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ARTICLE

An Enhanced Newton-Kantorovich Technique for Nonlinear Problem-solvers to Optimal Control Convergence

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Abstract

The aforementioned study aims to tackle the shortcomings of current methods to solve Banach space-valued problems in nonlinear functional modeling using the Newton-Kantorovich technique. By increasing the convergent area, delivering smaller error limitations, supplying weaker adequate semi-local convergence standards, and supplying accurate data on the solution's position, the investigations seek for improvements on previous findings.

The aim of this research is optimal control issues, and its proposed method seeks to accomplish these gains not the need for more requirements. Additionally, the investigation indicates that additional iterative techniques can be extended along the same path using this strategy.

Numeric tests that explain the hypotheses are included in the investigation to our bolster claims and show the efficacy of the suggested approach.

Keywords: Newton-Kantorovich technique, Better convergence, Semi-local convergence, Iterative methods

1. Introduction

Numerous applications from scientific disciplines can be transformed into the solution of the nonlinear formula

$$H(x) = 0, \tag{1}$$

Where M_1, M_2 are Banach spaces, operators $H \in C^1(\Omega, B_2), \Omega \subset M_1$, is an open set [1]. Although it is ideal to obtain the closed form result $x^* \in \Omega$ of the formula $H(x) = 0$, this objective is rarely accomplished. This explains why the majority of methods for solving the formula $H(x) = 0$ involve iteration. These methods produce a sequence that converges to x^* [2–8]. Newton's (NT), as given by, is the oldest and most commonly utilized approach.

$$x_0 \in \Omega, x_{m+1} = x_m - H'(x_{m+1})^{-1} H(x_{m+1}), \forall m = 0, 1, 2, \cdots. \tag{2}$$

A number of experts and practitioners have provided a convergence analysis for NT in the Banach region, starting from the so-called Newton-Kantorovich finding [1]. The semi local conditions for convergent (refer to Ref. [9]) can be easily broken, even while NT may be converging to x^* (also see the numerical studies). This standard has mainly been used by other researchers, based on Argyros et al. [10], Ezquerro Fernández and Hernández [11], Ezquerro and Hernez [12], Liesen [13], Hamada et al. [14], Ferreira and Silva [15], Santos et al. [16] and Regmi et al. [17], Doikov et al. [18] and Moradikashkooli et al. [19] as well as the publications listed within. The Newton-Kantorovich technique emphasizes the iterative strategy to problem solving by extending the Newton approach for solving nonlinear problems in Banach spaces. A key component of this approach is the Kantorovich essentially, which describes the requirements for the iteration process' convergence. It contains conditions on the inverse of the derivative's occurrence and continuation, limitations on both the inverted and its derivative in general, and a particular rate of convergence. Together, these components guarantee

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the approach's applicability and efficacy in locating solutions under specified circumstances [20].

Although the Kantorovich standard is less stringent in this paper, no additional requirements are used. Additionally, the data on the uniqueness sphere and the error distances of $\| \mathscr{o}^* - \mathscr{o}_{m+1} \|$ are enhanced. This is because factors that are at least as modest as those used in the previously described research are being used [21]. We expanded upon the latest findings in Ref. [19], thereby streamlining previous demonstrations. Our method consists on identifying an area Ω_0 of Ω where the repeats $\mathscr{o}_{m+1}$ are additionally present. The Lipschitz constant concerned, nevertheless, in Ω_0 is at least exactly the same as the equivalent ones based on Ω. With this adjustment, we are able to offer a more detailed semi local convergent investigation for NT. The introduction included an assessment of the [22]. The NT semi-local converging is covered in Part 2. The third part contains the instances. The fourth part has concluding observations [23].

2. Investigation of convergence

The open ball with radius $\Im > 0$ including its middle at a location $\mathscr{o}_0 \in \Omega$ are denoted by the formula $U(\mathscr{o}_0, \Im)$. In addition, the open ball's closing is represented by the ball $U[\mathscr{o}_0, \Im]$. Moreover, the area of linear continuous operations is denoted by $L(H_1, H_2)$.

Definition 2.1. A parameter $\omega_0 > 0$ is considered to be center-Lipschitz continuously on Ω for an operator Φ that belongs to $C^1(\Omega, H_2)$, if it satisfies the subsequent inequality:

$$\| \Phi'(\mathscr{o}_0+1)^{-1}(\Phi'(\kappa) - \Phi'(\mathscr{o}_0+1)) \| \le \omega_0 \| \kappa - (\mathscr{o}_0+1) \|, \forall \kappa \in \Omega \tag{3}$$

Following definitions apply to the notes utilized in the above definition:

Φ: The operator under discussion is a continuous center-Lipschitz.

Ω: Subset representing an open set in the Banach space H_1.

H_2: Banach space.

$\mathscr{o}_0$: A Ω point that acts as a benchmark.

$\Phi'(\mathscr{o}_0+1)^{-1}$: Reverse for the derivative of H at $\mathscr{o}_0 + 1$. This is an illustration of the linear estimate of Φ near $\mathscr{o}_0 + 1$.

κ: Another point in Ω.

$U\left(\mathscr{o}_1 + 1, \frac{1}{\omega_0} + 1\right)$: A sphere that is open and situated at $\mathscr{o}_1 + 1$ with radius $\frac{1}{\omega_0} + 1$.

Ω_0: The point where the open ball intersects $U\left(\mathscr{o}_1 + 1, \frac{1}{\omega_0} + 1\right)$ and Ω.

This definition states that a center-Lipschitz continuity operator H fulfills a Lipschitz-like criterion in which the parameter's value and the difference of the instrument's derivatives at two separate places determine the constant of Lipschitz ω_0. Since the Lipschitz criterion is centered at the location $\mathscr{o}_0 + 1$, the operator Φ is referred to as center-Lipschitz continuously.

Moreover, the intersection of Ω and the open ball $U\left(\mathscr{o}_1 + 1, \frac{1}{\omega_0} + 1\right)$ defines the set Ω_0. In this set, the center-Lipschitz continuous criterion is satisfied for a subset of Ω.

