

Effect of Post-Buckling on the Stiffness and Stress of Plate

Dr. Hani Aziz Ameen

Technical College , Dies and Tools Eng.Dept/ Baghdad.

E-mail:haniazizameen@yahoo.com

Dr. Kayser Aziz Ameen

Al-Esra'aUniversity College / Baghdad.

E-mail:kayseraziz@yahoo.com

Dr. Ibtissam Mahdi shihab

Mechanical Engineering Dept, Universit Technology /Baghdad.

E-mail:ems55_2005@yahoo.com

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ABSTRACT

This paper presents a theoretical investigation of post and pre buckling of the simply supported plate. The effect of post and pre buckling on the stiffness of plate is determined. The full derivation of the equation described the ratio of stiffness of plate in the post to pre buckling is derived and from this equation it can be deduced that the simply support plates loss about (3/5) of its initial compressional stiffness after buckling also it can be concluded that after buckling the maximum stress increases at almost four times the pre-buckling.

Keywords: Plates, buckling, post-buckling, compression, pre-buckling

دراسة تأثير ما بعد الانبعاج على جساءة واجهاد الصفائح

الخلاصة

تتضمن هذه الدراسة تحليل نظري لحالة ما بعد وبداية الابعاج لالواح مثبته تثبيت بسيط . حيث تم دراسة وحساب تأثير الانبعاج على صلابة الوح. حيث انه تم اشتقاق معادلات رياضية تبين نسبة تأثير الصلابة للوح في حالة ما بعد الانبعاج وبداية الانبعاج وقد تم استنتاج ان الصلابة تقل بنسبة (5/3) من قيمتها الابتدائية بعد الانبعاج وكذلك تم الاستنتاج ان اقصى اجهاد يزداد اربع مرات بعد الانبعاج عن بداية لانبعاج .

INTRODUCTION

Buckling is the general term frequently used to describe the failure structure between the stable and unstable case. When the magnitude of the load on a structure is such that the equilibrium is changed from stable to unstable, the load is called a critical load or (buckling load). Buckling means loss of the stability of an equilibrium configuration, without fracture or separation of the material or at least prior to it [1]. It is an important type of failure that occurs widely in many structural applications, it is characterized by an abrupt large deformation that occurs in a structure when the load that is applied reaches a certain critical value. One problem

faced in the design of structures is buckling, in which structural members collapse under compressive loads greater than the material can withstand. The nature of buckling pattern in plate not only depends upon the type of the applied loading but also on the shape (dimension) of the problem and the material properties and also upon the manner in which the edges are supported [2].

The compression of plate is discussed by Von Karman et al ,1932, [3]. Marguerre , 1938 [4] investigated the elastic plate post-buckling , this analysis aided to determine the in-plane strains and stresses from the exact compatibility conditions with the out-of plane deflections. Rhodes, 1968 [5], modified the explicit expression obtained from basis of a Marguerre model. Rhodes, 1982 [6] studied the post-buckling behavior of bending elements and in 2003 [7] studied the post-buckling analysis of plates and plate structure, under eccentric load also investigated the application of plate analysis to strut, beam and column design. Michael, 1996 [8], studied the effect of a cutout on the buckling and post-buckling behavior of rectangular plates made of advanced composite materials. Rakesh, 2010 [9], studied the elastic buckling of thin plates were studied with shell finite element eign-buckling analysis.

This paper presents a theoretical investigation of post and pre buckling of the simply supported plate, which is of strong interest in the design of structures. The prediction of the stiffness of plate and stress due to buckling is thus a challenging task. This paper has a novel discussion of the effect of post and pre buckling on the stiffness of plate.

Basic Equations

In order to study the behavior of plate after buckling, account must be taken of the effects of out – of – plane displacements on the middle surface strains of a buckled plate. Fig.(1) shown the rectangular plate.

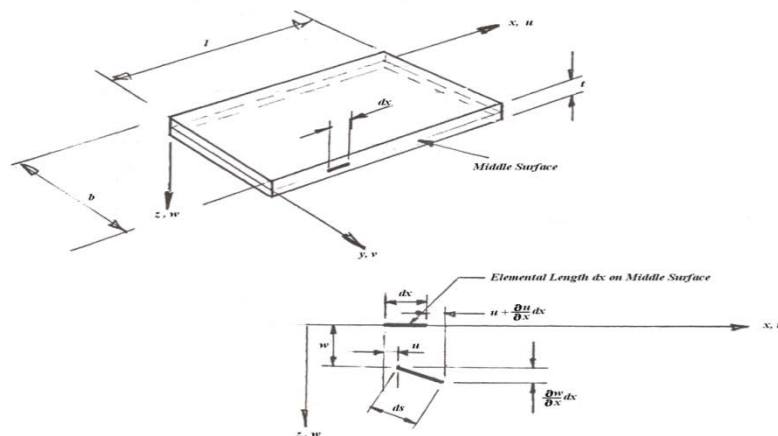


Figure. (1) Rectangular plate

for the middle strain in the x-direction , can be deduced that[2]

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \quad \dots (1)$$

$$\text{similarly, } \epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad \dots (2)$$

Also it can be shown with a slightly more complex geometrical construction that

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \quad \dots (3)$$

It can be shown that these three equations are obtained from the geometry of the displacements and constitute compatible requirements which must be adhered to if an energy method of analysis is to be used in the post- buckling range.

To eliminate the in-plane displacements, equation (1) differentiated twice with respect to y and equation (2) with respect to x , and equation (3) once with respect to x and again with respect to y and add, obtaining a single differential equation linking the mid- surface strains to the out – of – plane displacements. This is

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial y^2} \right) \quad \dots (4)$$

Now for a linear elastic material .The stresses σ_x , σ_y and τ_{xy} can be related by the introduction of a stress function $F(x,y)$ [7] .Substitution of these equations into equation(4) yields the final equation

$$\frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = \nabla^4 F = E \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial y^2} \right) \right] \quad \dots (5)$$

If an energy approach is to be used to investigate the post- buckling behavior of plates then equation (5) must be satisfied by the post- buckling stress variations. So,if the initial out of plane displacements w_o are present the equation then becomes

$$\nabla^4 F = E \left\{ \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial y^2} \right) \right] - \left[\left(\frac{\partial^2 w_o}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 w_o}{\partial x^2} \right) \left(\frac{\partial^2 w_o}{\partial y^2} \right) \right] \right\} \quad \dots (6)$$

