IRAQI STATISTICIANS JOURNAL

https://isj.edu.iq/index.php/isj

ISSN: (3007-1658) Online

Statistical Evaluation of the Gompertz-Fréchet distribution: Statistical Elasticity Analysis Using Simulation, Estimation and Application

Ola H. Hammoodi[1], Hiba H. Abdullah[2]

[1,2] Department of Mathematics, College of Education for Women, Tikrit University , 34001 Salahaddin, Iraq

ARTICLE INFO

Article history:

Received 16 December 2024
Revised 23 December 2024
Accepted 24 January 2025
Available online 24 January 2025

Keywords:

Gompertz family
Fréchet distribution
GoFr
WLSE
RMSE

ABSTRACT

This paper deals with the analysis of the Gompertz-Fréchet (GoFr) distribution, which combines the Gompertz family and the Fréchet distribution, where the statistical properties of proposed distribution are studied and accurate estimation methods for the distribution parameters are developed. Monte Carlo simulation was employed to evaluate the performance of three estimation methods: the Maximal likelihood method (MLE), the least squares method (LSE), and weighted least squares method (WLSE). The simulation included different sample sizes ranging from n=50 to 200, and the results were analyzed using criteria such as bias, mean square error (MSE), and root mean square error (RMSE). The results showed that the MLE method is the most accurate and efficient, as the bias and MSE decreased with the increase in sample size. The GoFr distribution was also applied to real data and compared with six other distributions using statistical accuracy criteria such as AIC and BIC. The results confirmed that the GoFr distribution is superior in its fit to the data compared to other models, which reflects its flexibility and effectiveness in analyzing data of complex nature.

1. Introduction

Statistical distributions are mathematical tools used to describe how the values of a particular random variable are distributed. These distributions play a vital role in mathematical statistics, as the contribute to understanding and interpreting the underlying patterns in data. Distributions can be divided into serval categories such as discrete and continuous probability distributions, based on the nature of random variables. The development of statistical distributions began in the eighteen century with the emergence of basic models such as the normal distribution.

This was followed by the emergence of a number of methods for generating statistical distributions and then families of statistical distributions. The most famous of these methods is the T−X method, which is a modern technique used to create new families of probability distributions by combining two or more distributions using specific transformation functions. This method is considered a natural extension of traditional approaches to designing distributions, as it aims to improve the ability to describe and analyze data more accurately. The T−X method provides a flexible way to design distributions that can adapt to data that are asymmetric in

* Corresponding author: E-**mail** address: ola.hammoodi509@st.tu.edu.iq
https://doi.org/10.62933/hgcbp449

nature or contain extreme properties (Outliers). This makes it useful in studying phenomena that are difficult to analyze using traditional distributions [1]. Among these families that have emerged are: MKi-G [2], GME family [3], MT-X [4], MOTL-G family [5], , logarithmic family [6], ITL-H family [7], SHE-G family [8], NOGEE−G family [9], WEE-X family [10], GOM-G family [11], and OL-G family [12], by same method founded the Gompertz family with respective probability mass and density functions defined as follows:

$$G_{Go}(x,\alpha,\beta,\phi)=1-e^{\frac{\alpha}{\beta}\left(1-e^{\beta\frac{D(x,\phi)}{1-D(x,\phi)}}\right)} \quad (1)$$

$$g_{Go}(x,\alpha,\beta,\phi)=\frac{\alpha d(x,\phi)e^{\beta\frac{D(x,\phi)}{1-D(x,\phi)}}}{\left(1-D(x,\phi)\right)^{2}}e^{\frac{\alpha}{\beta}\left(1-e^{\beta\frac{D(x,\phi)}{1-D(x,\phi)}}\right)} \quad (2)$$

Where $d(x,\phi)$, and $D(x,\phi)$ are pdf, and CDF of baseline distribution with ϕ paremeter.

The research gap is that the current literature lacks comprehensive studies to evaluate its performance in practical data analysis.

This research aims to fill this gap by combining theoretical and experimental analysis to improve the understanding of the GoFr distribution and its role in data analysis. The aim of the study is to explore the basic statistical properties of the GoFr distribution, including moments, skewness, and kurtosis, and evaluate the performance of three main estimation methods (MLE, LSE, WLSE) using different sample sizes, as well as compare the fit of the GoFr distribution with other distributions when applied to real data.

The first part included the introduction and the objective of the study, while the second part included finding the proposed distribution, while the third part included some statistical properties of the GoFr distribution. The fourth part included estimating the parameters for three different methods, while the fifth and sixth parts included Monte Carlo simulations to evaluate the performance and practical application, respectively.

2. Gompertz-Fréchet (GoFr) distribution

Using the Fréchet distribution as a baseline for proposed distribution which has a CDF function by form:

$$D(x,\gamma,\delta)=e^{-\left(\frac{\gamma}{x}\right)^{\delta}} \quad (3)$$

and pdf function

$$d(x,\gamma,\delta)=\delta\gamma^{\delta}x^{-(\delta+1)}e^{-\left(\frac{\gamma}{x}\right)^{\delta}} \quad (4)$$

$x,\gamma,\delta>0$ [13], and γ,δ are shape parameters.

The Gompertz-Fréchet (GoFr) distribution CDF function is obtained by combining equation (1) with (3) to obtain the equation in the form:

$$G_{GoFr}(x,\alpha,\beta,\gamma,\delta)=1-e^{\frac{\alpha}{\beta}\left(1-e^{\left(\beta\frac{e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}\right)}\right)},x,\alpha,\beta,\gamma,\delta>0 \quad (5)$$

Figure 1 represent the plot of the CDF function for GoFr distribution for different value of parameters.