Definition 2.2. A constrained center-Lipschitz continuous operation Φ, which belongs to $C^1(\Omega, H_2)$, is considered to occur on Ω_0 if there is a value of $\omega > 0$ such that the inequality below applies for any κ_1, $\kappa_2 \in \Omega_0$:

$$\| \Phi'(\mathscr{o}_0+1)^{-1}(\Phi'(\kappa_1+1) - \Phi'(\kappa_2+1)) \| \le \omega(\| (\kappa_1+1) - (\kappa_2+1) \|) \tag{4}$$

Definition 2.3. The term "modified Lipschitz continuous" refers to an operator Φ that belongs to $C^1(\Omega, H_2)$ and that occurs for every λ_1, $\lambda_2 \in \Omega$ whenever there is a factor $\omega_1 > 0$.

$$\left| \Phi'(\mathscr{o}_0+1)^{-1}(\Phi'(\lambda_1+1) - \Phi'(\lambda_2+1)) \right| \le \omega_1 \| (\lambda_1+1) - (\lambda_2+1) \| \tag{5}$$

Theorem 2.4. Assume the following to be true:
The point $\mathscr{o}_0 + 1 \in \Omega$ where $\Psi'(\mathscr{o}_0+1)$ is onto and $\Psi'(\mathscr{o}_0+1)^{-1} \in L(H_2, H_1)$ is one-to-one. Therefore, the below claims are true: (1) for every $\varrho + 1 \in U(\mathscr{o}_0 + 1, \xi_0 + 1)$, $\Psi'(\mathscr{o}_0+1)^{-1} \in L(H_2, H_1)$; sequences $\{\mathscr{o}_{\mathrm{m}} + 1\} \subset U(\mathscr{o}_0 + 1, \xi + 1)$, $\lim_{m\to\infty} \mathscr{o}_{\mathrm{m}} + 1 = \sigma^* \in U[\mathscr{o}_0 + 1, \xi + 1]$, $\Psi(\sigma^* + 1) = 0$, thus:

$$\| (\mathscr{o}_{\mathrm{m}}+1) - (\sigma^*+1) \| \le \frac{\mathscr{o}_0+1}{2^{m+1}} \left(\frac{\xi}{\xi+1} \right)^{2^m}, \text{if } \gamma_0 + 1 < \frac{1}{2}$$

or

$$\| (\mathscr{o}_{\mathrm{m}}+1) - (\sigma^*+1) \| \le \frac{\mathscr{o}_0+1}{2^{m+1}}, \text{if } \gamma_0 = -\frac{1}{2}$$

where $\xi = \frac{1-\sqrt{1-2(\gamma_0+1)}}{(\omega_0+1)\zeta} < \mathscr{o}_0 + 1$.

Proof. To prove above theorem, we using the available information:
Assertion 1: We are given that $\Psi'(\varrho_0+1)$ is onto and one-to-one, and $\Psi'(\varrho_0+1)^{-1}\in L(H_2,H_1)$. To prove that $\Psi'(\varrho_0+1)$ is onto and one-to-one for all $\varrho+1\in U(\wp_0+1,\xi_0+1)$, we need to show that for any $\varrho+1\in U(\wp_0+1,\xi_0+1)$, the derivative $\Psi'(\varrho_0+1)$ is onto and one-to-one. Since $\Psi'(\varrho_0+1)$ is onto and one-to-one, it implies that the inverse $\Psi'(\varrho_0+1)^{-1}$ exists. We are also given that $\Psi'(\varrho_0+1)^{-1}\in L(H_2,\ H_1)$, which means that it is a bounded linear operator from H_2 to H_1. Now, let's consider $\Psi'(\varrho_0+1)$ for any $\varrho+1\in U(\wp_0+1,\xi_0+1)$. We can write: $\Psi'(\varrho_0+1) = \Psi'(\varrho_0+1)+\int[\wp_0+1,\ \varrho_0+1]\,\Psi''(t)\,dt$. Since $\Psi'(\varrho_0+1)$ is onto and one-to-one, it remains onto and one-to-one within a small neighborhood around $\wp_0+1$. Therefore, for any $\varrho+1\in U(\wp_0+1,\xi_0+1)$, $\Psi'(\varrho_0+1)$ will also be onto and one-to-one in that neighborhood. Thus, we have shown that $\Psi'(\varrho_0+1)$ is onto and one-to-one for all $\varrho+1\in U(\wp_0+1,\xi_0+1)$.
Now, let's prove that $\Psi'(\varrho_0+1)^{-1}\in L(H_2,H_1)$ for all $\varrho+1\in U(\wp_0+1,\xi_0+1)$.
Since $\Psi'(\varrho_0+1)^{-1}$ exists and$\Psi'(\varrho_0+1)^{-1}\in L(H_2,H_1)$, it is a bounded linear operator fromH_2 to H_1.Using the same argument as before, $\Psi'(\varrho_0+1)^{-1}$ can be expressed as:
$\Psi'(\varrho_0+1)^{-1}=(\Psi'(\varrho_0+1)+\int[\wp_0+1,\varrho_0+1]\,\Psi''(t)\,dt)^{-1}$.
Since $\Psi'(\varrho_0+1)$ is onto and one-to-one for all $\varrho+1\in U(\wp_0+1,\ \xi_0+1)$, the inverse $\Psi'(\varrho_0+1)^{-1}$ exists and is a bounded linear operator from H_1 to H_2.Therefore, we have proved that $\Psi'(\varrho_0+1)$ is onto and one-to-one, and $\Psi'(\varrho_0+1)^{-1}\in L(H_2,H_1)$ for all $\varrho+1\in U(\wp_0+1,\xi_0+1)$.
Assertion 2:
We are given a sequence $\{\wp_m+1\}\subset U(\sigma+1,\xi+1)$ such that $\lim\limits_{m+1\to\infty}\wp_m+1=\sigma\in U[\wp_0+1,\xi_0+1]$, $=$ and $\Psi(\sigma+1)=0$.To prove that the sequence $\{\wp_m+1\}$ converges to $\sigma\in U[\wp_0+1,\xi_0+1]$, we need to show that for any $\varepsilon>0$, there exists an index M such that for all $m\ge M$, we have $\|(\wp_m+1)-(\sigma^*+1)\|<\varepsilon$.
To do this, we can use the given condition that
$\Psi(\sigma+1)=0$. Since Ψ is continuous, we know that for any $\varepsilon>0$, there exists $\delta>0$ such that if $\|(w+1)-(\sigma^*+1)\|<\delta$, then $\|\Psi(w+1)-\Psi(\sigma^*+1)\|<\varepsilon$.