Where

(w_o) is the local imperfection, since the middle surface stress are assumed to act uniformly through the thickness this gives the strain energy

$$V_M = \frac{t}{2} \iint (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy}) dx dy \quad \dots (7)$$

Or in other form

$$V_M = \frac{t}{2E} \iint \left\{ \frac{\partial^2 F}{\partial y^2} \left(\frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2} \right) + \frac{\partial^2 F}{\partial x^2} \left(\frac{\partial^2 F}{\partial x^2} - \nu \frac{\partial^2 F}{\partial y^2} \right) + 2(1 + \nu) \frac{\partial^2 F}{\partial x \partial y} \cdot \frac{\partial^2 F}{\partial x \partial y} \right\} dx dy \quad \dots (8)$$

Rearranging gives

$$V_M = \frac{t}{2E} \iint \left\{ \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right)^2 - 2(1 + \nu) \left(\frac{\partial^2 F}{\partial x^2} \cdot \frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y} \right)^2 \right) \right\} dx dy \quad \dots (9)$$

Post- Buckling Behavior of Plates with Simple Supported Loaded Edges

Consider a plate compressed in one direction to such an extent that buckling has occurred. The total compression displacement is $u = -\bar{u}$ and the stress system σ_x at the plate ends corresponding to this is as yet unknown.

It can be assumed the following deflection form for the post- buckling analysis

$$w = A.Y(y). \sin \frac{n\pi x}{l} \quad \dots (10)$$

Since the post-buckling is considered and it must satisfy equation (5) , Von Karmans compatibility equation[??].

Substituting for w into equation (5) gives

$$\nabla^4 F = E \left(\frac{n\pi}{l} \right)^2 A^2 \left\{ (Y')^2 \cos^2 \frac{n\pi x}{l} + Y Y'' \sin^2 \frac{n\pi x}{l} \right\} \quad \dots (11)$$

Rearranged gives

$$\nabla^4 F = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 \left\{ [(Y')^2 + Y Y''] + [(Y')^2 - Y Y''] \cos \frac{2n\pi x}{l} \right\} \quad \dots (12)$$

From this equation it can be seen that the stress function F may be considered in two parts, one part varying periodically with x and the other constant with respect to x .

$$\text{i.e. } F = F_1 + F_2 \cos \frac{2n\pi x}{l} \quad \dots (13)$$

where

$$\nabla^4 F_1 = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 [(Y')^2 + Y Y''] \quad \dots (14)$$

and

$$\nabla^4 \left(F_2 \cos \frac{2n\pi x}{l} \right) = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 [(Y')^2 - Y Y''] \cos \frac{2n\pi x}{l} \quad \dots (15)$$

Since

F_1 is a function of y only then $\nabla^4 F_1$ can be replaced by $\frac{\partial^4 F_1}{\partial y^4} = F_1''''$. equation (14)

then becomes;

$$F_1'''' = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 [(Y')^2 + Y Y''] \quad \dots (16)$$

Integrating gives

$$F_1''' = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 [Y Y'] + C_1 \quad \dots (17)$$

Integrating again gives

$$F_1'' = \frac{E}{4} \left(\frac{n\pi}{l} \right)^2 A^2 Y^2 + C_1 y + C_2 \quad \dots (18)$$

Since

F_1 is a function of y only then $\frac{\partial^2 F_1}{\partial x^2} = \frac{\partial^2 F_1}{\partial x \partial y} = 0$ and $\frac{\partial^2 F_1}{\partial y^2} = F_1''$ is the only derivative of F_1 which is of any consequence , since this constitutes a stress in the x direction. The constants of integration C_1 and C_2 are used to satisfy the displacement boundary conditions at the plate loaded ends, therefore,

$$F_1 = \iint \left\{ \frac{E}{4} \left(\frac{n\pi}{l} \right)^2 A^2 Y^2 + C_1 y + C_2 \right\} dy dy \quad \dots (19)$$

For F_2 equation(15) is used and obtained

$$\nabla^4 \left(F_2 \cos \frac{2n\pi x}{l} \right) = \left[F_2'''' - 2 \left(\frac{2n\pi}{l} \right)^2 F_2'' + \left(\frac{2n\pi}{l} \right)^4 F_2 \right] \cos \frac{2n\pi x}{l}$$

$$= \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 [(Y')^2 - Y Y''] \cos \frac{2n\pi x}{l}$$

hence

$$F_2'''' - 2 \left(\frac{2n\pi}{l} \right)^2 F_2'' + \left(\frac{2n\pi}{l} \right)^4 F_2 = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 [(Y')^2 - Y Y''] \quad \dots (20)$$

Putting $F_2 = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 \psi(y)$ and substituting in equation (20) gives

$$\psi'''' - 2 \left(\frac{2n\pi}{l} \right)^2 \psi'' + \left(\frac{2n\pi}{l} \right)^4 \psi = (Y')^2 - Y Y'' \quad \dots (21)$$

If the function ψ can be obtained from equation (21) the stress function F_2 is known. The function ψ is composed of two parts, the particular integral solution and the complementary function solution. The particular integral solution depends on the function Y and can only be obtained for a specified Y .

The complementary function solution is as follows:

$$\psi_{C.F} = B_1 \sinh \frac{2n\pi y}{l} + B_2 \cosh \frac{2n\pi y}{l} + B_3 y \sinh \frac{2n\pi y}{l} + B_4 y \cosh \frac{2n\pi y}{l} \dots (22)$$

The coefficients B_1, B_2, B_3, B_4 are used in conjunction with the particular integral solution to satisfy the boundary conditions on the unloaded edges.

assumed for the present that ψ can be obtained and hence F_2 using equations (20), (21) and (22) and continue with the analysis.

Boundary Conditions on the Loaded Ends

These are zero shear stress and uniform compression, i.e.

$$x = 0, l \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = 0 \quad \dots (23)$$

$$\left. \begin{array}{l} x = 0 \\ x = l \end{array} \right\} \begin{array}{l} u = 0 \text{ across the plate} \\ u = \text{constant} \end{array} \quad \dots (24)$$

using equation (13) for F gives

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = -\frac{\partial^2}{\partial x \partial y} \left[F_1(y) + F_2(y) \cos \frac{2n\pi x}{l} \right]$$

Hence

$$\tau_{xy} = F_2' \frac{2n\pi}{l} \sin \frac{2n\pi x}{l} \quad \dots (25)$$

Thus condition equation (23) is automatically satisfied.