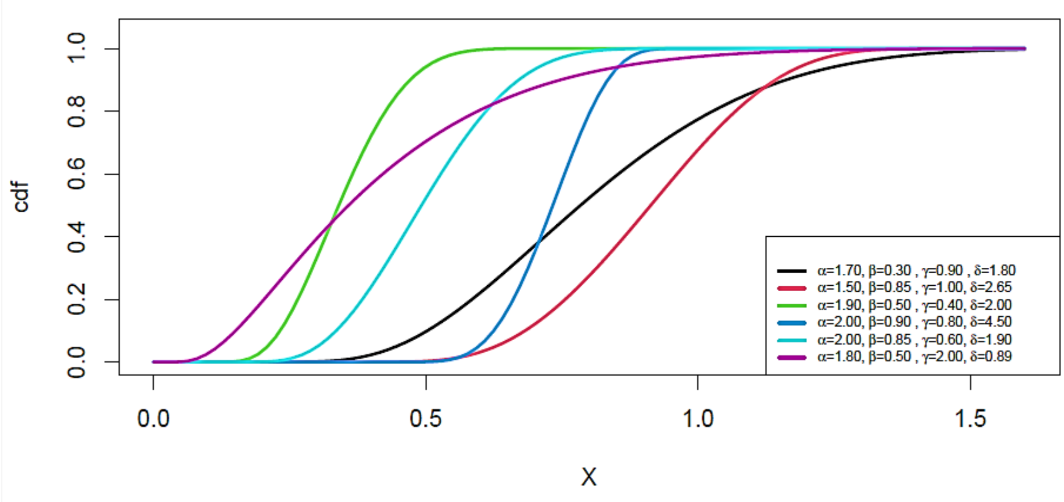


Figure 1. plot CDF function of GoFr distribution

Figure 1represents the change in the CDF of GoFr distribution at different value of parameters. The results show the change in the CDF shape with parameters changes, reflecting the flexibility of distribution in dealing with different data.

The pdf function for GoFr is obtained by combining equations (2) and (3) with equation (4) to form the equation in form:

$$g_{GoFr}(x,\alpha,\beta,\phi)=\frac{\alpha\delta\gamma^{\delta}x^{-(\delta+1)}e^{-\left(\frac{\gamma}{x}\right)^{\delta}}e^{\left(\beta\frac{e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}\right)}}{\left(1-e^{-\left(\frac{\gamma}{x}\right)^{\delta}}\right)^{2}}e^{\frac{\alpha}{\beta}\left(1-e^{\left(\beta\frac{e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}\right)}\right)} \tag{6}$$

As in CDF function, pdf function are also plotted with different values of parameters.

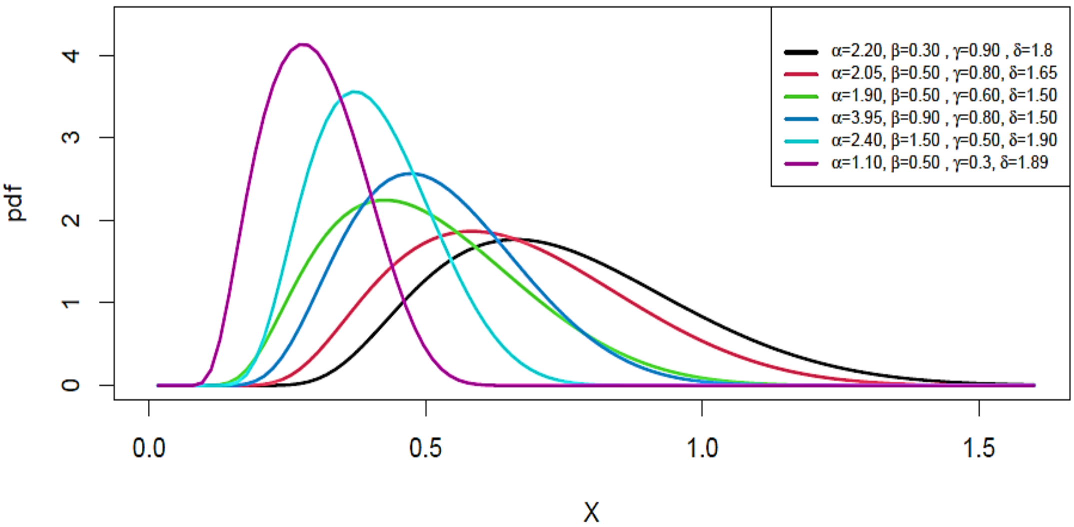


Figure 2. plot pdf function of GoFr distribution

The figure 2 show the pdf of GoFr distribution at multiple values of the parameters. The results reflect the large differences between the probability distribution in different case, indicating that the distribution is consistent with data of a diverse nature.

The Survival function for GoFr distribution can be computed using the formula [14]:

$$S_{GoFr}(x) = e^{\frac{\alpha}{\beta}\left(1-e^{\left(\beta\frac{e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}\right)}\right)} \tag{7}$$

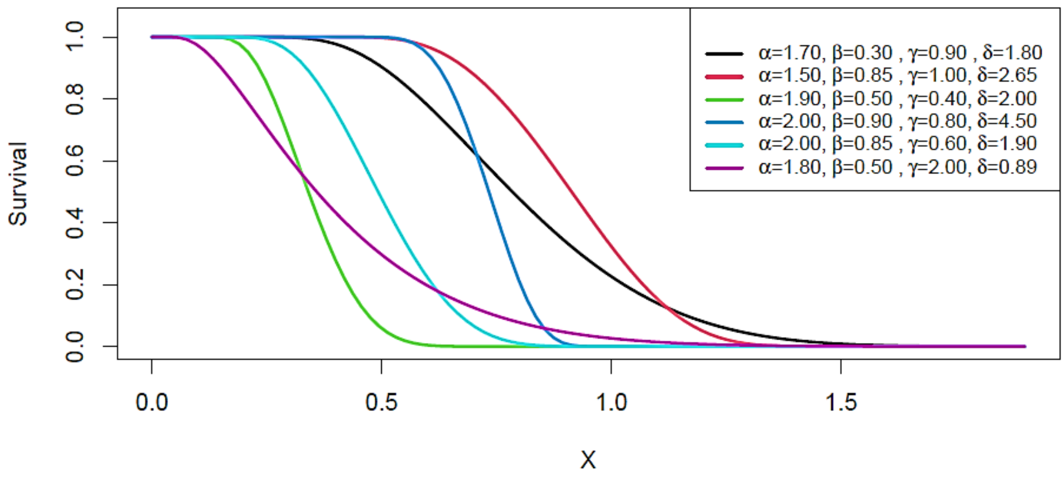


Figure 3. Survival for GoFr distribution with different values of parameters

This figure shows how the survival function changes as the values of the parameters change. The figure confirms the model's ability to represent data associated with both long and short survival phenomena.