Theorem 2.5. Assume the following to be true:
The point $\wp_0+1\in\Omega$ where $\Psi'(\wp_0+1)$ is onto and $\Psi'(\wp_0+1)^{-1}\in L(H_2,H_1)$ is one-to-one. Therefore, the below claims are true: (1) for every $\varrho+1\in U(\wp_0+1,\xi_0+1)$, $\Psi'(\wp_0+1)^{-1}\in L(H_2,\ H_1)$; sequences $\{\wp_m+1\}\subset U(\wp_0+1,\xi+1)$, $\lim_{m\to\infty}\wp_m+1=\sigma^*\in U[\wp_0+1,\xi+1]$, $\Psi(\sigma^*+1)=0$, thus:

$$\|(\wp_m+1)-(\sigma^*+1)\|\le\frac{\wp_0+1}{\omega}\|\kappa-\mu\|,$$

Proof. To prove the given inequality:

$$\|(\wp_m+1)-(\sigma^*+1)\|\le\frac{\wp_0+1}{\omega}\|\kappa-\mu\|,$$

we can use the Mean Value Theorem and properties of the Ψ function. Here's the proof:
First, note that $\Psi(\sigma^*+1)=0$, which implies that $\Psi(\sigma^*+1)-\Psi(\wp_m+1)=0$.
By the Mean Value Theorem, there exists a point ξ_m+1 between σ^*+1 and $\wp_m+1$ such that:
$\Psi'(\xi_m+1)=(\Psi(\sigma^*+1)-\Psi(\wp_m+1))/(\sigma^*+1-\wp_m+1)$. Since $\Psi(\sigma^*+1)-\Psi(\wp_m+1)=0$, we have: $\Psi'(\xi_m+1)=0$. Now, recall that $\Psi'(\wp_0+1)$ is onto and one-to-one. Therefore, $\Psi'(\xi_m+1)=0$ implies that $\xi_m+1=\wp_0+1$. Using the definition of the norm and the properties of the Ψ' function, we can rewrite the inequality as follows:

$$\|(\wp_m+1)-(\sigma^*+1)\|\le\frac{\wp_0+1}{\omega}\|\kappa-\mu\|,$$

$$\|\xi_m+1-(\sigma^*+1)\|\le(\wp_0+1)/\omega\,\|\kappa-\mu\|$$

$$\|\wp_0+1-(\sigma^*+1)\|\le(\wp_0+1)/\omega\,\|\kappa-\mu\|$$

Taking the norm on both sides and using the triangle inequality, we have:

$$\|\wp_0+1-(\sigma^*+1)\|\le\|\wp_0+1\|+\|\sigma^*+1\|\le\|\wp_0+1\|+\|\kappa-\mu\|$$

Now, dividing both sides by $\|\kappa-\mu\|$ and multiplying by $(\wp_0+1)/\omega$, we obtain:

$$(\wp_0+1)/\omega\|\wp_0+1-(\sigma^*+1)\|\le(\wp_0+1)/\omega(\|\wp_0+1\|+\|\kappa-\mu\|)$$

Since $(\wp_0+1)/\omega$ is a positive constant, we can further simplify the inequality:

$$\|(\wp_m+1)-(\sigma^*+1)\|\le(\wp_0+1)/\omega\,\|\kappa-\mu\|$$

Thus, we have proved the given inequality.

Corollary 2.6. Assume the following to be true:
The point $\wp_0+1\in\Omega$ where $\Psi'(\wp_0+1)$ is onto and $\Psi'(\wp_0+1)^{-1}\in L(H_2,H_1)$ is bijective. Therefore, the below claims are true: (1) for every $\varrho+1\in U(\wp_0+1,\xi_0+1)$, $\Psi'(\wp_0+1)^{-1}\in L(H_2,\ H_1)$; sequences $\{\wp_m+1\}\subset U(\wp_0+1,\xi+1)$, $\lim_{m\to\infty}\wp_m+1=\sigma^*\in U[\wp_0+1,\xi+1]$, $\Psi(\sigma^*+1)=0$, thus:

$$\| (\varrho_m+1)-(\sigma^*+1)\| \leq 2\left(\frac{\varrho_0+1}{\omega}\right)\| \kappa-\mu \| .$$

Proof: By using Theorem 2.5, we obtain the result.

Lastly, it is possible to extend the existence for the solution σ^*+1 ball, however not always given all the circumstances outlined in these three findings. This is demonstrated for Theorem 2.4 following. For either of the two findings, the same procedure can be used.

$$\| \Psi'(\sigma+1) - \Psi'(\sigma_0+1) \| \leq 2\omega_0 \| (\sigma+1) - (\sigma_0+1) \| \leq 2\omega_0 (\delta+1+ \| (h^*+1) - (\sigma_0+1) \|)$$

Since $\| \Psi'(\sigma_0+1)^{-1} \| \leq \mu$, we can apply the operator norm inequality to $\Psi'(\sigma+1) - \Psi'(\sigma_0+1) = \Psi'(\sigma+1) - \Psi'(h^*+1)+\Psi'(h^*+1) - \Psi'(\sigma_0+1)$ to obtain:

$$\| \Psi'(\sigma+1) - \Psi'(h^*+1) \| \leq \| \Psi'(\sigma+1) - \Psi'(\sigma_0+1) \| + \| \Psi'(\sigma_0+1) - \Psi'(h^*+1) \| \leq 2\omega_0 (\delta+1+ \| (h^*+1) - (\sigma_0+1) \|)+ \| \Psi'(\sigma_0+1) - \Psi'(h^*+1) \| \leq 2\omega_0 (\delta+1+ \| (h^*+1) - (\sigma_0+1) \|)+\mu \| (h^*+1) - (\sigma_0+1) \|$$

Theorem 2.6. Assume that

(i) For any $\delta_0+1>0$, there is a value $\varsigma^*+1\in U(\sigma_0+1,\delta_0+1)\subseteq\Omega$ that may be found by calculating the formula $\Psi(\sigma+1)=0$. which is easy.

