P- ISSN 1991-8941 E-ISSN 2706-6703
2024,(18), (02):248 – 256
Journal of University of Anbar for Pure Science (JUAPS) Open Access

Solve the Schrödinger Fractional-Order Boundary Value Problem by Laplace Transformation Method

Shwan Swara Fatah[1], Hozan Hilmi [2*], Karwan H.F. Jwamer[2]

[1] Department of Physics, College of Science, Charmo University, Chamchamal,46023, Iraq.
[2] Department of Mathematics, College of Science, University of Sulaimani, Sulaymaniyah city, 46001, Iraq.
hozan.mhamadhilmi@univsul.edu.iq

ARTICLE INFO

Received: 10 / 03 /2024
Accepted: 25/ 04 /2024
Available online: 26/ 12 /2024

10.37652/juaps.2024.147572.1213

Keywords:
Schrodinger operator; Fractional derivative; Fractional integral; Fractional boundary problem; Fractional Laplace transformation; Caputo derivative; Fractional differential equations.

ABSTRACT

Asymmetry plays a remarkable role in the transmission dynamics of novel fractional calculus. Only a few studies have mathematically modeled such asymmetry properties, and none has developed Schrödinger models that incorporate different symmetry developmental stages. Laplace transform can be used to discover the analytic (exact) solution to linear fractional differential equations. This study recommends Laplace transform as a technique to resolve the fractional-order Schrödinger equation with boundary conditions, where the fractional derivatives of Caputo and Riemann–Liouville are applied. It can be used to resolve fractional and ordinary differential equations. Afterward, the precise solution to a specific fractional differential equation example is determined. Results show that when novel Laplace transform is applied to the provided fractional differential equation boundary value problem, it yields accurate solutions without the need for lengthy calculations. In addition, we investigate a class of fractional boundary value problems with two boundary value conditions, namely, $\sigma \in (2,3]$ and $x \in (0, a]$, that involve orders of the fractional derivative. We show our primary findings through several cases. We provide multiple examples to highlight our main conclusions and prove the solution by employing Laplace transform after ascertaining solution's existence via fractional integral and integral operator methods.

1. Introduction

The Schrödinger equation of the fractional-order boundary value problem has been studied in the context of general Schrödinger equations with super-linear nonlinearity, fractional q-difference Schrödinger equations, and time-fractional subdiffusion equations with arbitrary elliptic differential operators. Various studies have addressed the solvability and existence of solutions for these types of equations by using different mathematical methods, such as variational approaches, meromorphic solutions, and iterative positive solutions. Although much remains to be learned about fractional Schrödinger equations, recently published literature on the subject offers insights into the mathematical strategies applied to resolve these problems [1],[5].

*Corresponding author at: Department of Mathematics, College of Science, University of Sulaimani, Sulaymaniyah city,46001, Iraq.
ORCID: https//https://orcid.org/0009-0007-5226-180X,
Tel: +964 7736990981
Email: hozan.mhamadhilmi@univsul.edu.iq

Our search results include a large number of studies describing common Schrödinger equations with super-linear nonlinearity, time-fractional subdiffusion equations with arbitrary elliptic differential operators, and boundary value problems for fractional q-difference Schrödinger equations. These studies have explored the solvability and existence of solutions for the presented problems by using various mathematical methodologies. Research on fractional Schrödinger equations is still in progress, and extant literature on the subject offers insights into how these problems are treated mathematically [4]–[6]. Fractional calculus is a useful tool when examining the memory and inherited features of various materials and processes [7]–[9]. Aside from its adoption in scientific and engineering domains, it is used in finance, biomedical engineering, signal processing, seismology, control systems, viscoelastic materials, and electrochemistry.

These applications involve a set of singularity-containing integro-differential equations, including fractional differential equations [10]–[12]. Some analytical and numerical methods, including those in [11], [13], and [14], have been introduced to resolve fractional differential equations. For example, a Schrodinger formula with Regge conditions, along with existence and uniqueness theorems, has been established for fractional-order differential equations. The Schrödinger equation with two specific boundary conditions is known as the T. Regge problem, which was explored in [6], [15], [16]. Prior studies, including [14], [17], and [18], have recommended some analytical or numerical methods for resolving fractional differential equations. With regard to resolving issues in engineering, social science, and science, researchers prefer to use integral transform instead of other mathematical techniques because of integral transform's main advantages, namely, simplicity, accuracy, and the ability to produce results without the need for laborious calculation [11], [19], [20]. Mathematical modeling and engineering benefit from fractional calculus [17], [21], [22]. For instance, certain novel integral transformations are intended to address differential and fractional differential equations. Kamal transformation [23] is one of the latest developments in fractional differential equations and integral transforms [24], [25]. Other developments include Sawi [26]–[28], Aboodh [29], Sumudu [30], and Rishi transformations [25], [31].