The following formula is used to derive the hazard functions for GoFr distribution [14]:

$$h_{GoFr}(x) = \frac{\alpha\delta\gamma^{\delta}x^{-(\delta+1)}e^{-\left(\frac{\gamma}{x}\right)^{\delta}}e^{\left(\beta\frac{e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}\right)}}{\left(1-e^{-\left(\frac{\gamma}{x}\right)^{\delta}}\right)^{2}} \tag{8}$$

While the odd ratio of GoFr distribution given by formula [15]:

$$O_{GoFr}(x) = \frac{1-e^{\frac{\alpha}{\beta}\left(1-e^{\left(\beta\frac{e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}\right)}\right)}}{e^{\frac{\alpha}{\beta}\left(1-e^{\left(\beta\frac{e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x}\right)^{\delta}}}\right)}\right)}} \tag{9}$$

3. Statistical Properties of GoFr distribution

3.1 Useful representations

This section's goal is to expand and improve the pdf and CDF suitability for the GoFr distribution. Thus, the CDF for GoFr distribution may be derived from equation (3) as follows:

$$G_{GoFr}(x) = 1 - \mathrm{K}x_{N}^{-v\delta} \tag{10}$$

Where

$$\mathrm{K} = \sum_{i=j=k=s=v=0}^{\infty}\frac{(-1)^{i+j+k+v}\Gamma(k+s)}{i!k!s!v!\Gamma(k)}\binom{i}{j}\alpha^{i}\beta^{k-v}j^{k}\gamma^{v\delta}(k+s)^{v}$$

Similarly, the G^{λ}_{GoFr} may be obtained in the following form by an equivalent extension of the CDF:

$$G^{\lambda}_{GoFr}(x) = Tx^{-q\delta} \quad (11)$$

Where

$T = \sum_{l=w=u=z=p=q=0}^{\infty} \frac{(-1)^{l+w+u+z+q}\Gamma(z+p)}{w!z!p!q!\Gamma(z)} \binom{\lambda}{l} \binom{w}{u} \alpha^{w} l^{w} \beta^{z-w} u^{z} \gamma^{q\delta} (z+p)^{q}$

Equation (4) yields the pdf using the same technique for extending the CDF, as follows:

$$g_{GoFr}(x) = \Upsilon x^{-\delta(1+w)-1} \quad (12)$$

Where

$\Upsilon = \sum_{i=j=t=v=w=0}^{\infty} \frac{(-1)^{i+j+t+w}\Gamma(t+2+v)}{i!\gamma^{i}t!v!w!\Gamma(t+2)} \binom{i}{j} \alpha^{i+1}\beta^{t} (j+1)^{t}\delta\gamma^{(w+1)\delta}(t+v+1)^{w}$

Similarly, using the previously described expansion procedure, the $g^{\theta}_{GoFr}(x)$ is obtained as follows:

$$g^{\theta}_{GoFr}(x) = \mathrm{E}x^{-\delta(\theta+l)-\theta} \quad (13)$$

Where

$\mathrm{E} = \sum_{k=q=z=\varepsilon=l=0}^{\infty} \frac{(-1)^{k+q+z+\varepsilon+l}}{k!\,l!\beta^{k}(z+\varepsilon+\theta)^{-l}} \binom{k}{q} \alpha^{k+\theta}\theta^{k} (q+\theta)^{z}\delta^{\theta}\gamma^{(\theta+l)\delta}$

3.2 Quintile function

A quintile function is a statistical tool used to describe data and interpret its distribution it can be defined as the inverse of CDF [16]. In other words, if $F(x)$ is the CDF that gives the probability that the value of the random variable X is less than or equal to x, then the quintile function $Q(u)$ gives the value of x associated with a certain probability u. The quintile function of GoFr distribution is given by the equation below:

$$Q_x = Q_u \left[\left\{ \frac{-\gamma^{\delta}}{\ln\left[\frac{ln\left[1-\frac{\beta}{\alpha}ln(1-u)\right]}{\beta+ln\left[1-\frac{\beta}{\alpha}ln(1-u)\right]}\right]} \right\}^{\frac{1}{\delta}} \right] \quad (14)$$

The table below illustrates the values of the quintile function for different parameter.

Table 1: Explanation of the Quantile function for particular parameter values of the GoFr distribution

	(ω,δ,β,a)				
u	**(1.3,0.2,0.8,0.5)**	**(0.1,0.4,1.2,0.6)**	**(0.4,0.5,0.7,0.9)**	**(0.5,0.8,1.2,0.2)**	**(0.8,0.4,0.2,0.3)**
0.1	**0.1185164**	**1.897602**	**0.4102002**	**0.06109438**	**0.01501964**
0.2	**0.2136199**	**3.984172**	**0.6237769**	**0.34040431**	**0.04497292**
0.3	**0.3300605**	**6.225698**	**0.8309047**	**1.08949824**	**0.09904995**
0.4	**0.4785219**	**8.586882**	**1.0414116**	**2.70741440**	**0.19050850**
0.5	**0.6735323**	**11.091948**	**1.2616929**	**5.84674144**	**0.34115685**
0.6	**0.9392640**	**13.809421**	**1.4993268**	**11.65769987**	**0.58969864**
0.7	**1.3219798**	**16.873507**	**1.7664693**	**22.42631232**	**1.01353593**
0.8	**1.9290337**	**20.572532**	**2.0879481**	**43.66284355**	**1.79973650**
0.9	**3.1179288**	**25.755615**	**2.5354384**	**94.66707668**	**3.60118168**

This table shows the quantile function values o f the GoFr distribution at a set of parameters an d at different values.

- Variation between parameters: At small values of the quantile, the effect of a an d b increases. At large values, the effect of c and d appears.
- Nonlinear behavior: It is clear from the values that there are significant variatio ns, reflecting the nonlinear nature of the distribution.
- Comparison between models: Different parameters greatly affect the spread of t he distribution and its concentrations at different values.

The quantile function shows the abilit y of NGoFr to model different data through sim ple modification of the parameter.