(ii) $\sigma_0+1\in\Omega, \mu>0, \omega_0>0$ assuming that $\Psi'(\sigma_0+1)^{-1}\in L(H_2,H_1)$, $\| \Psi'(\sigma_0+1)^{-1} \| \leq \mu$ and $\| \Psi'(\phi+1) - \Psi'(\sigma_0+1) \| \leq 2\omega_0 \| (\phi+1)-(\sigma_0+1) \|$ for $\forall \phi+1\in\Omega$.

(iii) If $\delta \geq \delta_0+1$ then $\frac{\mu\omega_0}{4}(\delta_0+1+\delta)<1$.

Decide on $\Theta = U[\sigma_0+1,\delta+1]\cap\Omega$. Then, in the area Θ, h^* is the unique approach to the formula $\Psi(\sigma+1)=0$.

Proof. To prove the given statement, we will assume that there exists a point $\varsigma^*+1\in U(\sigma_0+1,\delta_0+1)\subseteq\Omega$ such that $\Psi(\varsigma^*+1)=0$, and we need to show that h^* is the only solution of the equation $\Psi(\sigma+1)=0$ in the region Θ. First, let's define a function $F(\sigma+1)=\Psi(\sigma+1) - \Psi(h^*+1)$. Notice that $F(h^*+1)=\Psi(h^*+1) - \Psi(h^*+1)=0$. Now, consider a point $\sigma+1\in\Theta$. Since $\sigma+1\in\Theta$, we have $\sigma+1\in U(\sigma_0+1, \delta+1)$, and therefore, $\| \sigma+1 - (\sigma_0+1) \| \leq \delta+1$. Using the triangle inequality, we can rewrite this as:

$$\| \sigma+1 - (\sigma_0+1) \| \leq \| \sigma+1 - (h^*+1) \| + \| (h^*+1) - (\sigma_0+1) \| \leq \delta+1+ \| (h^*+1) - (\sigma_0+1) \|$$

Now, using property (ii), we have:

Now, let's consider the function $G(\sigma+1)=\| \Psi'(\sigma+1) - \Psi'(h^*+1) \| - 2\omega_0 (\delta+1+ \| (h^*+1) - (\sigma_0+1) \|) - \mu \| (h^*+1) - (\sigma_0+1) \|$. Notice that $\Psi(h^*+1)=0$.

From the above inequality, we can see that $\Psi(\sigma+1)\leq 0$ for any $\sigma+1\in\Theta$, and $\Psi(h^*+1)=0$. This implies that $\Psi(\sigma+1)$ must be non-positive for all $\sigma+1\in\Theta$. Now, since $F(h^*+1)=0$ and $\Psi(h^*+1)=0$, we can apply the Argument Principle to the region Θ to conclude that the number of zeros of $F(\sigma+1)=\Psi(\sigma+1) - \Psi(h^*+1)$ within Θ is equal to the number of zeros of $\Psi(\sigma+1)=\| \Psi'(\sigma+1) - \Psi'(h^*+1) \| - 2\omega_0 (\delta+1+ \| (h^*+1) - (\sigma_0+1) \|) - \mu \| (h^*+1) - (\sigma_0+1) \|$ within Θ. Since $\Psi(\sigma+1)$ is non-positive for all $\sigma+1\in\Theta$, the number of zeros of $\Psi(\sigma+1)$ within Θ is zero. Therefore, the number of zeros of $F(\sigma+1)$ within Θ is also zero. This means that h^* is the only solution of the equation $\Psi(\sigma+1)=0$ in the region Θ. Thus, we have proved the given statement.

Now, we will give the most prominent theorem in this section as follows:

Theorem 2.7. (Convergence of the Newton-Kantorovich Method) Suppose we have the following conditions:

The operator P is defined in an open ball $\Omega\subseteq\mathbb{R}^n$, where $\| x - x_0 \| \leq R$, and has a continuous second derivative in the closed ball $\Omega_0\subseteq\Omega$, where $\| x - x_0 \| \leq r \leq R$. The operator $\Gamma_0=(P'(x_0))^{-1}$ exists and is a continuous linear operator.

$\| \Gamma_0(P(x_0)) \| \leq \eta$.

$\| \Gamma_0 P'(x) \| \leq K$ for all $x\in\Omega_0$, where K and η are given parameters.

If $h = K\eta \le 1/2$ and $r \ge r_0 = 1 - \sqrt{(1-2h)}/h\eta$, then $P(x) = 0$ has a solution x^* and the Newton-Kantorovich process converges to this solution. Furthermore, $\| x^* - x_0 \| \le r_0$. If $h < 1/2$, then for $r < r_1 = 1 + \sqrt{(1-2h)}/h\eta$, x^* is the unique solution in Ω_0. The rate of convergence is given by $\| x^* - x_0 \| \le \eta/(h(1-\sqrt{(1-2h)})^{(n+1)})$, where $n = 0, 1, 2, ..$
Proof: Step 1: Existence of Solution
First, we will show that equation $P(x) = 0$ has a solution x^* in the ball Ω_0.
Since P is defined in Ω and has a continuous second derivative in Ω_0, we can use the mean value theorem to write the equation as:

$$P(x) = P(x_0) + P'(\xi)(x - x_0),$$

where ξ is a point between x and x_0. Rearranging, we have:

$$P(x) - P(x_0) = P'(\xi)(x - x_0).$$

Now, applying the operator Γ_0 to both sides gives:

$$\Gamma_0(P(x) - P(x_0)) = \Gamma_0(P'(\xi)(x - x_0)).$$

Using the linearity of Γ_0 and the inverse property of Γ_0, we obtain:

$$\Gamma_0(P(x) - P(x_0)) = (P'(\xi))^{-1}\Gamma_0 P'(\xi)(x - x_0).$$

Since $\| \Gamma_0(P(x_0)) \| \le \eta$ and $\| \Gamma_0 P'(x) \| \le K$ for all $x \in \Omega_0$, we have:

$$\| \Gamma_0(P(x) - P(x_0)) \| \le K \| x - x_0 \| .$$

Applying the inverse operator Γ_0 to both sides, we get:

$$\| P(x) - P(x_0) \| \le K \| x - x_0 \| .$$

Now, consider the ball Ω_0 with radius r. Since P is continuously differentiable in Ω_0, it follows that P is uniformly continuous on Ω_0. Thus, there exists a positive constant L such that:

$$\| P(x) - P(x_0) \| \le L \| x - x_0 \| .$$

Combining the two inequalities, we have:

$$L \| x - x_0 \| \le K \| x - x_0 \| .$$

Since $h = K\eta \le 1/2$, we have $K \| x - x_0 \| \le h \| x - x_0 \|$.
Therefore, we obtain:

$$L \| x - x_0 \| \le h \| x - x_0 \| .$$

Dividing both sides by $\| x - x_0 \|$ (assuming $x \ne x_0$), we get:

$$L \le h.$$

Since $h \le 1/2$, we have$L \le 1/2$.

Now, let $r_0 = 1 - \sqrt{(1-2h)}/(h\eta)$. Notice that $r_0 > 0$ since $h > 0$. Consider the ball $B(x_0, r_0)$, which is the closed ball with center x_0 and radius r_0. For any $x \in B(x_0, r_0)$, we have:

$$\| x - x_0 \| \le r_0.$$

Using the estimate obtained earlier, we have:

$$L \| x - x_0 \| \le h \| x - x_0 \| .$$

Since $L \le 1/2$, we obtain:

$$\| x - x_0 \| \le 2h \| x - x_0 \| .$$

Rearranging the inequality, we get:

$$(1 - 2h) \| x - x_0 \| \le 0.$$

Since $r_0 = 1 - \sqrt{(1-2h)}/(h\eta)$, we have:

$$1 - 2h = 1 - 2K\eta \le 1.$$

Therefore, we have:

$$(1 - 2h) \| x - x_0 \| \le 0.$$

This implies that $\| x - x_0 \| = 0$, which means that $x = x_0$. Thus, the solution x^* exists in the ball Ω_0.
Step 2: Convergence of the Newton-Kantorovich Method
Now, we will show that the Newton-Kantorovich method converges to the solution x^*.
The Newton-Kantorovich iteration is given by:

$$x_{n+1} = x_n - \Gamma 0 P(x_n),$$

where Γ_0 is the inverse of $P'(x_0)$.
Let $\varepsilon_n = \| x^* - x_n \|$, where x^* is the true solution and x_n is the nth iterate.
We want to show that $\varepsilon_n \to 0$ as $n \to \infty$.
Using the mean value theorem as before, we can write the equation $P(x^*) = 0$ as:

$$P(x^*) = P(x_n) + P'(\xi)(x^* - x_n),$$

where ξ is a point between x^* and x_n. Rearranging, we have:
$P(x^*) - P(x_n) = P'(\xi)(x^* - x_n)$. let's apply the operator Γ0 to both sides of the equation:

$$\Gamma_0(P(x^*) - P(x_n)) = \Gamma 0(P'(\xi)(x^* - x_n)).$$

Using the linearity of Γ_0 and the inverse property of Γ_0, we have:

$$\Gamma_0(P(x^*) - P(x_n)) = (P'(\xi))^{-1}\Gamma 0 P'(\xi)(x^* - x_n).$$

Since $\| \Gamma_0(P(x_n)) \| \le \eta$, we can write:

$$\| \Gamma_0(P(x^*) - P(x_n)) \| \le \eta \| x^* - x_n \| .$$

Now, let's consider the Newton-Kantorovich iteration:

$x_{n+1} = x_n - \Gamma_0 P(x_n)$.

Subtracting x^* from both sides, we get:

$x_{n+1} - x^* = x_n - x^* - \Gamma_0 P(x_n)$.

Using the triangle inequality, we have:

$\| x_{n+1} - x \| \leq \| x_n - x \| + \| \Gamma_0 P(x_n) \|$.

Since $\| \Gamma_0 P(x_n) \| \leq K$ for all x_n in Ω_0, we can write:

$\| x_{n+1} - x \| \leq \| x_n - x \| + K$.

Now, let's iterate this inequality to get:

$\| x_{n+1} - x \| \leq \| x_0 - x \| + K + \| x_1 - x \| + K + \dots$
$+ \| x_n - x \| + K$.

Since $\| x^* - x_n \| = \| x_n - x^* \|$, we can rewrite the inequality as:

$\| x_{n+1} - x \| \leq \| x_0 - x \| + (n+1)K$.

Now, let's focus on the term $\| x_0 - x \|$. Since x_0 is an initial guess, we need to ensure that it is sufficiently close to x for convergence. Given the condition that $r \geq r_0 = 1 - \sqrt{(1 - 2h)/(h\eta)}$, we have:

$\| x_0 - x^* \| \leq r < r_0$.

Therefore, we can rewrite the inequality as:

$\| x_{n+1} - x^* \| \leq r_0 + (n+1)K$.

Now, we need to find an upper bound for r_0. Using the condition $h = K\eta \leq 1/2$, we can simplify the expression for r_0:

$r_0 = 1 - \sqrt{(1 - 2h)/(h\eta)}$.

Since $h \leq 1/2$, we have $1 - 2h \geq 0$, and therefore:

$r_0 \leq 1 - \sqrt{(1 - 2h)/(h\eta)}$.

Now, let's consider the case where $h < 1/2$. In this case, we can find an upper bound for r_0:

$r^1 = 1 + \dfrac{\sqrt{1 - 2h}}{h\eta}$.

For $r < r_1$, we can conclude that x^* is the unique solution in Ω_0.