The study of the Schrodinger operator on half-axis R^+, with potential q compactly supported in the interval $[0, a]$, is the focus of this research. Specifically, we examine solutions for the fractional boundary value problem identified in [15]. The formula used in this study is

$$-{}^{LC}_{0}D^{\alpha}_{x}y(x) + q(x)y(x) = \lambda^2 p(x)y(x);\ x \in [0, a],\quad 2 < \alpha \le 3, \tag{1.1}$$

$$y(0) = y'(0) = 0, \qquad y(a) = y'(a), \tag{1.2}$$

in such a way that $q(x), p(x) \in L^+\,[0, a]$, where $L^+\,[0, a]$ comprises every integrable function. $f(x)$ is on $[0, a]$, $0 < n \le f(x) \le N < \infty$, $\alpha \in (2,3]$, and λ is a spectral parameter.

In this study, we work on the Schrodinger equation for the fractional-order boundary value problem for $2 < \alpha \le 3$

2. Materials and Methods

Preliminaries

This section provides several definitions, lemmas, and theorems that are essential to comprehending our results.

Definition 2.1 [11], [32]. The integral formula defines the gamma function.

$$\Gamma(z) = \int_0^\infty s^{z-1}\, e^{-s}\, ds$$

The integral completely converges when $Re(z) > 0$.

Definition 2.2 [6], [11] (**Fractional Integral [FI] of Order $\boldsymbol{\sigma}$**). For every $\sigma > 0$ and local integrable function $g(\rho)$, the right FI of order σ is defined as follows:

$${}_aI^{\sigma}_t\, g(\rho) = \frac{1}{\Gamma(\sigma)}\int_a^t (\rho - t)^{\sigma-1} g(t)dt, -\infty \le a < \rho < \infty.$$

Alternatively, it can be defined for the left FI as

$${}_tI^{\sigma}_b\, g(\rho) = \frac{1}{\Gamma(\sigma)}\int_t^b\ (t - \rho)^{\sigma-1} g(t)dt,\ -\infty < \rho \le b \le \infty.$$

Definition 2.3 [8], [11], [32] (**Fractional Derivative of Order $\boldsymbol{\sigma}$**). For any σ and $m = \lceil \sigma \rceil$, the Riemann–Liouville derivative of order σ has the following definition:

$${}_aD^{\sigma}_t h(t) = \frac{1}{\Gamma(m-\sigma)}\frac{d^m}{dt^m}\int_a^t (t - s)^{m-\sigma-1} h(s)ds.$$

Definition 2.4 [11], [22] . Assume that $\sigma > 0$, where $m = \lceil\sigma\rceil$. When $f(t)$ is an m-times differentiable function and the Liouville–Caputo derivative operator of order σ is applied, $t > a$ is defined as

$${}^{LC}_{a}D^{\sigma}_{t} f(u) = \frac{1}{\Gamma(m-\sigma)}\int_a^t (u - x)^{m-\sigma-1}\left(\frac{d}{dx}\right)^m f(x)dx,$$

or $\ {}^{LC}_{a}D^{\sigma}_{t} f(u) = \frac{1}{\Gamma(m-\sigma)}\int_\sigma^t \frac{f^m(x)}{(u-x)^{\sigma-m+1}}dx$. For $a = 0$, we introduce the notation ${}^{C}D^{\sigma}_{t} f(u) = {}^{C}D^{\sigma} f(u)$.

Remark 2.1 ([6], [11], [22], [32]). The following text shows some fundamental characteristics of fractional calculus. The fractional (integral and differential) operator is linear.