3.3 Moments

As shown in equation (12), let x be a random variable described by the pdf function. equation (12) may be used to represent the $m-th$ moment of GoFr distribution as follows [16]:

$$\mu'_{m_N} = \Upsilon \int_0^\infty x^{-\delta(1+w)+m-1} dx$$

$$\mu'_m = \frac{\Upsilon}{m-\delta(1+w)} \tag{15}$$

Then the μ'_1, μ'_2, μ'_3, and μ'_4 has forms:

$$\mu'_1 = \frac{\Upsilon}{1-\delta(1+w)} \tag{16}$$

$$\mu'_2 = \frac{\Upsilon}{2-\delta(1+w)} \tag{17}$$

$$\mu'_3 = \frac{\Upsilon}{3-\delta(1+w)} \tag{18}$$

$$\mu'_4 = \frac{\Upsilon}{4-\delta(1+w)} \tag{19}$$

The formula $\sigma^2 = \mu'_2 - {\mu'_1}^2$ [17], used to calculate the variance for the GoFr distribution, production the following expression:

$$\sigma_N^2 = \frac{\Upsilon}{2-\delta(1+w)} - \left[\frac{\Upsilon}{1-\delta(1+w)}\right]^2 \tag{20}$$

The following mathematical formulas be used to calculate skewness (S) and kurtosis (K) of GoFr distribution [18]:

$$S = \frac{\frac{\Upsilon}{3-\delta(1+w)}}{\left(\frac{\Upsilon}{2-\delta(1+w)}\right)^{\frac{3}{2}}} \tag{21}$$

$$K = \frac{\frac{\Upsilon}{3-\delta(1+w)}}{\left(\frac{\Upsilon}{2-\delta(1+w)}\right)^{2}} - 3 \tag{22}$$

Table 2 presents a range of different values for the moment, variance, skewness, and kurtosis.

Table 2. Some Moments value of GoFr distribution

a	b	c	d	μ_1	μ_2	μ_3	μ_4	var	S	K
0.6	**0.4**	**1.2**	**0.1**	**88.04237**	**34721.02**	**582953**	**4829672**	**26969.56**	**0.090104**	**0.004006**
			0.2	**82.82036**	**73716.1**	**1.64E+08**	**6.39E+11**	**66856.89**	**8.216348**	**117.5012**
		1.4	**0.3**	**15.31193**	**1060.305**	**140470.1**	**27635104**	**825.8498**	**4.068531**	**24.581**
			0.4	**6.83686**	**135.2733**	**4417.629**	**196658.2**	**88.53065**	**2.807832**	**10.74702**
	0.6	**1.6**	**0.5**	**4.079129**	**32.86546**	**375.97**	**5372.086**	**16.22617**	**1.995463**	**4.973516**
			0.6	**3.280982**	**18.2613**	**135.9802**	**1222.315**	**7.496457**	**1.742521**	**3.665386**
		1.8	**0.7**	**3.198191**	**15.60747**	**97.46464**	**716.2559**	**5.379044**	**1.580696**	**2.940378**
			0.8	**2.893477**	**11.82927**	**59.77864**	**348.5024**	**3.457061**	**1.469295**	**2.490519**

Table 2 shows the instant values of th e first four statistical moments, and properties s uch as variance, skewness, and kurtosis. The ta ble shows the effect of changing the basic para meters on these properties.

- Statistical moments: The first moment (μ_1) represents the mean, and increases o r decreases according to the value of the parameter c. The higher moments (μ_2, μ_3, μ_4) indicate the dispersion of the values, their concentration, and their co nsistency with the different parameters.
- Variance: The variance values increase with increasing values of c and d, indica ting that the data becomes more dispers ed under these conditions.
- Skewness: Low positive skewness indic ates that the data is not far from symmet ry.

- Kurtosis: Low kurtosis values (close to 3) indicate that the distribution is close to the normal distribution.

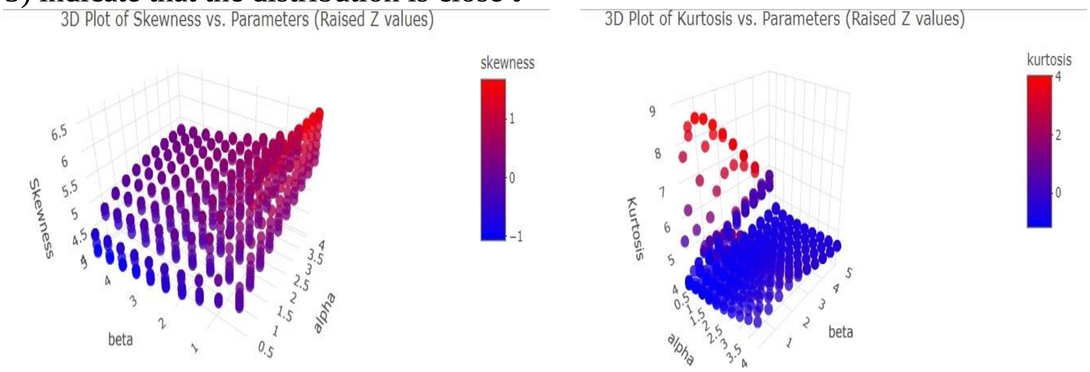


Figure 4. 3D plot of Skewness, and Kurtosis

Figure 4 shows a 3D relationship between skewness and kurtosis of a distribution. Low positive skewness indicates that the data is close to symmetry, while low kurtosis indicates that the data is close to normal distribution. Therefore, the variance will be increasing with specific parameters such as γ, and δ, indicating increased dispersion.

3.4 Moment Generating Function

The moment-generating function (MGF) for the GoFr distribution can be derived from Equation (12), as given by [18]:

$$M_x'(y)_{GoFr} = \sum_{m=0}^{\infty} \frac{y^m}{m!}\left[\frac{\Upsilon}{m-\delta(1+w)}\right] \tag{23}$$

3.5 Incomplete Moments

The following formula is used to calculate a random variable X incomplete moments [19-20]:

$$\grave{\mu}_r(y) = \int_0^y x^r g_{GoFr}(x)\, dx$$

The $g_{GoFr}(x)$ is generated by replacing the GoFr distribution from equation (12) into the previous equation:

$$\grave{\mu}_r(y) = \Upsilon \int_0^y x^r x^{-\delta(1+w)-1} dx$$

Next, we have:

$$\grave{\mu}_r(y) = \frac{\Upsilon y^{r-\delta(1+w)}}{r-\delta(1+w)} \tag{24}$$

If $r-\delta(1+w) \neq 0$, the integral may be solved as follows, if $r-\delta(1+w) = 0$ the integral can be simplified to:

$$\grave{\mu}_r(y) = \ln(y) \tag{25}$$

3.6 Rényi Entropy

The following formula can be used to get the Rényi entropy for the GoFr distribution [20]:

$$I_R(\theta)_{GoFr} = \frac{1}{1-\theta}\log\left[\mathrm{E}\int_0^{\infty} x^{-\delta(\theta+l)-\theta} dx\right]$$