To examine condition equation (24) equation (1) is recalled.

gives

$$u = \int_0^x \left[\epsilon_x - \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] dx \quad \dots (26)$$

Substituting for w and ϵ_x gives

$$u = \int_0^x \left\{ \frac{1}{E} \left(\frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2} \right) - A^2 Y^2 \cos^2 \frac{n\pi x}{l} \left(\frac{n\pi}{l} \right)^2 \right\} dx \quad \dots (27)$$

Substituting for F from equation (13) gives

$$u = \int_0^x \left\{ \frac{1}{E} \left[F_1'' + F_2'' \cos \frac{2n\pi x}{l} + v \left(\frac{2n\pi}{l} \right)^2 F_2 \cos \frac{2n\pi x}{l} \right] - \frac{1}{2} A^2 Y^2 \cos^2 \frac{n\pi x}{l} \left(\frac{n\pi}{l} \right)^2 \right\} dx \quad \dots (28)$$

at $x = 0$, $u = 0$ and

$$\text{at } x = l \text{ , } u = -\bar{u} = \frac{l}{E} \left[F_1'' - \frac{E}{4} A^2 Y^2 \left(\frac{n\pi}{l} \right)^2 \right] \quad \dots (29)$$

substituting for F_1'' from equation (18) gives

$$-\bar{u} = \frac{l}{E} \left\{ \frac{E}{4} A^2 Y^2 \left(\frac{n\pi}{l} \right)^2 + C_1 Y + C_2 - \frac{E}{4} A^2 Y^2 \left(\frac{n\pi}{l} \right)^2 \right\} \text{ i.e. } -\bar{u} = \frac{l}{E} (C_1 Y + C_2) \quad \dots (30)$$

$$\text{since } \bar{u} \text{ is constant across the plate then , } C_1 = 0 \text{ and } C_2 = -\frac{E\bar{u}}{l} \quad \dots (31)$$

F_1'' is now fully known in terms of the end displacement $\bar{u}$ and the out – of – plane deflection coefficient A. this is

$$F_1'' = -\frac{E\bar{u}}{l} + \frac{E}{4} A^2 Y^2 \left(\frac{n\pi}{l} \right)^2 \quad \dots (32)$$

On the assumption that ψ is also known it can be evaluated the strain energy of the middle surface stresses V_M from equation (9). This is

$$V_M = \frac{t}{2E} \int_0^b \int_0^l \left\{ \left[-\left(\frac{2n\pi}{l} \right)^2 F_2 \cos \frac{2n\pi x}{l} + F_1'' + F_2'' \cos \frac{2n\pi x}{l} \right]^2 - 2(1+v) \left[-\left(\frac{2n\pi}{l} \right)^2 F_2 \cos \frac{2n\pi x}{l} \left(F_1'' + F_2'' \cos \frac{2n\pi x}{l} \right) - \left(\frac{2n\pi}{l} \right)^2 (F_2')^2 \sin^2 \frac{2n\pi x}{l} \right] \right\} dx dy \quad \dots (33)$$

Integrating in the x-direction gives

$$V_M = \frac{t l}{4E} \int_0^b \left\{ \left[F_2'' - \left(\frac{2n\pi}{l} \right)^2 F_2 \right]^2 + 2(F_1'')^2 + 2(1+v) \left(\frac{2n\pi}{l} \right)^2 [F_2 F_2'' + (F_2')^2] \right\} dy \quad \dots (34)$$

Substituting for F_1'' from equation (32) and using $F_2 = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 \psi$, gives

$$V_M = \frac{t l}{4E} \int_0^b \left\{ \frac{E^2}{4} \left(\frac{n\pi}{l} \right)^4 A^4 \left[\psi'' - \left(\frac{2n\pi}{l} \right)^2 \psi \right]^2 + \frac{E^2}{8} \left(\frac{n\pi}{l} \right)^4 A^4 Y^4 - \frac{E^2}{l} \left(\frac{n\pi}{l} \right)^2 A^2 Y^2 \bar{u} + 2 \frac{E^2}{l^2} \bar{u}^2 + 2(1+v) \left(\frac{2n\pi}{l} \right)^2 \frac{E^2}{4} \left(\frac{n\pi}{l} \right)^4 A^4 [\psi \psi'' + (\psi')^2] \right\} dy \quad \dots (35)$$

From a consideration of zero shear stress on the unloaded edges of the plate it can be get

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 \psi' \sin \frac{2n\pi x}{l} \cdot \left(\frac{2n\pi}{l} \right) = 0 \text{ , at } y = 0, b. \text{ If this is true for all } x \text{ then}$$

ψ' must equal zero on the boundaries $y=0, b$

Rearranging equation (35) gives the strain energy of middle a surface stresses finally as

$$V_M = \frac{ltE}{32} \left(\frac{n\pi}{l} \right)^4 \int_0^b A^4 \left[Y^4 + 2 \left(\psi'' - \left(\frac{2n\pi}{l} \right)^2 \psi \right)^2 \right] dy + \int_0^b \frac{E\bar{u}^2 t}{2l} dy - \frac{Et}{4} \left(\frac{n\pi}{l} \right)^2 \int_0^b A^2 \bar{u} Y^2 dy \quad \dots (36)$$

Thus, the strain energy of the plate bending can be deduced

$$V_B = \frac{eD}{4} A^2 \left\{ \int_0^b \left[Y'' - \left(\frac{2n\pi}{l} \right)^2 Y \right]^2 dy + 2(1-\nu) \left(\frac{n\pi}{l} \right)^2 [YY']_0^b \right\} \text{Where } D = \frac{Et^3}{12(1-\nu^2)} \quad \dots (37)$$

