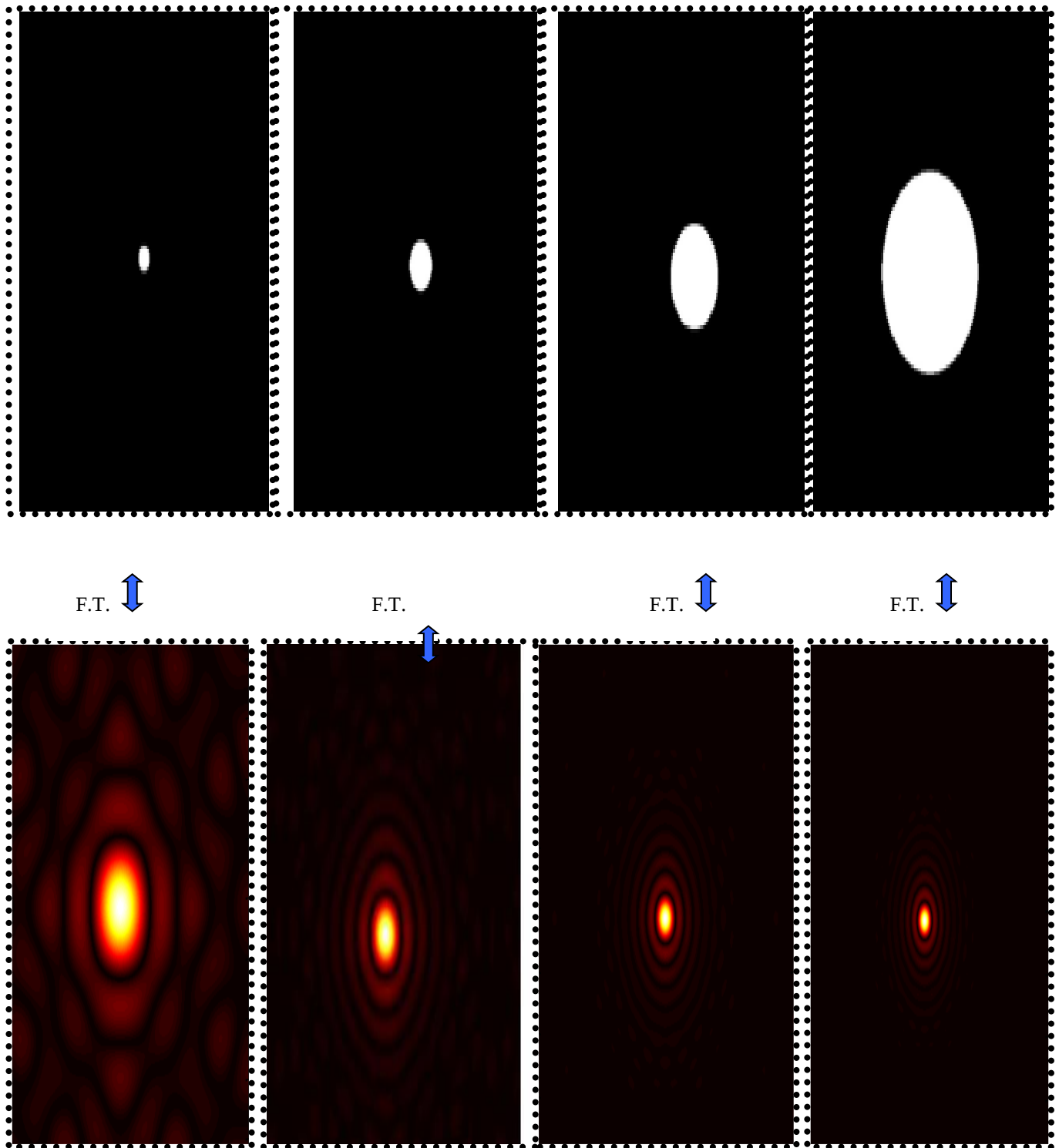


Fig . 8: Bitmap images and its transform generated by the m-file in the Matlab software for different size elliptical apertures.

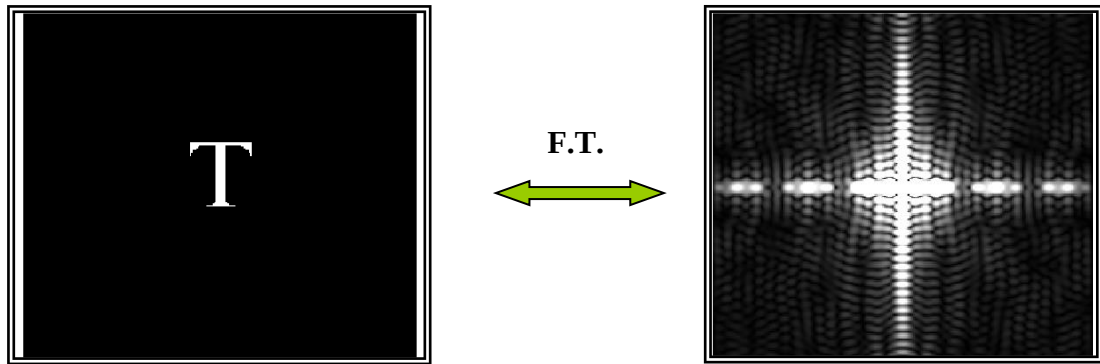


Fig . 9: Diffraction pattern of the letter(T- aperture) made with times new roman fonts .