The GoFr Rényi entropy is then shown in the following form:

$$I_R(\theta)_{GoFr} = \frac{1}{1-\theta}\log\left[\frac{-\mathrm{E}}{\delta(\theta+l)+\theta}\right] \tag{26}$$

4 Estimation

4.1 Maximum Likelihood Estimation

The maximal likelihood estimation (MLE) approach used to estimate the parameters of the GoFr distribution. The log-likelihood function is calculated for a random sample of data point $x_1, x_2, \ldots, x_n$ in order to do this. For the GoFr distribution, the pdf is constant throughout these data point [21-22].

$$L(\vartheta, x_i) = \prod_{i=1}^{n} \frac{\alpha\delta\gamma^{\delta} x_i^{-(\delta+1)} e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}} e^{\left(\beta \frac{e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}\right)}}{\left(1 - e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}\right)^2} e^{\frac{\alpha}{\beta}\left(1-e^{\left(\beta \frac{e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}\right)}\right)}$$

The log-likelihood function L, which can be written as follows, is then modified to include the distribution's parameters, represented by ϑ:

$$L = nlog(\alpha) + nlog(\delta) + n\delta log(\gamma) - (\delta + 1)\sum_{i=1}^{n} \log x$$

$$-2\sum_{i=1}^{n} \log\left[1 - e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}\right] + \beta \sum_{i=1}^{n} \frac{e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}{1 - e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}} + \frac{\alpha}{\beta}\sum_{i=1}^{m}\left(1 - e^{\left(\beta \frac{e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}\right)}\right) \quad (27)$$

4.2 Least square estimation

The least squares estimation (LSE) approach, which applies the following equation, can also be used for the parameter estimation [23]:

$$\varphi(\vartheta) = \sum_{i=1}^{n}\left[G_{GoFr}(x) - \frac{i}{n+1}\right]^2$$

$$\varphi(\vartheta) = \sum_{i=1}^{n}\left[1 - e^{\frac{\alpha}{\beta}\left(1-e^{\left(\beta \frac{e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}\right)}\right)} - \frac{i}{n+1}\right]^2 \quad (28)$$

4.3 Weighted Least square estimation

The following formula is used to produce the weighted least squares estimators [24]:

$$\mathrm{W}\,(\vartheta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[G_{GoFr}(x_i) - \frac{i}{n+1}\right]^2$$

[*] Corresponding author: E-**mail** address: ola.hammoodi509@st.tu.edu.iq
https://doi.org/10.62933/hgcbp449

$$W(\vartheta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-e^{\frac{\alpha}{\beta}\left(1-e^{\left(\beta\frac{e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}{1-e^{-\left(\frac{\gamma}{x_i}\right)^{\delta}}}\right)}\right)}-\frac{i}{n+1}\right]^2 \qquad (29)$$

The functions shown in equations (27), (28) and (29) are partially derived with respect to the model parameters and then set equal to zero to obtain estimates for the for model parameters. Due to the difficulty of above equation, computer programs are actually used for this purpose, such as the R programming language or others.

5 Simulation

Monte Carlo simulation is statistical technique that relies on generating a large number of random samples to simulate and estimate the performance of a mathematical system or model. This method is widely used in statistics, finance, engineering, and physics to study system that are too complex to solve analytically. The basic steps of Monte Carlo simulation are [25]:

- Identify the system or model.
- Generate random samples.
- Implement the model.
- Analyze the results.

In this study, we used Monte Carlo simulation to evaluate the performance of three parameter estimation methods (MLE, LSE, and WLSE) for the GoFr distribution using different sample sizes, ranging from n=50 to n=200. The results are analyzed based on mean values, bias, mean squared error (MSE) [25-26], and root mean squared error (RMSE) [27], for value of parameters:

- $\alpha=0.6,\ \beta=0.5,\ \gamma=0.9,\ \delta=0.7$
- $\alpha=0.4,\ \beta=0.8,\ \gamma=1,\ \delta=0.6$

Table 3 : Monte Carlo simulations conducted for the NGoFr distribution

$\boldsymbol{\alpha=0.6}$, $\boldsymbol{\beta=0.5}$, $\boldsymbol{\gamma=0.9}$, $\boldsymbol{\delta=0.7}$					
n	**Est.**	**Est. par.**	**MLE**	**LSE**	**WLSE**
٥٠	MEAN	$\hat{\alpha}$	0.4668235	0.605984358	0.596004693
		$\hat{\beta}$	0.2236034	0.3701572	0.3363766
		$\hat{\gamma}$	0.6919922	0.7990268	0.81336277
		$\hat{\delta}$	0.87149492	0.73070053	0.74533545
	MSE	$\hat{\alpha}$	0.5712059	0.787634939	1.092857333
		$\hat{\beta}$	0.1984880	0.1765709	0.1196291
		$\hat{\gamma}$	0.8474612	0.7758614	1.70508498
		$\hat{\delta}$	0.04991637	0.02807967	0.02091436
	RMSE	$\hat{\alpha}$	0.7557817	0.88748799	1.04539817
		$\hat{\beta}$	0.4455199	0.4202034	0.3458744
		$\hat{\gamma}$	0.9205766	0.8808300	1.30578903
		$\hat{\delta}$	0.22341973	0.16756990	0.14461797
	BAIS	$\hat{\alpha}$	0.1331765	0.00598435	0.00399530
		$\hat{\beta}$	0.2763966	0.1298428	0.1636234
		$\hat{\gamma}$	0.2080078	0.1009732	0.08663723
		$\hat{\delta}$	0.17149492	0.03070053	0.04533545
١٠٠	MEAN	$\hat{\alpha}$	0.4406333	0.51996186	0.4964701
		$\hat{\beta}$	0.2743074	0.37123751	0.3360059
		$\hat{\gamma}$	0.6902034	0.7577956	0.7303712
		$\hat{\delta}$	0.84900845	0.74257052	0.75959288
	MSE	$\hat{\alpha}$	0.2256672	0.47369178	0.1521556
		$\hat{\beta}$	0.1782398	0.09643233	0.1289889
		$\hat{\gamma}$	0.3998772	0.9262214	0.3031071
		$\hat{\delta}$	0.04025793	0.01908300	0.01237564
	RMSE	$\hat{\alpha}$	0.4750444	0.68825270	0.3900712
		$\hat{\beta}$	0.4221845	0.31053555	0.3591502
		$\hat{\gamma}$	0.6323584	0.9624040	0.5505516
		$\hat{\delta}$	0.20064379	0.13814123	0.11124586
	BAIS	$\hat{\alpha}$	0.1593667	0.08003814	0.1035299
		$\hat{\beta}$	0.2256926	0.12876249	0.1639941
		$\hat{\gamma}$	0.2097966	0.1422044	0.1696288
		$\hat{\delta}$	0.14900845	0.04257052	0.05959288
١٥٠	MEAN	$\hat{\alpha}$	0.4277493	0.4837080	0.4588061
		$\hat{\beta}$	0.2846431	0.37464959	0.3341065
		$\hat{\gamma}$	0.6816965	0.7240664	0.7140231