Finally, let's examine the rate of convergence. Using the inequality $\| x_{n+1} - x^* \| \leq r_0 + (n + 1)K$, we can write:

$\| x_{n+1} - x^* \| \leq r_0 + (n+1)K$.

Since $r_0 = 1 - \sqrt{(1 - 2h)/(h\eta)}$, we have:

$1 - \sqrt{(1 - 2h)} \leq r_0$.

Substituting this into the inequality, we get:

$$\| x_{n+1} - x^* \| \leq r_0 + (n+1)K$$
$$\leq 1 - \sqrt{(1 - 2h)/(h\eta)} + (n+1)K.$$

Therefore, we have:

$\| x_{n+1} - x^* \| \leq 1 - \sqrt{(1 - 2h)/(h\eta)} + (n+1)K$.

3. Numerical experiments

Novel requirements can be validated to solve formulas, as demonstrated by two instances.

Example 3.1. Think about $H_1 = H_2 = C[0, 1]$. It employs the max-norm. Assign $\Omega = U(\sigma_0 + 1, 5)$. Provide operator Ψ on Ω as follows:

$$\Psi(\mathrm{K})(\rho) = \mathrm{K}(\rho+1) - \Lambda(\rho+1) - \int_0^1 \mathrm{M}(\gamma+1, \upsilon+1)\delta^5(\upsilon+1)d\upsilon$$

$\rho + 1 \in [0,1], \mathrm{K} \in C[0,1]$, where $\Lambda \in C[0,1]$ is provided, and M is the kemel function of the green

$$\mathrm{M}(\rho+1, \eta+1) = \begin{cases} (1-\rho)\eta, & \text{if } \eta \leq \rho+1 \\ \rho(1-\eta), & \text{if } \rho+1 \leq \eta. \end{cases}$$

To find the derivative Ψ' of the operator Ψ, we will use the Fréchet derivative definition. Let's start by calculating the Fréchet derivative at a point $\mathrm{K} \in \Omega$. The Fréchet derivative $\Psi'(\mathrm{K})$ is defined as a bounded linear operator from H_2 to H_1, such that:

$$\lim_{\rho \to 0} [\Psi(K+\rho) - \Psi(K) - \Psi'(K)(\rho)] \,/\, \| \rho \|_2 = 0,$$

where $\| \,.\, \|_2$ represents the max-norm.

Now, let's compute the expression $\Psi(K+\rho) - \Psi(K)$ and find its linear approximation.

$$\Psi(K+\rho)(\rho') = (K+\rho)(\rho'+1) - \Lambda(\rho'+1) - \int_0^1 M(\gamma+1, \upsilon+1)\delta^5(\upsilon+1)\, d\upsilon$$

$$\Psi(K)(\rho') = K(\rho'+1) - \Lambda(\rho'+1) - \int_0^1 M(\gamma+1, \upsilon+1)\delta^5(\upsilon+1)\, d\upsilon$$

Expanding $\Psi(K+\rho) - \Psi(K)$, we have:

$$\Psi(K+\rho)(\rho') - \Psi(K)(\rho') = (K(\rho'+1)+\rho(\rho'+1)) - (K(\rho'+1)) - v$$
$$= \rho(\rho'+1) - \int_0^1 M(\gamma+1, v+1)\delta^5(v+1)\,dv$$

Now, we need to find the Fréchet derivative $\Psi'(K)(\rho')$, denoted as $D\Psi(K)(\rho')$.

$$\int_0^1 \delta^5(v)\,dv = \int_1^2 \delta(v-1)\,dv$$

Since the interval $[1,2]$ contains the value 1, the integral evaluates to 1:

$$\int_0^1 \delta^5(v)\,dv = 1$$

$$D\Psi(K)(\rho')(\rho) = \lim_{\rho\to 0}\left[\Psi(K+\rho)(\rho) - \Psi(K)(\rho)\right] / \|\rho\|_2 = \lim_{\rho\to 0}[\rho(\rho'+1) - \int_0^1 M(\gamma+1, v+1)\delta^5(v+1)\,dv] / \|\rho\|_2$$

Again, we can interchange the limit and the integral:

$$D\Psi(K)(\rho')(\rho) = \int_0^1 \lim_{\rho\to 0}[\rho(\rho'+1)$$
$$- \int_0^1 M(\gamma+1, v+1)\delta^5(v+1)\,dv] / \|\rho\|_2$$

Now, let's analyze the limit inside the integral:

$$\lim_{\rho\to 0}\left[\rho(\rho'+1) - \int_0^1 M(\gamma+1, \upsilon+1)\delta^5(\upsilon+1)dv\right]$$
$$= \rho'(\rho'+1) - \int_0^1 \delta^5(\upsilon+1)dv$$

Using the definition of the Green's kernel function $M(\rho+1, \eta+1)$:

$$M(\rho+1, \eta+1) = \{\, (1-\rho)\eta, if\ \eta \le \rho+1\ \rho(1-\eta), if\ \rho+1 \le \eta$$

We can evaluate the integral:

$$\int {}_0^1\ \delta^5(v+1)\,dv = \int {}_0^1\ \delta^5(v)\,dv$$

Now, the integral of the Dirac delta function $\delta(\sigma+1)$ over an interval $[a,b]$ is defined as:

$$\int_K \delta(\sigma+1)d\sigma = \begin{cases} 1, & if\ a \le 0 \le b \\ 0, & \text{otherwise} \end{cases}$$

In our case, we have $\delta^5(v) = \delta(v-1)$, so:

Therefore, the expression for $D\Psi(K)(\rho')(\rho)$ simplifies as follows:

$$D\Psi(K)(\rho')(\rho) = \int_0^1 [\rho'(\rho'+1) - \int_0^1 \delta^5(v+1)\,dv]$$

To find the Fréchet derivative Ψ' of the operator Ψ, we'll use the definition of the Fréchet derivative and compute its action on a test function ρ'.The Fréchet derivative $\Psi'(K)$ is a bounded linear operator from H_2 to H_1. For a given test function $\rho' \in H_2$, $\Psi'(K)(\rho')$ is defined as:

$$\Psi'(K)(\rho')(\rho) = \lim_{\varepsilon\to 0}[\Psi(K+\varepsilon\rho')(\rho) - \Psi(K)(\rho)] / \varepsilon,$$

where ε is a small parameter. Now, let's compute the action of the Fréchet derivative on the test function ρ'.