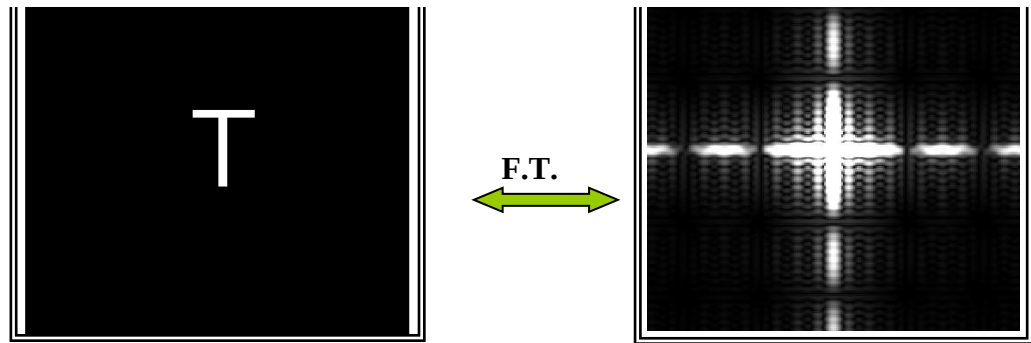


Fig . 10: Diffraction pattern of the letter (T- aperture) made with an arial font .

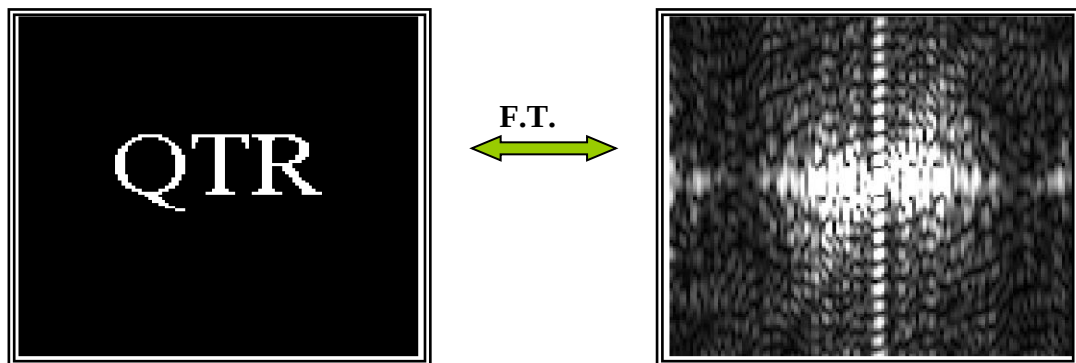


Fig . 11: Diffraction pattern of an aperture constructed with (QTR) letters.

4- Experimental Results

The experimental results include investigation of the Fourier transform by a convex lens for different diffraction objects in a (2F set-up) system . As shown in figure (12).



Fig . 12 : Experimental (2F set-up) system .

The diffraction pattern of a rectangular aperture by using a digital camera and CCD camera as shown in figures (13,14,15,16) .

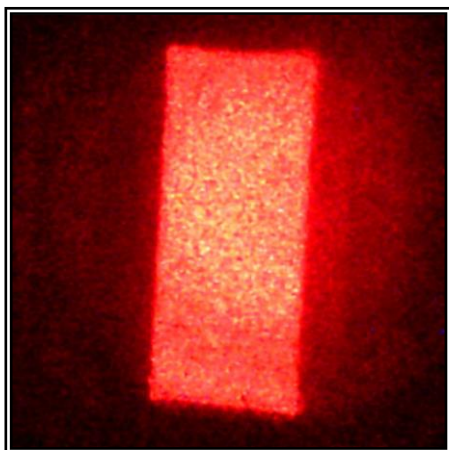


Fig . 13 : Rectangular aperture recorded with a digital camera .

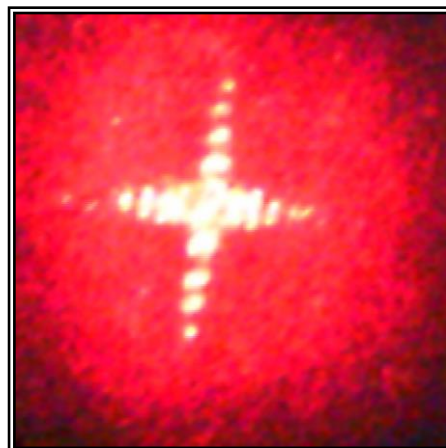


Fig . 14 : Diffraction pattern of rectangular recorded with a digital camera .