The total strain energy of the compressed plate is obtained from the sum of equation (35) and equation (37)

$$\text{i.e } V_T = V_M + V_B \quad \dots (38)$$

applying the principle of minimum strain energy and differentiate V_T with respect to A This gives

$$\frac{\partial V_T}{\partial A} = 0 = \psi'' \frac{ltE}{8} \left(\frac{n\pi}{l} \right)^4 A^3 \int_0^b \left[\left(Y^4 + 2 \left[\psi'' - \left(\frac{2n\pi}{l} \right)^2 \psi \right]^2 \right) \right] dy - \frac{ltE}{2} \left(\frac{n\pi}{l} \right)^2 A \int_0^b \frac{\bar{u}}{l} Y^2 dy + \frac{lD}{2} A \cdot f_1^2 \cdot (y) \quad \dots (39)$$

rearranging equation (39)

$$A^2 = \frac{4 \left(\frac{l}{n\pi} \right)^2 \bar{u} \int_0^b Y^2 dy - \frac{4D}{Et} \left(\frac{l}{n\pi} \right)^4 f_1(y)}{\int_0^b \left[Y^4 + 2 \left[\psi'' - \left(\frac{2n\pi}{l} \right)^2 \psi \right]^2 \right] dy} \quad \dots (40)$$

it is convenient to use is a non-dimensionalised form of this equation by putting

$$\frac{\bar{u}}{l} = u^* \cdot \frac{\pi^2 D}{b^2 t} \cdot \frac{1}{E} \quad \dots (41)$$

and using the relationship of D to get

$$\left(\frac{A}{t} \right)^2 = \frac{\frac{1}{3(1-\nu^2)} \left(\frac{l}{nb} \right)^2 2[u^* \int Y^2 dy - \left(\frac{lb}{n} \right)^2 f_1 / \pi^4]}{\int \left(Y^4 + 2 \left(\bar{\psi} - \left(\frac{2n\pi}{l} \right)^2 \psi \right)^2 \right) dy} \quad \dots (42)$$

Thus for any specified deflection function Y the deflection magnitude A corresponding to any end displacement u^* is obtained from (42). it can be shown that if $u^* \int Y^2 dy < \left(\frac{lb}{n} \right)^2 \frac{f_1}{\pi^4}$ then A/t , is imaginary and this the plate has not buckled.

Buckling is initiated when $u^* \int Y^2 dy < \left(\frac{lb}{n} \right)^2 \frac{f_1}{\pi^4}$

$$\text{i.e when } u^* = \left(\frac{lb}{n} \right)^2 \frac{f_1}{\pi^4 \int Y^2 dy}$$

This value of u^* is identical with the non-dimensional buckling stress (K). The load on the plate corresponding to the end displacement u^* is obtained by integrating the stresses across the plate, the stress at any point is given by

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2}{\partial y^2} \left[F_1 + F_2 \cos \frac{2n\pi x}{l} \right] \quad \dots (43)$$

$$\text{i.e. } \sigma_x = F_1'' + F_2'' \cos \frac{2n\pi x}{l}$$

Substituting for F_1'' from equation (32) and using

$$F_2 = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 \psi \text{ gives}$$

$$\sigma_x = -\frac{Eu}{l} + \frac{E}{4} \left(\frac{n\pi}{l} \right)^2 A^2 \left[Y^2 + \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 \psi'' \cos \frac{2n\pi x}{l} \right] \quad \dots (44)$$

$$\text{i.e. } \sigma_x = -\frac{Eu}{l} + \frac{E}{4} \left(\frac{n\pi}{l} \right)^2 A^2 \left[Y^2 + 2\psi'' \cos \frac{2n\pi x}{l} \right] \quad \dots (45)$$

σ_x can be non-dimensional by writing $\sigma_x = -\sigma_x^* \pi^2 D / b^2 t$ hence

$$\sigma_x^* = -\frac{Eu}{l} \left(-\frac{b^2 t}{\pi^2 D} \right) + \frac{E}{4} \left(\frac{n\pi}{l} \right)^2 A^2 \left[Y^2 + 2\psi'' \cos \frac{2n\pi x}{l} \right] \left(-\frac{b^2 t}{\pi^2 D} \right) \quad \dots (46)$$

$$\text{i.e. } \sigma_x^* = u^* - \left(\frac{nb}{l} \right)^2 \cdot 3(1 - \nu^2) \left(\frac{A}{t} \right)^2 \left[Y^2 + 2\psi'' \cos \frac{2n\pi x}{l} \right] \quad \dots (47)$$

The total load on the plate is obtained from the integral

$$P = \int_0^b \sigma_x t \, dy$$

non-dimensionalising by writing $P = -\frac{P^* \pi^2 D}{b}$ gives

$$P^* = \int_0^b \frac{\sigma_x^*}{b} \, dy \quad \dots (48)$$

Substituting for σ_x^* from equation (48) and integrating gives

$$P^* = u^* - \left(\frac{nb}{l} \right)^2 3(1 - \nu^2) \left(\frac{A}{t} \right)^2 \left\{ \frac{1}{b} \int_0^b \left[Y^2 + 2\psi'' \cos \frac{2n\pi x}{l} \right] dy \right\} \quad \dots (49)$$

It can be shown the integral of ψ'' is simply ψ' and since this is zero on the boundaries $y=0, b$ due to the requirement of zero shear stress at the boundaries then equation (49) can be written

$$P^* = u^* - \left(\frac{nb}{l} \right)^2 3(1 - \nu^2) \left(\frac{A}{t} \right)^2 \cdot \frac{1}{b} \int_0^b Y^2 \, dy \quad \dots (50)$$

Substituting for $\left(\frac{A}{t} \right)^2$ from equation (4) gives

$$P^* = u^* - \frac{\left(\frac{1}{b} \int_0^b Y^2 \, dy \left[u^* \int Y^2 - \left(\frac{lb}{n} \right)^2 \frac{f_1}{\pi^4} \right] \right)}{\int \left[Y^4 + 2 \left(\psi'' - \left(\frac{2n\pi}{l} \right)^2 \psi \right)^2 \right] dy} \quad \dots (51)$$

The stiffness of the plate against end compression after buckling is given by $\frac{\partial P}{\partial u}$ or

in Mon-dimensional form by $\frac{\partial P^*}{\partial u^*}$, this gives

$$\frac{\partial P^*}{\partial u^*} = \left[1 - \frac{1}{b} \left[\frac{\int Y^2 \, dy}{f_3} \right]^2 \right] \quad \dots (52)$$