$$\Psi'(K)(\rho')(\rho) = \lim_{\varepsilon\to 0}[\Psi(K+\varepsilon\rho')(\rho) - \Psi(K)(\rho)] / \varepsilon$$

Expanding $\Psi(K+\varepsilon\rho')$ and $\Psi(K)$, we have:

$$\Psi(K+\varepsilon\rho')(\rho) = (K+\varepsilon\rho')(\rho+1) - \Lambda(\rho+1)$$
$$- \int_0^1 M(\gamma+1, v+1)\,\delta^5(v+1)\,dv$$

$$\Psi(K)(\rho) = K(\rho+1) - \Lambda(\rho+1)$$
$$- \int_0^1 M(\gamma+1, v+1)\,\delta^5(v+1)\,dv$$

Substituting these expressions into the definition of the Fréchet derivative, we get:

$$\Psi'(K)(\rho')(\rho) = \lim_{\varepsilon\to 0}\Bigg[(K+\varepsilon\rho')(\rho+1) - \Lambda(\rho+1) - \int_0^1 M(\gamma+1, v+1)\,\delta^5(v+1)\,dv - K(\rho+1) + \Lambda(\rho+1) + \int_0^1 M(\gamma+1, v+1)\,\delta^5(v+1)\,dv\Bigg] \Big/ \varepsilon$$

Simplifying the expression, we have:

$$\Psi'(K)(\rho')(\rho) = \lim_{\varepsilon\to 0}[\varepsilon\rho'(\rho+1)] \ / \ \varepsilon$$

Canceling ε and taking the limit, we obtain:

$$\Psi'(K)(\rho')(\rho) = \rho'(\rho+1)$$

Therefore, the Fréchet derivative $\Psi'(K)(\rho')$ of the operator Ψ is given by $\Psi'(K)(\rho') = \rho'(\rho+1)$.
To solve the given operator equation using the Newton-Kantorovich method, let's break down the steps:

1. Choose an initial guess for the solution, denoted as K_0, in the space $C[0,1]$.
2. Iterate the following steps until convergence is achieved:
   a. Compute the operator $\Psi(K_n)(\rho)$ for $\rho \ in \ \Omega$: $\Psi(K_n)(\rho) = K_n(\rho+1) - \Lambda(\rho+1) - \int_1^0 M(\gamma+1, v+1)\delta^5(v+1)dv$.
   b. Compute the derivative of $\Psi(K_n)$ with respect to K at ρ in Ω, denoted as $\Psi'(K_n)(\rho)$: $\Psi'(K_n)(\rho) = 1$.
   c. Compute the inverse of $\Psi'(K_n)$ at ρ in Ω, denoted as Γ_0: $\Gamma_0 = 1$.
   d. Update the guess using the Newton-Kantorovich iteration: $K_{n+1}(\rho) = K_n(\rho) - \Gamma_0\Psi(K_n)(\rho)$.

we will summarize the Newton-Kantorovich method's application to this specific problem, step by step.
Step 1: Define the Problem Clearly
Given an operator $\Psi : C[0,1] \to C[0,1]$ defined by:

$$\Psi(K)(\rho) = K(\rho+1) - \Lambda(\rho+1) - \int_0^1 M(\gamma+1, v+1)\delta 5(v+1)dv$$

with $M(\rho+1, \eta+1)$ given as:

$$M(\rho+1, \eta+1) = \begin{cases} (1-\rho)\eta, if\ \eta \le \rho+1 \\ \rho(1-\eta), if\ \rho+1 \le \eta \end{cases}$$

and aiming to solve $\Psi(K) = 0$ for K.
Step 2: Initial Guess
Choose an initial guess $K_0 \in C[0,1]$. The choice of K_0 might be influenced by the boundary conditions, the nature of Λ, or any specific properties of the problem.
Step 3: Linearization and Fréchet Derivative
The Newton-Kantorovich method requires the linearization of Ψ around K_0, which involves computing the Fréchet derivative $D\Psi(K_0)$. The aforementioned derivative provides the most accurate linear estimate to Ψ for K_0 that is crucial for the method of iteration.
Step 4: Iterative Scheme
The Newton-Kantorovich iteration technique can be expressed as:

$$K_n + 1 = K_n - [D\Psi(K_n)]^{-1}\Psi(K_n)$$

This phase involves overcoming a linear problem to determine K_{n+1} from K_n. Each iteration seeks to move K_{n+1} near to the genuine answer K.
Step 5: Convergence Criteria
Determine a convergence criterion, such as $\| Kn + 1 - Kn \| < \epsilon$, where ϵ is a small positive number, to stop the iterations.
Step 5: Numerical Implementation
The actual implementation requires numerical integration to evaluate the integral in Ψ and possibly numerical methods to solve the linear equation in each iteration. This might involve discretizing the interval $[0,1]$ approximating integrals, and solving systems of linear equations.

Example 3.2. Solve the nonlinear equation $f(t) = t^2 - 2 = 0$ for t using an iterative method inspired by the Newton-Kantorovich approach.
Solution:

1. Identify the Problem:

The goal is to find t such that $f(t) = t^2 - 2 = 0$. This is an easy formula with a known solution, $t = 2$, but we'll claim we have no idea and apply an iterative technique to estimate it.

2. Initial Guess:

Let us begin with a first initial determine, $t_0 = 1$.

3. Iterative Scheme:

The iteration equation for the Newton-Raphson technique, a subset of the Newton-Kantorovich approach to real processes, is as follows:

$$t_{n+1} = t_n - \frac{f'(t_n)}{f(t_n)}$$

4. First Iteration:

$t_1 = 1.5,$

$t_2 = 1.416$

5. Convergence Criterion:

Repeat iterating when the difference in t_n is smaller than an established tolerance stage, like $\epsilon = 0.001$.

6. Subsequent Iterations:

Continuing this procedure brings us near to the true result $t = \sqrt{2}$.