1. The definition of the composition between two Riemann–Liouville integrations of orders $\alpha\ and\ \beta$ is ${}_aI^{\alpha}_t\ {}_aI^{\beta}_t f(t) = {}_aI^{\beta}_t\ {}_aI^{\alpha}_t f(t) = I^{\beta+\alpha}_t f(t)$.
If $k \ge \alpha$, for $f(t) \in C[a, b]$ and at every point $t \in [a, b]$, ${}^{RL}_{a}D^{k}_{t}(\ {}_aI^{\alpha}_t f(t)) = {}^{RL}_{a}D^{k-\alpha}_{t} f(t)$.
2. The definition of the composition between the fractional (differentiation and integration) of the Liouville–Caputo operator of order α is
${}^{LC}_{a}D^{\alpha}_{t}({}_aI^{\alpha}_t f(t)) = f(t)$.
3. The definition of the composition between $m = \lceil\alpha\rceil$ and the fractional (integration and differentiation) of the Liouville–Caputo operator of order α is

$${}_aI^{\alpha}_t(\ {}^{LC}_{a}D^{\alpha}_{t} f(t)) = f(t) - \sum_{k=0}^{m-1}\frac{(t-a)^k}{k!} f^{(k)}(a).$$

In general, ${}^{LC}_{a}D^{k}_{t}({}_aI^{\alpha}_t f(t)) \ne {}_aI^{\alpha}_t\big(\ {}^{LC}_{a}D^{k}_{t} f(t)\big)$.

The Liouville differential and fractional integral are utilized. For function $t^n, n \ge 0$, and the Caputo operator yields ${}_aI^{\alpha}_t t^n = \frac{\Gamma(1+n)}{\Gamma(1+n+\alpha)} t^{n+\alpha}$ and

$${}^{LC}_{a}D^{k}_{t} t^n = \frac{\Gamma(1+n)}{\Gamma(1+n-\alpha)} t^{n-\alpha}.$$

Remark 2.2 (Association between the Caputo σ Order

Derivative and the Riemann–Liouville σ Order Derivative) [12, 29]. Assume that $n \in \mathbb{N}, \sigma \in [n-1, n)$ and suppose that $f(t)$ is a function, such that ${}^{c}D_{a}^{\sigma} f(t)$ and $D_{a}^{\sigma} f(t)$ exist. The relationship between the Riemann–Liouville and Caputo derivatives is

$$ {}^{c}D_{a}^{\sigma} f(t) = D_{a}^{\sigma} f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-\sigma}}{\Gamma(k+1-\sigma)} f^{(k)}(a). $$

Definition 2.5 [10], [33]. The definitions of the one- and two-parameter Mittag–Leffler functions are as follows: $E_{a}(s) = \sum_{n=0}^{\infty} \frac{s^n}{\Gamma(an+1)}$, $a > 0$, $s \in \mathbb{C}$, $E_{a,b}(s) = \sum_{n=0}^{\infty} \frac{s^n}{\Gamma(an+b)}$, $a > 0, b > 0, s \in \mathbb{C}$.

3. **Laplace Transform** [34], [35]

Let $x \in \mathbb{C}$. Typically, Laplace transform (LT) is described as $F(x) = L\{f(p)\} = \int_0^{\infty} e^{-xp} f(p) dp$.

Theorem 3.1. Multiplication Theorem (Convolution) [34], [35].

The convolution product of functions $g(t)$ and $y(t)$ is designated by the symbol *. We have

$(y * g)(t) = \int_0^t y(\tau) g(t-\tau) d\tau = Y(s) G(s)$,

where $Y(s) = L[y(t)]$, $G(s) = L[g(t)]$.

Property 3.1.Differentiationand Integration Theorems [34]

i. LT of the derivative of order k from $f(t)$ yields

$L[f^{(k)}(t)] = s^k F(s) - [s^{k-1} f(0) + s^{k-2} f'(0) + \cdots + f^{(k-1)}(0)] = s^k F(s) - \sum_{i=0}^{k-1} s^{k-i-1} f^{i}(0)$.

ii. **(Integration of an Original)**

We obtain $L\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}$.

3.1 Laplace Transform of the Fractional Integrals and Derivatives [7], [22], [32], [34]

1. Fractional Integrals (FIs)

If $\alpha \in [n-1, n)$, the Riemann–Liouville and Caputo FIs are the same for both cases.

$$ Q = I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} f(y) dy $$

Using the LT of the convolution product formula, we derive $L[Q] = \frac{1}{\Gamma(\alpha)} L[t^{\alpha-1}] L[f(t)] = \frac{F(s)}{s^{\alpha}}$.