Where

$$f_3 = \int \left[Y^4 + 2 \left(\psi'' - \left(\frac{2n\pi}{l} \right)^2 \psi \right)^2 \right] dy \quad \dots (53)$$

before buckling $P^* = u^*$ and $\frac{\partial P^*}{\partial u^*} = 1$

The Ratio of Post-Buckling Compressional Stiffness to Pre-Buckling Stiffness

From the above analysis, it can be deduced

$$\frac{\text{Post Buckling stiffness}}{\text{Pre Buckling stiffness}} = \frac{\left[1 - \frac{1}{b} \frac{(\int Y^2 dy)^2}{f_3}\right]}{1} = 1 - \frac{(\int Y^2 dy)^2}{b f_3} \quad \dots (54)$$

Since the plate stiffness before buckling is related to Young's modulus, E, then the plate stiffness after buckling can be related to same fictitious modulus E^* and it can be deduced

$$\frac{E^*}{E} = 1 - \frac{\int Y^2 dy}{b \int \left[Y^4 + 2 \left(\psi'' - \left(\frac{2n\pi}{l} \right)^2 \psi \right)^2 \right] dy} \quad \dots (55)$$

It will be found that equation (55) is identical to what was given in Rhodes, 2003 [7] with $n=1$ and with different way.

The load –displacement curve of plate is given by the graph shown in Fig.(2). This consists of two lines one of slope proportional to E and the other of slope proportional to E^* . Both lines interest at the load.

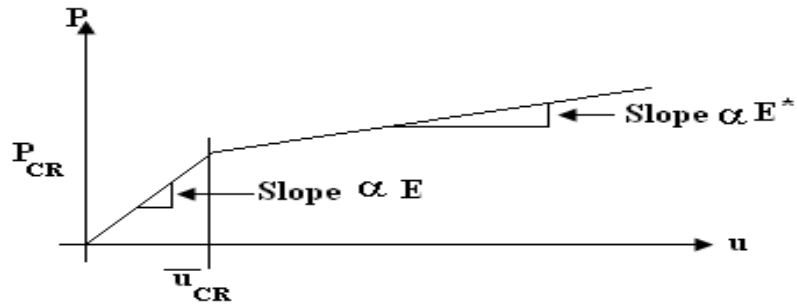


Figure (2) Load – displacement curve of plate

To obtain the stress at any point, substituting for $\left(\frac{A}{t}\right)^2$ from equation (42) in equation (47) and obtain

$$\sigma_x^* = u^* \cdot \left[\frac{[Y^2 + 2\psi'' \cos \frac{2n\pi x}{l}] [u^* \int Y^2 dy - \frac{(lb)^2}{\pi^4} f_1]}{f_3} \right] \quad \dots (56)$$

$$\text{and } \frac{\partial \sigma^*}{\partial u^*} = 1 - \frac{[Y^2 + 2\psi'' \cos \frac{2n\pi x}{l}] \int Y^2 dy}{f_3} \quad \dots (57)$$

This indicates that the stress at any point on the plate changes linearly in the post buckling range. The rate of change depend on the particular point chosen on the plate

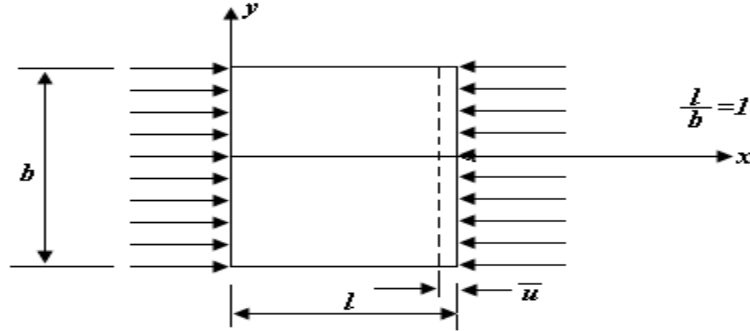
. Before buckling $\frac{\partial \sigma^*}{\partial u^*} = 1$

Therefore,

$$\frac{\left(\frac{\partial \sigma^*}{\partial u^*}\right)_{\text{Post-buckling}}}{\left(\frac{\partial \sigma^*}{\partial u^*}\right)_{\text{Pre-buckling}}} = 1 - \frac{[Y^2 + 2\psi'' \cos \frac{2n\pi x}{l}] \int Y^2 dy}{f_3} \quad \dots (58)$$

Results and Discussion

In this paper, the post-buckling behavior of a uniformly compressed, square, simply supported plate with the unloaded edges free from normal and shear stresses is examined as shown in Fig.(3) to show the effect of buckling on the plates stiffness.



Figure(3)Effect of buckling on the plates stiffness.

to simply the analysis of the y- direction is taken as the center line of the plate and assume the buckled form

$$Y = \cos \frac{\pi y}{b}$$

This is exactly the same deflected form of initial buckling analysis by specifying $Y = \sin \frac{\pi y}{b}$ and measuring y from the plate edge.

It can be deduced therefore state values for the bulking stress and for f_1 from this work or alternating f_1 and σ_{cr} can be evaluated using the following equations with the limits of integration begin $\mp \frac{b}{2}$

$$f_1(y) = \int_0^b \left\{ \left[Y'' - \left(\frac{n\pi}{l} \right)^2 Y \right] + 2(1 - \nu) \left(\frac{n\pi}{l} \right)^2 \left[Y Y'' + (\dot{Y})^2 \right] \right\} dy \quad \dots (59)$$

$$f_2(y) = \int_0^b \left\{ \left[\left(\frac{n\pi}{l} \right)^2 Y^2 \right] \right\} dy \quad \dots (60)$$

$$\text{and } \sigma_{cr} = \frac{D}{t} \frac{f_1}{f_2} \quad \dots (61)$$

Hence

for $n=1$ and $l=b$ it can be get $u_{cr}^* = \sigma_{cr}^* = P_{cr}^* = K=4$ and $f_1 = 2 \cdot \frac{\pi^4}{b^3}$

now evaluate the function $\psi(y)$ using equation.(21)

$$\text{i.e. } \psi'''' - 2(2n\pi/l)^2 \psi'' + (2n\pi/l)^4 \psi = (Y')^2 - Y Y''$$

$$\text{now } (Y')^2 - Y Y'' = \left(\frac{\pi}{b} \right)^2 \sin^2 \frac{\pi y}{b} + \left(\frac{\pi}{b} \right)^2 \cos^2 \frac{\pi y}{b} = \left(\frac{\pi}{b} \right)^2$$

the particular integral solution for ψ is there for easily obtained as

$$\psi_{P.I} = \left(\frac{\pi}{b} \right)^2 / \left(\frac{2n\pi}{l} \right)^4 = \frac{l^4}{16\pi^2 b^2} \quad (\text{using } n=1)$$