		$\hat{\delta}$	0.84245038	0.75110397	0.77500241
	MSE	$\hat{\alpha}$	0.2085744	0.1342748	0.1096349
		$\hat{\beta}$	0.1751099	0.08981887	0.1129413
		$\hat{\gamma}$	0.3750142	0.2111017	0.4569372
		$\hat{\delta}$	0.03725755	0.01366701	0.01466980
	RMSE	$\hat{\alpha}$	0.4566995	0.3664353	0.3311117
		$\hat{\beta}$	0.4184614	0.29969796	0.3360673
		$\hat{\gamma}$	0.6123840	0.4594580	0.6759713
		$\hat{\delta}$	0.19302215	0.11690600	0.12111896
	BAIS	$\hat{\alpha}$	0.1722507	0.1162920	0.1411939
		$\hat{\beta}$	0.2153569	0.12535041	0.1658935
		$\hat{\gamma}$	0.2183035	0.1759336	0.1859769
		$\hat{\delta}$	0.14245038	0.05110397	0.07500241
٢٠٠	MEAN	$\hat{\alpha}$	0.4266480	0.4699361	0.4570897
		$\hat{\beta}$	0.2920214	0.35540910	0.32687910
		$\hat{\gamma}$	0.6912495	0.7062622	0.7032891
		$\hat{\delta}$	0.83469707	0.75110506	0.76893322
	MSE	$\hat{\alpha}$	0.1674008	0.1756773	0.1713289
		$\hat{\beta}$	0.2098486	0.06148699	0.08470531
		$\hat{\gamma}$	0.4901291	0.3291574	0.4676291
		$\hat{\delta}$	0.03415035	0.01197046	0.01129350
	RMSE	$\hat{\alpha}$	0.4091464	0.4191388	0.4139190
		$\hat{\beta}$	0.4580924	0.24796571	0.29104176
		$\hat{\gamma}$	0.7000922	0.5737224	0.6838341
		$\hat{\delta}$	0.18479813	0.10940962	0.10627088
	BAIS	$\hat{\alpha}$	0.1733520	0.1300639	0.1429103
		$\hat{\beta}$	0.2079786	0.14459090	0.17312090
		$\hat{\gamma}$	0.2087505	0.1937378	0.1967109
		$\hat{\delta}$	0.13469707	0.05110506	0.06893322

$\alpha = 0.4, \quad \beta = 0.8, \quad \gamma = 1, \quad \delta = 0.6$

n	Est.	Est. par.	MLE	LSE	WLSE
٥٠	MEAN	$\hat{\alpha}$	0.32253009	0.34502396	0.2944900
		$\hat{\beta}$	0.3237421	0.4937618	0.5106748
		$\hat{\gamma}$	0.7568672	0.7660055	0.6795732
		$\hat{\delta}$	0.81085336	0.68394968	0.69538747
	MSE	$\hat{\alpha}$	0.64541234	0.23488964	0.1034086
		$\hat{\beta}$	0.4700053	0.2438520	0.3466985
		$\hat{\gamma}$	2.0616894	1.0549593	0.4405287
		$\hat{\delta}$	0.07053411	0.05016262	0.05261244
	RMSE	$\hat{\alpha}$	0.80337559	0.48465414	0.3215721
		$\hat{\beta}$	0.6855693	0.4938137	0.5888111
		$\hat{\gamma}$	1.4358584	1.0271121	0.6637234
		$\hat{\delta}$	0.26558258	0.22397013	0.22937401
	AI	$\hat{\alpha}$	0.07746991	0.05497604	0.1055100
		$\hat{\beta}$	0.4762579	0.3062382	0.2893252
		$\hat{\gamma}$	0.2431328	0.2339945	0.3204268
		$\hat{\delta}$	0.21085336	0.08394968	0.09538747
١٠٠	MEAN	$\hat{\alpha}$	0.25832569	0.27768611	0.28346250
		$\hat{\beta}$	0.3865865	0.4966368	0.5272982
		$\hat{\gamma}$	0.6635398	0.6439571	0.6939616
		$\hat{\delta}$	0.76685122	0.66886881	0.68435487
	MSE	$\hat{\alpha}$	0.05523625	0.04873272	0.03340065
		$\hat{\beta}$	0.3985013	0.1983337	0.2706856
		$\hat{\gamma}$	0.3852853	0.2731130	0.2222290
		$\hat{\delta}$	0.04575629	0.02704583	0.02865307
	RMSE	$\hat{\alpha}$	0.23502394	0.22075490	0.18275844
		$\hat{\beta}$	0.6312696	0.4453467	0.5202745
		$\hat{\gamma}$	0.6207135	0.5226021	0.4714117
		$\hat{\delta}$	0.21390720	0.16445617	0.16927219
	BAIS	$\hat{\alpha}$	0.14167431	0.12231389	0.11653750
		$\hat{\beta}$	0.4134135	0.3033632	0.2727018
		$\hat{\gamma}$	0.3364602	0.3560429	0.3060384
		$\hat{\delta}$	0.16685122	0.06886881	0.08435487
١٥٠	MEAN	$\hat{\alpha}$	0.27914309	0.27962543	0.29620665
		$\hat{\beta}$	0.4210713	0.4595168	0.5245053
		$\hat{\gamma}$	0.7239636	0.6564439	0.7251003
		$\hat{\delta}$	0.75227004	0.68415462	0.68191261
	MSE	$\hat{\alpha}$	0.06635575	0.03110097	0.07107528
		$\hat{\beta}$	0.4238516	0.1860909	0.2580081
		$\hat{\gamma}$	0.7234035	0.1989818	0.4567322
		$\hat{\delta}$	0.04026867	0.02140481	0.02123047
	RMSE	$\hat{\alpha}$	0.25759610	0.17635468	0.26659947
		$\hat{\beta}$	0.6510389	0.4313825	0.5079450
		$\hat{\gamma}$	0.8505313	0.4460738	0.6758197
		$\hat{\delta}$	0.20067056	0.14630382	0.14570679
	BAIS	$\hat{\alpha}$	0.12085691	0.12037457	0.10379335
		$\hat{\beta}$	0.3789287	0.3404832	0.2754947
		$\hat{\gamma}$	0.2760364	0.3435561	0.2748997
		$\hat{\delta}$	0.15227004	0.08415462	0.08191261
٢٠٠	MEAN	$\hat{\alpha}$	0.27018472	0.28495225	0.27801178
		$\hat{\beta}$	0.4074922	0.4682458	0.4659608
		$\hat{\gamma}$	0.6854168	0.6825083	0.6761051
		$\hat{\delta}$	0.74034861	0.68868269	0.69545327
	MSE	$\hat{\alpha}$	0.04421267	0.05285078	0.03488754
		$\hat{\beta}$	0.3280637	0.1896101	0.2068356
		$\hat{\gamma}$	0.3615284	0.3428117	0.2133993
		$\hat{\delta}$	0.03201499	0.02265666	0.02155232
	MS	$\hat{\alpha}$	0.21026808	0.22989298	0.18678206