2. Fractional Derivatives (FDs)

i. LT of Riemann–Liouville FD yields

$L[{}_{a}^{LR}D_{t}^{k} f(t)] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{k} D^{\alpha-k-1} f(0)$.

ii. LT of Caputo FD produces

$L[{}_{a}^{LC}D_{t}^{k} f(t)] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$.

3.2. Inverse of Laplace Transform [22], [35]

The equivalent opposite LT is

$f(t) = \frac{1}{2\pi i} \lim_{t\to\infty} \int_{\gamma-it}^{\gamma+it} F(s) e^{st} dt = L^{-1}[F(s)]$,

$\gamma \in R$, $i = \sqrt{-1}$.

3.3. Some Properties of Inverse Laplace Transform [22] [36]

1. $L^{-1}\left[\frac{s^{-(\alpha-\rho)}}{s^{\rho}-a}\right] = t^{\alpha-1} E_{\rho,\alpha}(at^{\rho})$, $\alpha, \rho > 0, s^{\alpha} > |a|$,
2. $L^{-1}\left[\frac{s^{-(\alpha-1)}}{s-a}\right] = t^{\alpha-1} E_{1,\alpha}(at) = E(t, \alpha-1, a)$,
3. $L^{-1}\left[\frac{s^{-\alpha}}{(s-a)^2}\right] = tE(t, \alpha, a) - \alpha E(t, \alpha+1, a)$,
4. $L^{-1}\left[\frac{s^{-\rho}}{(s-a)^3}\right] = \frac{1}{2} t^2 E(t, \rho, a) - \rho t E(t, \rho+1, a) + \frac{\rho(\rho+1)}{2} E(t, \rho+2, a)$,
5. $L^{-1}\left[\frac{1}{(s^{\alpha}+as^{\rho})^{n+1}}\right] = t^{\alpha(n+1)-1} \sum_{k=0}^{\infty} \frac{(-a)^k \binom{n+k}{k}}{\Gamma[k(\alpha-\rho)+(n+1)\alpha]} t^{k(\alpha-\rho)}$,where $0 \le \rho \le \alpha$,
6. $L^{-1}\left\{\frac{s^{\alpha}}{\lambda+s^{2\alpha}}\right\} = \frac{1}{t^{1-\alpha}} E_{2\alpha,\alpha}(-\lambda t^{2\alpha})$,
7. $L^{-1}\left\{\frac{1}{\lambda+s^{\alpha}}\right\} = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha})$,
8. $L^{-1}\left[\frac{s^{\gamma}}{s^{\alpha}+a\rho+b}\right] = t^{\alpha-\gamma-1} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-b)^n (-a)^p \binom{n+p}{p}}{\Gamma[p(\alpha-\rho)+(n+1)\alpha-\gamma]} t^{p(\alpha-\rho)+n\alpha}$,

Where $\rho \le \alpha, \gamma < \alpha, a \in \mathbb{R}$ or $|a| < s^{\alpha-\rho}, |b| < |s^{\alpha} + as^{\rho}|$.

4. Main Object

Lemma 4.1 [37]. Given $\varepsilon > 0$ and $n = [\varepsilon] + 1$, the equation's solution, ${}_{0}^{k}D_{S}^{\varepsilon} y(s) = 0$, is given by $y(s) = k_0 + k_1 s + k_2 s^2 + \cdots + k_{n-1} s^{n-1}$, where $k \in R, i = 0,1,2, \ldots, n-1$ are some constants. If $h \in k^n[0, a]$ is assumed, then

$I^{\varepsilon}\, {}_{0}^{k}D_{S}^{\varepsilon} y(s) = y(s) + k_0 + k_1 s + k_2 s^2 + \cdots + k_{n-1} s^{n-1}$

for some constants $k_i \in R$, $i = 0,1,2, \ldots, n-1$.

4.1 Solution of the Fractional-order Boundary Value Problem by Laplace Transform

In this section, we use LT to find the solution to our problem in the following cases. The fractional order is defined as

$-{}_{0}^{C}D_{x}^{\alpha} y(x) + q(x) y(x) = \lambda^2 p(x) y(x)$; $x \in [0, a], 2 \le \alpha \le 3$

$y(0) = y'(0) = 0$, $y(a) = y'(a)$.