The complementary function solution is given by equation (22). Therefore with n set equal to one the complete solution is

$$\psi = \psi_{P.I} + \psi_{C.F}$$

hence

$$\psi = \frac{l^4}{16\pi^2 b^2} + B_1 \sinh \frac{2\pi y}{b} + B_2 \cosh \frac{2\pi y}{b} + B_3 y \sinh \frac{2\pi y}{b} + B_4 y \cosh \frac{2\pi y}{b}$$

Since the symmetry is existed about the x-axis i.e. about the line $y=0$ then anti-symmetrical terms in the expression for ψ is disregard, that is, $B_1 = B_4 = 0$ since, $\sinh \frac{2\pi y}{b}$ and $\cosh \frac{2\pi y}{b}$ are anti-symmetrical.

to evaluate B_2 and B_3 the boundary condition of stress free edges, is used

$$\text{i.e. } \tau_{xy} = \sigma_y = 0 \text{ at } y = \pm \frac{b}{2}$$

$$\begin{aligned} \text{now } \tau_{xy} &= -\frac{\partial^2 F}{\partial x \partial y} = -\frac{\partial^2}{\partial x \partial y} [F_1 + F_2 \cos \frac{2n\pi x}{l}] \\ &= F_2' \left(\frac{2\pi}{l} \right) \sin \frac{2\pi x}{l} \quad (\text{for } n = 1) \end{aligned}$$

Since

$$F_2 = \frac{E}{2} \left(\frac{n\pi}{l} \right)^2 A^2 \psi$$

$$\text{Then } \tau_{xy} = \frac{E}{2} \left(\frac{\pi}{l} \right)^2 A^2 \left(\frac{2\pi}{l} \right) \psi' \sin \left(\frac{2\pi x}{l} \right) \quad (\text{for } n=1)$$

If $\tau_{xy} = 0$ at $y = \pm \frac{b}{2}$ for all x , then $\psi' = 0$ at $y = \pm \frac{b}{2}$

$$\psi' = \frac{d\psi}{dy} = B_2 \frac{2\pi}{b} \sinh \frac{2\pi y}{b} + B_3 \left(y \frac{2\pi}{b} \cosh \frac{2\pi y}{b} + \sinh \frac{2\pi y}{b} \right) = 0$$

at $y = b/2$

$$\therefore B_2 \left(\frac{2\pi}{b} \right) \sinh \pi + B_3 \left(\frac{b}{2} \cdot \frac{2\pi}{b} \cdot \cosh \pi + \sinh \pi \right) = 0$$

Hence

$$B_2 = -0.661 b B_3$$

$$\text{Also } \sigma_y = \frac{\partial^2}{\partial x^2} (F_1 + F_2 \cos \frac{2n\pi x}{l})$$

$$= -\frac{E}{2} \left(\frac{\pi}{l} \right)^2 A^2 \psi \left(\frac{2\pi}{l} \right)^2 \cos \frac{2n\pi x}{l} \quad (\text{for } n = 1)$$

Therefore if $\sigma_y = 0$ at $y = \pm b/2$ for all x , then $\psi = 0$ at $y = \pm \frac{b}{2}$

Hence

$$\frac{l^4}{16\pi^2 b^2} + B_2 \cosh \frac{2\pi y}{b} + B_3 y \sinh \frac{2\pi y}{b} = 0 \quad \text{at } y = \frac{b}{2}$$

$$\therefore \frac{b^2}{16\pi^2} - 0.661 b B_3 \cosh \pi + B_3 \frac{b}{2} \sinh \pi = 0 \quad (\text{for } l = b)$$

hence

$$B_3 = \frac{b^2}{16\pi^2} / (0.661 b \cosh \pi - 0.5 b \sinh \pi)$$

$$\text{i.e. } B_3 = \frac{0.0331 b}{\pi^2}$$

$$\text{and } B_2 = -\frac{0.02188 b^2}{\pi^2}$$

the equation for ψ is now

$$\psi = \frac{b^2}{16\pi^2} - 0.02188 \frac{b^2}{\pi^2} \cosh \frac{2\pi y}{b} + 0.0331 \frac{by}{\pi^2} \sinh \frac{2\pi y}{b}$$

In order to obtain f_3 the term

$$\int_{-b/2}^{b/2} \left[\psi'' - \left(\frac{2\pi}{l} \right)^2 \psi \right]^2 dy$$

is required to obtain

$$\psi'' = -0.02188 \frac{b^2}{\pi^2} \left(\frac{2\pi}{b} \right)^2 \cosh \frac{2\pi y}{b} + 0.0331 \frac{b}{\pi^2} \left[\left(\frac{2\pi}{b} \right)^2 y \sinh \frac{2\pi y}{b} + 2 \left(\frac{2\pi}{b} \right) \cosh \frac{2\pi y}{b} \right]$$

$$\text{and} (\psi'' - (\frac{2\pi}{l})^2 \psi)^2 = (-\frac{1}{4} + 2 * \frac{0.0662}{\pi} \cosh(\frac{2\pi y}{b}))^2$$

$$\therefore \int_{-b/2}^{b/2} [\psi'' - (\frac{2\pi}{l})^2 \psi]^2 dy$$

$$= \int_{-b/2}^{b/2} [\frac{1}{16} - \frac{0.0662}{\pi} \cosh \frac{2\pi y}{b} + \frac{4}{\pi^2} (0.0662)^2 \cosh^2 \frac{2\pi y}{b}] dy$$

this integral can easily be evaluated giving

$$\int_{-b/2}^{b/2} \left[\psi'' - \left(\frac{2\pi}{l} \right)^2 \psi \right]^2 dy = 0.02377b$$