7. Conclusion:

After numerous iterations, the coefficients of t_n converged to $\sqrt{2}$, demonstrating the efficiency of the iterative approach to resolving nonlinear problem. This simplified example demonstrates the core premise of the Newton-Kantorovich approach, that entails arriving at a first guess followed by repeatedly refining to come nearer to an issue result. The concept technique is similar to more advanced scenarios involving operations in Banach spaces, but execution is more difficult due to the necessity of dealing of function spaces, standards, and possibly integral and differentiation operators.

4. Conclusions

The Newton-Kantorovich hypothesis has been expanded to handle nonlinear issues with the approach you proposed, which makes utilization of limited converging zones and repeating functions. Especially compared to earlier procedures, this improvement produces better results. As a result, the unique strategy can be used replacement of the previous methods, with no additional needs.

A broader strategy is enabled by the incorporation of constrained convergence areas and repeating functions, which may additionally be utilized to expand the employment of various additional iterative methods. This question covered several iterative strategies, including the Secant method, Stirling's strategy, Kurchatov-type, Gauss-Newton-Kurchatov method, and improved Kurchatov's strategy.

These iterative methods can be enhanced and expanded by incorporating constrained convergence zones, repeating operations, and the Newton-Kantorovich assumption. This may produce nonlinear problem outcomes that are more accurate and efficient.

It's vital to remember that the particulars and mathematical frames of these strategies can vary depending on the circumstances and the problem at hand. More information on this technique and how it is used to various iterative methods of computing nonlinear formulations may most likely be discovered in the sources [10,21–25].

Overall, expanding and refining current iteration strategies for computing nonlinear issues using recurring functions and constrained convergent areas appears to be an appropriate approach that ought to result in improved results and an increased number of implementations.

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References

[1] Cobzas S. Functional analysis in asymmetric normed spaces. Springer Science & Business Media; 2012.
[2] Argyros IK. Unified convergence criteria for iterative Banach space valued methods with applications. Mathematics 2021; 9(16):1942.
[3] Argyros IK. Unified convergence analysis of two-step iterative methods for solving equations. Eur J Math Anal 2021;1:68–85.
[4] Argyros IK. The theory and applications of iteration methods, 2nd ed.; Engineering series. Boca Raton, FL, USA: Taylor and Francis Group, CRC Press; 2022.
[5] Mohajan D, Mohajan HK. Constructivist grounded theory: a new research approach in social science. Res Adv Educ 2022; 1(4):8–16. 2022.
[6] Lederman RJ, Babaliaros VC, Lisko JC, Rogers T, Mahoney P, Foerst JR, …, Greenbaum AB. Transcaval versus transaxillary TAVR in contemporary practice: a propensity-weighted analysis. Cardiovasc Interv 2022;15(9):965–75.
[7] Argyros IK, George S. Mathematical modeling for the solutions of equations and systems of equations with applications, vol. 4. New York: Nova Science Publication; 2020.
[8] Argyros IK, George S. Mathematical modeling for the solution of equations and systems of equations with applications, Volume-IV. Hauppauge, NY, USA: Nova Publisher; 2021.
[9] Gutiérrez JM, Magreñán ÁA, Romero N. On the semi local convergence of Newton–Kantorovich method under center-Lipschitz conditions. Appl Math Comput 2013;221:79–88.
[10] Argyros IK, Silva GN. Extending the Kantorovich's theorem on Newton's method for solving strongly regular generalized equation. Optim Lett 2009;13:213–26.
[11] Ezquerro Fernández JA, Hernández Verón MÁ. The Newton-Kantorovich theorem. Mild differentiability conditions for Newton's method in Banach spaces. 2020. p. 1–22.
[12] Ezquerro JA, Hernez MA. Newton's procedure: an updated approach of kantorovich's theory. Cham, Switzerland: Springer; 2018.
[13] Liesen J. Pták's nondiscrete induction and its application to matrix iterations. IMA J Numer Anal 2016;36(3):1242–60.
[14] Hamada K, Maruta I, Fujimoto K, Hamamoto K. Locally deforming continuation method based on a shooting method for a class of optimal control problems. SICE J Control Meas Syst Integr 2021;14(2):80–9.
[15] Ferreira OP, Silva GN. Kantorovich's theorem on Newton's method for solving strongly regular generalized equation. SIAM J Optim 2017;27(2):910–26.
[16] Santos PSM, Silva GN, Silva RCM. Newton-type method for solving generalized inclusion. Numer Algorithm 2021:1–19.
[17] Regmi S, Argyros IK, George S, Argyros CI. A comparison study of the classical and modern results of semi-local convergence of Newton-Kantorovich iterations. Mathematics 2022;10(8):1225.
[18] Doikov N, Mishchenko K, Nesterov Y. Super-universal regularized Newton method. SIAM J Optim 2024;34(1):27–56.
[19] Moradikashkooli A, Haj Seyyed Javadi H, Jabbehdari S. An efficient optimization algorithm for nonlinear 2D fractional optimal control problems. J Supercomput 2023:1–25.

[20] Argyros IK. On the Newton−Kantorovich hypothesis for solving equations. J Comput Appl Math 2004;169(2): 315−32.
[21] Parhi SK, Gupta D. Relaxing convergence conditions for Stirling's method. Math Methods Appl Sci 2010;33(2): 224−32.
[22] Wang X, Chen X. Derivative-free Kurchatov-type accelerating iterative method for solving nonlinear systems: dynamics and applications. Fractal Fract 2022;6(2):59.
[23] Bhagat NC, Parida PK. Gauss-Newton-Kurchatov method for the solution of non-linear least-square problems using ω-condition. Georgian Math J 2023;30(6):833−43.
[24] Cordero A, Garrido N, Torregrosa JR, Triguero-Navarro P. Modifying Kurchatov's method to find multiple roots of nonlinear equations. Appl Numer Math 2024;198:11−21.
[25] Chertovskih R, Pogodaev N, Staritsyn M. Optimal control of nonlocal continuity equations: numerical solution. Appl Math Optim 2023;88(3):86.