		$\hat{\beta}$	0.5727685	0.4354424	0.4547918
		$\hat{\gamma}$	0.6012724	0.5855012	0.4619516
		$\hat{\delta}$	0.17892734	0.15052129	0.14680708
	BAIS	$\hat{\alpha}$	0.12981528	0.11504775	0.12198822
		$\hat{\beta}$	0.3925078	0.3317542	0.3340392
		$\hat{\gamma}$	0.3145832	0.3174917	0.3238949
		$\hat{\delta}$	0.14034861	0.08868269	0.09545327

The table shows the results of Monte Carlo simulations using different estimation techniques on different sample sizes.

- MLE often provides the least bias and highest accuracy compared to LSE and WLSE.
- As the sample size increases (from 50 to 200), the mean square error (MSE) and bias decrease, indicating increasing accuracy.
- RMSE reflects that the estimates improve with larger samples, making them more stable and reliable.

MLE is the most accurate technique, and the simulation results support its use when analyzing data associated with the GoFr distribution.

6 Application on Real Data

To demonstrate the effectiveness of the GoFr distribution, it is applied to real data and to demonstrate its suitability in practical application. For this purpose, it is applied to data representing 100 observations on the fracture stress of carbon fibers (in GB) [28], the values shown below are values of some of the metrics for the data set.

Var	N	Mean	SD	Median	Trimmed	Mad	Min	Max	Range	SK	KU	Se
1	100	2.62	1.01	2.7	2.58	0.99	0.39	5.56	5.17	0.36	0.04	0.1

Where it is compared with six other distributions, which are:

i. Beta Fréchet (BeFr) distribution (new)
ii. Kumaraswamy Fréchet (KuFr) distribution (new)
iii. Exponential generalized Fréchet (EGFr) distribution (new)
iv. Odd Lomax Fréchet (LoFr) distribution (new)
v. [0,1] truncated Nadarajah Haghighi Fréchet ([0,1]NHFr) distribution (new)
vi. Fréchet (Fr) distribution (new)

For this comparison, four accuracy criteria are used: AIC, BIC, CAIC, HQIC [29]. In addition to six four statistical measures, which are: Cramér-von Mises (W), Anderson- Darling (A), Kolmogorov-Smirnov (KS), and p-value [15]

Table 4 represents the comparision values of the statistical distributions using the accuracy criteria, table 5 represents the comparision values of the statistical measures, and table 6 represents the values of the estimated for each distribution using MLE method.

Table 4: Estimates of models for data

Dist.	-2L	AIC	CAIC	BIC	HQIC
GoFr	142.2724	292.5448	292.9659	302.9655	296.7623
BeFr	143.8644	295.7288	296.1499	306.1495	299.9463
KuFr	143.6921	295.3842	295.8053	305.8049	299.6017
EGFr	143.8444	295.6889	296.11	306.1096	299.9063
LoFr	142.7129	293.4259	293.8469	303.8466	297.6433
[0,1]NHFr	145.6973	299.479	299.9001	309.8997	303.6965
Fr	173.144	350.2879	350.4116	355.4982	352.3966

The table provides a comparison between several distributions using different metrics

- GoFr values are lower compared to other models, indicating that it is the best fit for modeling the data.
- In terms of variances between distributions, the LoFr distribution is ranked second in terms of accuracy performance, while the Fr distribution shows poor performance based on these criteria.

GoFr distribution is the best in terms of efficiency and accurate modeling of the data.

Table 5: Evaluate statistical metrics for the data

Dist.	W	A	K-S	p-value
GoFr	0.08369044	0.5325165	0.0674418	0.75329
BeFr	0.1394041	0.7386192	0.09756772	0.297012
KuFr	0.1534269	0.7911772	0.09921322	0.2785612
EGFr	0.1427558	0.7519503	0.09660952	0.3081466
LoFr	0.1244131	0.6405652	0.08894331	0.4074871
[0,1]NHFr	0.1865064	0.988263	0.1147742	0.1434392
Fr	0.7565907	4.312066	0.1777059	0.003614754

The table shows the results of statistical tests to evaluate the fit of the models to the data. From this we can see that the distribution GoFr shows high p values, which means that it fits the data well. The K-S test reflects that GoFr outperforms the data representation compared to other distributions. Thus, NGoFr provides the best fit to the data compared to other models.