$$\text{From equation (53)} f_3 = \int_{-b/2}^{b/2} Y^4 dy + 2 * 0.02377b$$

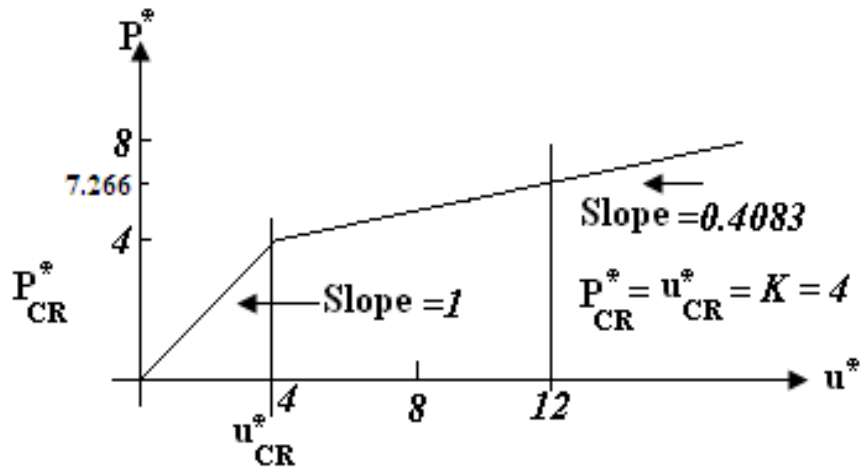
$$\text{and} \int_{-b/2}^{b/2} Y^4 dy = \int_{-b/2}^{b/2} \cos^4 \frac{\pi y}{b} dy = \frac{3}{8}b$$

the ratio of post – buckling to pre-buckling compressional stiffness can be obtained from equation (54)

$$\text{i.e.} \frac{E^*}{E} = 1 - \frac{\left(\frac{b}{2}\right)^2}{b\left(\frac{3}{8}b + 2 * 0.02377b\right)} = 1 - 0.5917$$

$$\therefore \frac{E^*}{E} = 0.4083$$

Thus the plate loses about 3/5 the of its initial compressional stiffness after buckling and the load –end displacement curve is shown in Fig.(4).



Figure(4) Load- end displacement curve of the plate

To examine the growth of out - of- plane deflections equation (42) is used to obtain

$$\left(\frac{A}{t}\right)^2 = \frac{1}{3(1-v^2)} \cdot \frac{u^* \cdot \frac{b}{2} - \left(\frac{2\pi}{b^3}\right)^4 \cdot \left(\frac{b^4}{\pi^4}\right)}{\left(\frac{3}{8}b + 2.0.02377b\right)}$$

Taking $v=0.3$ gives

$$\left(\frac{A}{t}\right)^2 = \frac{1}{2.73} \cdot \frac{0.5u^* - 2}{0.42254} = 0.4334(u^* - 4)$$

hence

$$\left(\frac{A}{t}\right)^2 = 0.4334(u^* - u_{cr}^*)$$

Since

$$(u^* - u_{cr}^*) = 1/0.4083 (P^* - P_{cr}^*)$$

It can be written

$$\left(\frac{A}{t}\right)^2 = \frac{0.4334}{0.4083} (P^* - P_{cr}^*) = 1.0615(P^* - P_{cr}^*)$$

This relationship is plotted in Fig.(5).

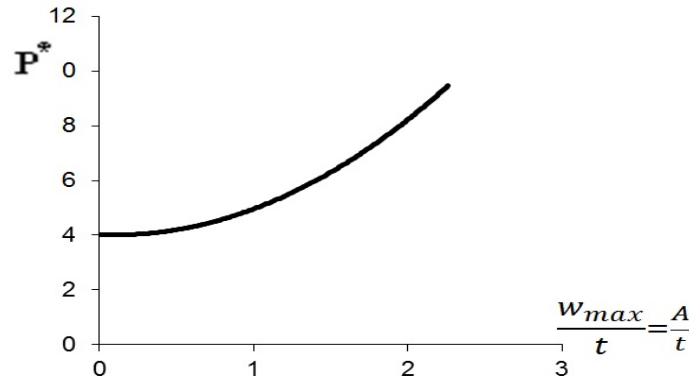


Figure (5) relation between A/t with load

The membrane stress σ_x at any point on the plate can be obtained in its non-dimensional form, σ_x^* , from equation (47). i.e.

$$\sigma_x^* = u^* - 3(1 - \nu^2) \left(\frac{A}{t}\right)^2 \left[Y^2 + 2\psi'' \cos \frac{2\pi x}{l} \right] \quad (\text{for } l=b \text{ and } n=1)$$

substituting for $\left(\frac{A}{t}\right)^2$, Y^2 and ψ'' gives

$$\sigma_x^* =$$

$$u^* - 2.73 * 1.0615(P^* - P_{cr}^*) \left\{ \cos^2 \frac{\pi y}{b} + 2(-0.2188) * 4 \cosh \frac{2\pi y}{b} \right. \\ \left. + \frac{0.0331}{\pi^2} \left[4\pi^2 \frac{y}{b} \sinh \frac{2\pi y}{b} + 4\pi \cosh \frac{2\pi y}{b} \right] * \cos \frac{2\pi x}{l} \right\}$$

This can be evaluated for any values of x and y if u^* is given

Note that at $x = \frac{l}{4}, \frac{3l}{4}, \cos \frac{2\pi x}{l} = 0$

$$\text{And since } u^* = u_{cr}^* + \frac{1}{0.4083} (P^* - P_{cr}^*)$$

Then the stress distribution across the plate at $x = l/4, 3l/4$, is

$$i.e. \sigma_x^* = u_{cr}^* + \frac{P^* - P_{cr}^*}{0.4083} - 2.898(P^* - P_{cr}^*) \left(\cos \frac{\pi y}{b} \right)^2$$

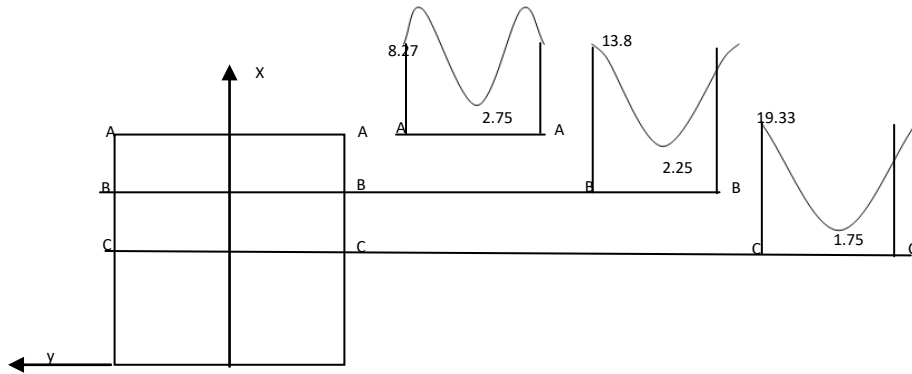
Also the edge stress is obtained by substituting $y=b/2$ in the general expression for σ_x^* giving

$$\sigma_x^*|_{edge} = u_{cr}^* + \frac{P^* - P_{cr}^*}{0.4083} - 2.898(p^* - p_{cr}^*) * 0.477 \cos \frac{2\pi x}{l}$$