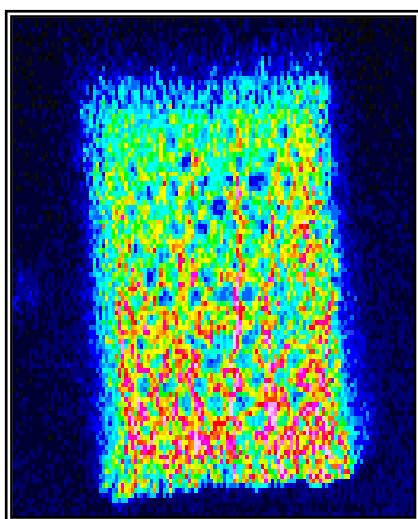


Fig . 15 : Power distribution in rectangular aperture recorded with a CCD camera .

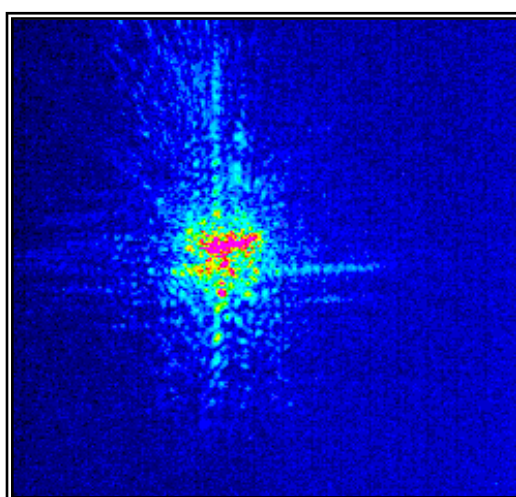


Fig . 16 : Diffraction pattern of rectangular recorded with a CCD camera .

The diffraction pattern of a triangular aperture by using a digital camera and CCD camera as shown in figures (17,18,19,20) .

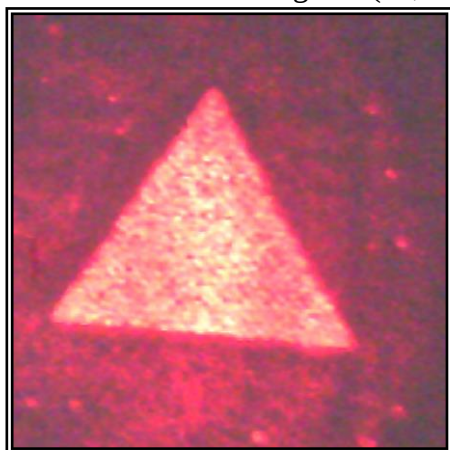


Fig . 17 : The power distribution in the triangle aperture as shown by a digital camera HD.



Fig . 18 : Diffraction pattern of triangle aperture as shown by a digital camera HD.

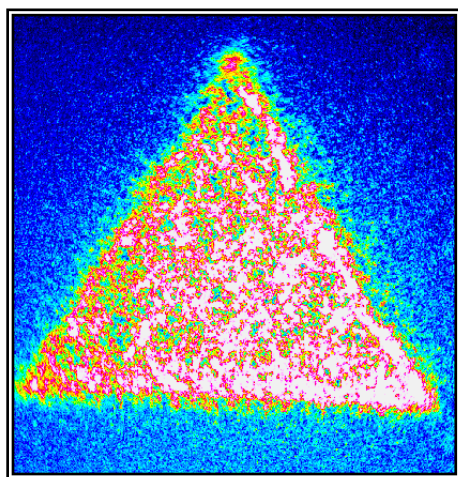


Fig . 19 : The power distribution in the triangle aperture as shown by a CCD camera .

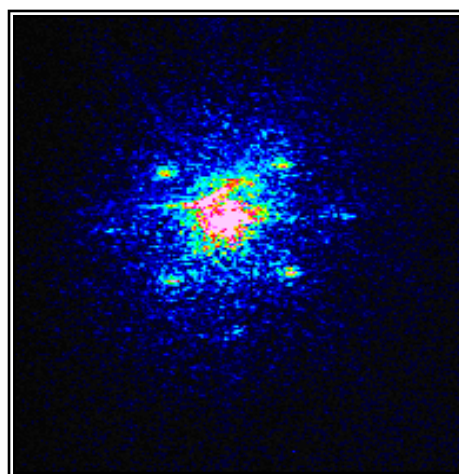


Fig . 20 : Diffraction pattern of triangle aperture as shown by a CCD camera .

5- Conclusions:

We conclude the diffraction pattern which we calculated with fft2 method, the number of pixels in the diffraction pattern is equal to the number of pixels in the initial image, as shown in figures (4,6,8,9,10,11). obviously, the total extent of the image is increased without changing the slit or hole width by adding space around the image, which is called zero padding. In figures (13,14,17,18) these figures taken with spatial filter, recorded by a digital camera, from this figures we notice that the intensity pattern observed varies with the distance from the aperture. In figures (5,16,19,20), recorded by a CCD camera, as seen the diffraction pattern depends on the size of aperture or element size. So, the experimental results agree with the theoretical calculations.

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