Table 6: parameter estimators by MLE for the data

Dist.	$\boldsymbol{\theta}$	$\boldsymbol{\beta}$	$\boldsymbol{\eta}$	$\boldsymbol{\kappa}$
GoFr	12.0674158	13.1825678	17.9332335	0.5954393
BeFr	0.4092416	60.2554834	21.3743239	0.8268424
KuFr	4.2197503	146.286275	3.9320372	0.5075761
EGFr	64.2719230	0.4989272	21.2486654	0.7837266
LoFr	0.05485881	37.6303695	51.85882873	0.40832281
[0,1]NHFr	3.8905939	1.7193904	6.6117464	0.8991111
Fr	---	---	1.891567	1.769019

Displays the estimates of the basic parameters of each model using MLE. GoFr shows accurate and consistent parameters, reflecting the stability and flexibility in data estimation.

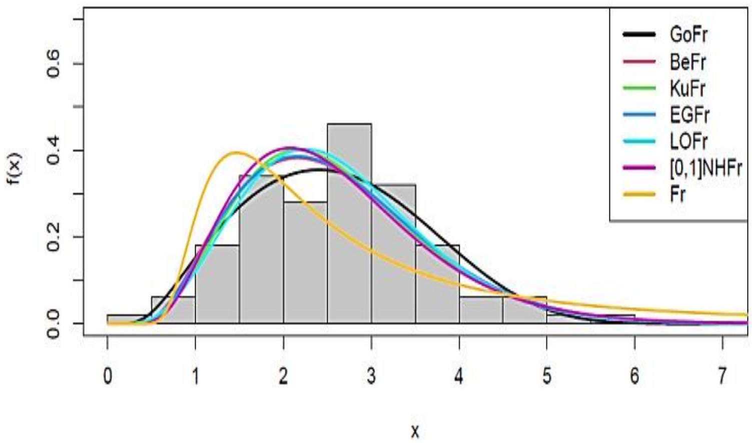
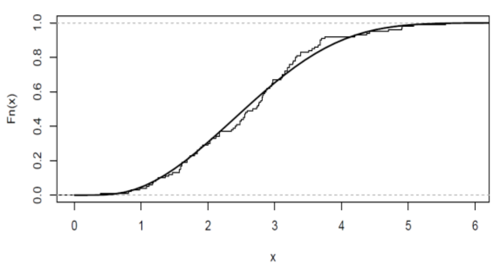
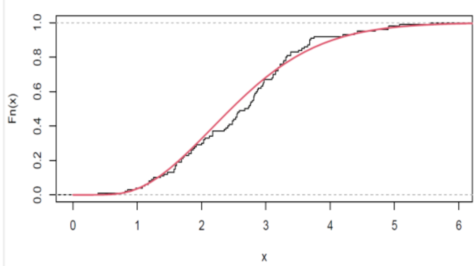


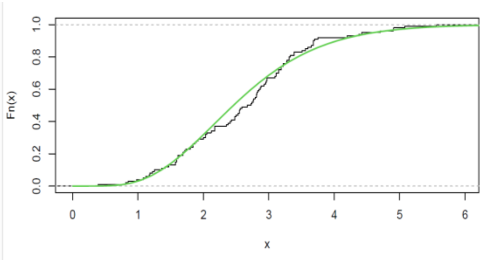
Figure 5: Fitted densities for Data



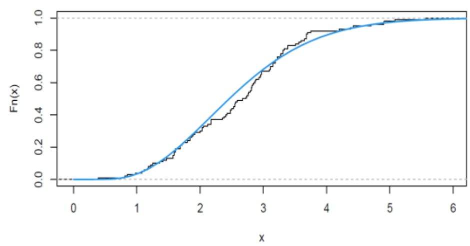
Empirical CDF for GoFr



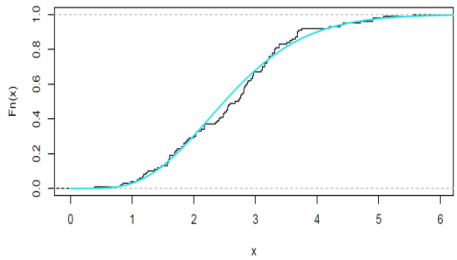
Empirical CDF for BeFr



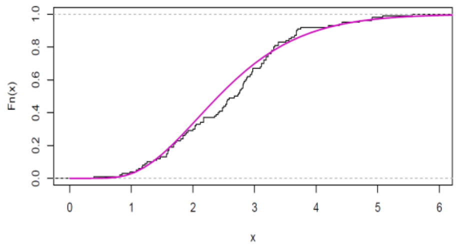
Empirical CDF for KuFr



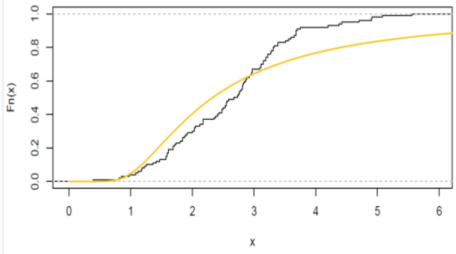
Empirical CDF for EGFr



Empirical CDF for LoFr



Empirical CDF for [0,1]NHFr



Empirical CDF for Fr

Figure6. Empirical CDF for Data

The figure 5 shows the distribution of experimental densities versus the different models. GoFr shows a better match with the experimental density. Some models such as Fr and BeFr show large discrepancies. Figure 6 shows the agreement of the cumulative distribution function of the different models with the experimental data. Again, GoFr shows a good agreement, confirming the results from the tables.

Conclusion

The GoFr distribution has proven to be highly flexible in dealing with diverse data, showing superiority over competing distributions in terms of statistical accuracy criteria (AIC, BIC). As for the simulation, it has been proven that the maximum likelihood (MLE) method was the most accurate and efficient, especially with large sample sizes, as it reduced bias and square error, while the LSE and WLSE methods provided acceptable performance but were less efficient compared to MLE. It was also noted that increasing the sample size leads to improving the accuracy of the estimation and reducing the error. Large sample sizes (n=200) gave more stable results. The GoFr distribution showed a high agreement with real data compared to six other distributions, making it a suitable choice for analyzing complex phenomena. The GoFr distribution can be used in applications that require high flexibility and accuracy in describing data, such as engineering, finance, and data science. Its parameter estimation methods can also be improved using more advanced analysis techniques.

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