$\sigma_x^*|_{edge}$ is maximum when $\cos \frac{2\pi x}{l} = -1$ i.e. when $x=l/2$

$$i.e \sigma_x^*|_{edge(max)} = u_{cr}^* + 3.8315(p^* - p_{cr}^*)$$

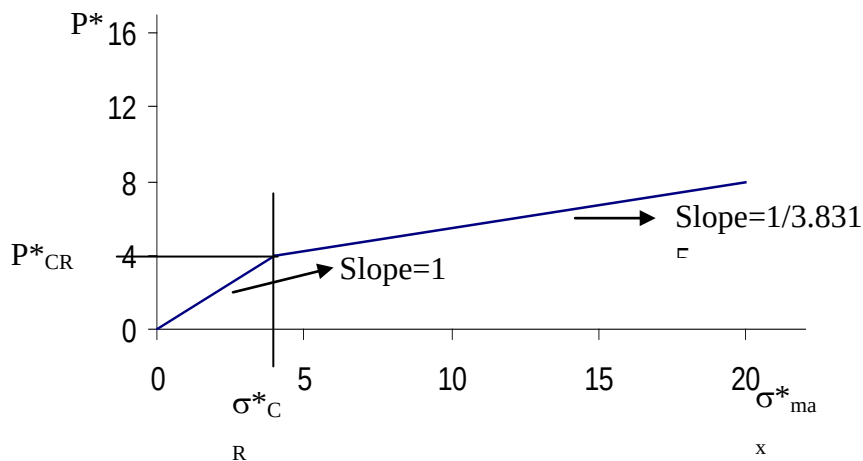
Stress distributions for a load of twice the buckling load are plotted as shown in Fig.(6).



Figure(6) Stress distribution of the load

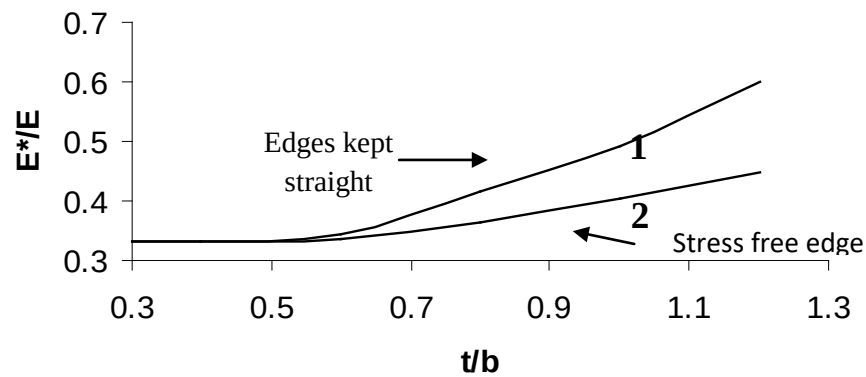
twice the buckling load is about the limited of accuracy of the one term solution derived, for further loading more terms are required in the solution to take into account the changes in deflected form that occur after buckling. As can be seen the maximum membrane stress incurred in the plate edge in line with the crest of the buckle. This is $\sigma_{x^*edge(max)}$. Using $\sigma_{cr}^* = u_{cr}^*$ It can be written

$\sigma_{max}^* - \sigma_{cr}^* = 3.8315(P^* - P_{cr}^*)$, therefore, after buckling the maximum stress increases at almost 4 times the pre-buckling rate with respect to load as in Fig.(7).



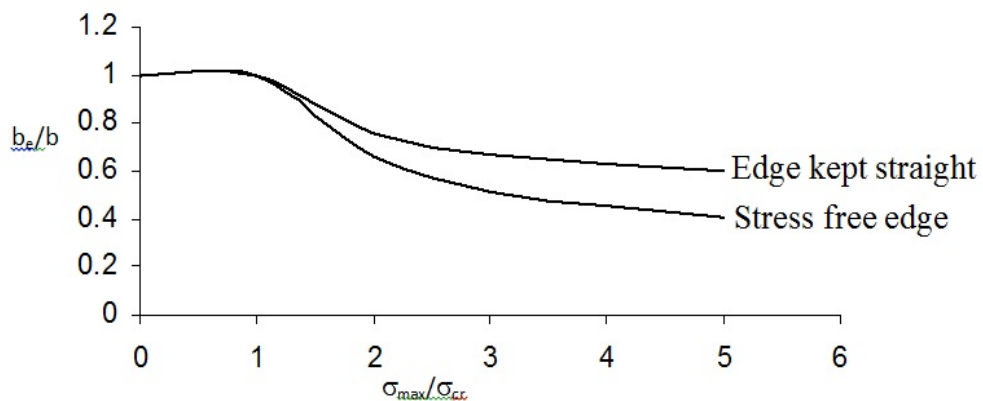
Figure(7) Critical load vs. max. stress due to buckling

Fig.(8) shows the variation of E^*/E with variation in plate length is shown above for the cases of straight edges and stress – free edges ,showing that for $l/b \rightarrow 0$ $E^*/E \rightarrow 1/3$ for both cases and as l/b increases the case of stress-free edges has lower stiffness after buckling. The curve(1) could easily be draw using equation (9) whereas the curve(2) required computer analysis.



Figure(8) Variation of E^*/E with variation in plate length

Fig.(9) shows the variation in $\frac{b_e}{b}$ with variation in σ_{max} for both types of conditions. As can be seen the stress- free condition gives lower effective widths and hence lower ultimate loads.



Figure(9) Variation in $\frac{b_e}{b}$ with variation in σ_{max}

CONCLUSIONS

The following points can be concluded:

- 1- In this paper the stiffness ratio of post buckling to pre- buckling is investigated.
- 2- The simply support plates loss about (3/5) of its initial compressional stiffness after buckling.
- 3- After buckling the maximum stress increases at almost four times the pre-buckling.

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