We solve the fractional-order problem in this section via LT if it exists and if the conditions of the original function hold.

Case 1: Constant Coefficients

In this case, we suppose that $q(x) = M = p(x)$.

The fractional differential equation is

$-{}_0^C D_x^\alpha y(x) + My(x) = \lambda^2 My(x)$,

$\rightarrow {}_0^C D_x^\alpha y(x) + M(\lambda^2 - 1)y(x) = 0$.

By applying LT to both sides, we have

$L\{{}_0^C D_x^\alpha y(x)\} + L\{M(\lambda^2 - 1)y(x)\} = L\{0\}$.

Let $L\{y(x)\} = \int_0^\infty e^{-st} y(t)dt = Y(s)$ and $L\{0\} = 0$.

On the basis of Proposition 5 and the properties of LT, the LT of Caputo FD is

$L[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$.

We have $\alpha \in (2,3]$, so

$L[D^\alpha y(x)] = s^\alpha Y(s) - \sum_{k=0}^{2} s^{\alpha-k-1} y^{(k)}(0) = s^\alpha Y(s) - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) - s^{\alpha-3}y''(0)$.

The initial condition is $y(0) = y'(0) = 0$, so $L[D^\alpha y(x)] = s^\alpha Y(s) - s^{\alpha-3}y''(0)$.

Then,

$L\{{}_0^C D_x^\alpha y(x)\} + L\{M(\lambda^2 - 1)y(x)\} = L\{0\}$,

$s^\alpha Y(s) - s^{\alpha-3}y''(0) + M(\lambda^2 - 1)Y(s) = 0$,

$s^\alpha Y(s) + M(\lambda^2 - 1)Y(s) = s^{\alpha-3}y''(0)$,

$Y(s)\left(s^\alpha + M(\lambda^2 - 1)\right) = As^{\alpha-3}$,

where $A = y''(0)$. Thus, $Y(s) = \frac{As^{\alpha-3}}{s^\alpha - M(1-\lambda^2)}$.

The equivalent inverse LT is

$$f(x) = \frac{1}{2\pi i} \lim_{n\to\infty} \int_{\gamma-in}^{\gamma+in} F(s)e^{sx}dx = L^{-1}[F(s)],$$

$$\gamma \in R, i = \sqrt{-1}.$$

If it exists, we can use inverse LT to obtain the two sides.

$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{A\,s^{\alpha-2}}{s^\alpha - M(1-\lambda^2)}\right\}$. Given that $L\{y(x)\} = Y(s) \rightarrow y(x) = L^{-1}\{Y(s)\}$, the other side is related to the Mittag–Leffler function.

Two-parameter Mittag–Leffler functions are defined as

$E_{a,b}(s) = \sum_{n=0}^\infty \frac{s^n}{\Gamma(an+b)}$, $a > 0, b > 0, s \in C$.

On the basis of Section 3.3, we derive

$L^{-1}\left[\frac{s^{-(\beta-\alpha)}}{s^\alpha - a}\right] = t^{\beta-1}E_{\alpha,\beta}(at^\alpha)$, $a \in R$, $\alpha, \beta > 0, s^\alpha > |a|$.

Hence, we have $y(x) = Ax^2 E_{\alpha,3}(M(1-\lambda^2)x^\alpha)$, where A *refers to constant* $\alpha \in (2,3]$ and λ *is a complex number*.

It exists if $\alpha, \beta > 0, s^\alpha > |a|$, but in our case, $\beta = 3 > 0$ *and* $\alpha \in (2,3]$. Thus, $\alpha > 0$. The only condition for inverse LT to exist is $s^\alpha > M|(1-\lambda^2)|$.

The solution to the initial fractional value problem provided by 1.1 and 1.2 is

$$y(x) = Ax^2 E_{\alpha,3}(M(1-\lambda^2)x^\alpha),$$

$$x \in [0, a] \quad , \quad \alpha \in (2,3],$$

such that $A = y''(0)$ is any nonzero constant.

If $\alpha = 2$, $y(x) = gx^2E_{2,3}(M(1-\lambda^2)x^2) = Ax^2 \sum_{n=0}^\infty \frac{(M(1-\lambda^2)x^2)^n}{\Gamma(2n+3)} = \frac{A}{M(1-\lambda^2)} \sum_{n=0}^\infty \frac{\left(\sqrt{M(1-\lambda^2)}\,x\right)^{2n+2}}{(2n+2)!} = \frac{A}{M(1-\lambda^2)}\left(\cosh\left(\sqrt{M(1-\lambda^2)}\,x\right) - 1\right)$.

If $\alpha = 3$, $y(x) = Ax^2E_{3,3}(M(1-\lambda^2)x^3) = Ax^2 \sum_{n=0}^\infty \frac{(M(1-\lambda^2)x^3)^n}{\Gamma(3n+3)} = \frac{A}{(M(1-\lambda^2))^{\frac{2}{3}}} \sum_{n=0}^\infty \frac{\left(\sqrt[3]{M(1-\lambda^2)}x\right)^{3n+2}}{(3n+2)!}$.

Case 2: Variable Coefficients

In this case, we assume that weight function $p(x) = 1$ and that $q(x)$ is any continuous function and let $q(x)y(x) = f(x)$.

The fractional differential equation becomes

$-{}_0^C D_x^\alpha y(x) + f(x) = \lambda^2 y(x)$

$\rightarrow {}_0^C D_x^\alpha y(x) + \lambda^2 y(x) = f(x)$.

By applying LT to both sides, we derive

$L\{{}_0^C D_x^\alpha y(x)\} + L\{\lambda^2 y(x)\} = L\{f(x)\}$.

Let $L\{y(x)\} = \int_0^\infty e^{-st}y(t)dt = Y(s)$ and $L\{f(x)\} = \int_0^\infty e^{-st}f(t)dt = F(s)$.

From the properties of LT for $\alpha \in (2,3]$, we have

$$L[D^\alpha y(x)] = s^\alpha Y(s) - \sum_{k=0}^{2} s^{\alpha-k-1}y^{(k)}(0)$$

$= s^\alpha Y(s) - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) - s^{\alpha-3}y''(0)$.

The initial condition is $y(0) = y'(0) = 0$, so $L[D^\alpha y(x)] = s^\alpha Y(s) - s^{\alpha-3}y''(0)$.

Then, $L\{{}_0^C D_x^\alpha y(x)\} + L\{\lambda^2 y(x)\} = L\{f(x)\}$ and $s^\alpha Y(s) - s^{\alpha-3}y''(0) + \lambda^2 Y(s) = F(s)$, where $L\{f(x)\} = F(s) \rightarrow Y(s)(s^\alpha + \lambda^2) = c_1 s^{\alpha-3} + F(s)$ and $c_1 = y''(0) \rightarrow Y(s) = \frac{c_1 s^{\alpha-3}}{(s^\alpha+\lambda^2)} + \frac{F(s)}{(s^\alpha+\lambda^2)}$.

If it exists, we apply LT to both sides and obtain $L\{y(x)\} = Y(s) \rightarrow y(x) = L^{-1}\{Y(s)\}$ and $L\{f(x)\} = F(s) \rightarrow f(x) = L^{-1}\{F(s)\}$,

$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{c_1 s^{\alpha-3}}{(s^\alpha+\lambda^2)}\right\} + L^{-1}\left\{\frac{F(s)}{(s^\alpha+\lambda^2)}\right\}$.

On the basis of Section 3.3, we derive

$L^{-1}\left[\frac{s^{\alpha-3}}{s^\alpha+\lambda^2}\right] = x^2E_{\alpha,2}(-\lambda^2x^\alpha), s^\alpha > |\lambda^2|$, and $L^{-1}\left[\frac{1}{s^\alpha+\lambda^2}\right] = x^{\alpha-1}E_{\alpha,\alpha}(-\lambda^2x^\alpha)$.

This condition implies that $y(x) = c_1 x E_{\alpha,3}(-\lambda^2 x^\alpha) + x^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 x^\alpha) * f(x)$.

From the properties of the Mittag–Leffler function, we have $E_{\alpha,\alpha}(s) = \frac{1}{s} E_{\alpha,0}(s)$.

Hence,

$x^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 x^\alpha) = x^{\alpha-1} \frac{1}{-\lambda^2 x^\alpha} E_{\alpha,0}(-\lambda^2 x^\alpha) = \frac{-1}{\lambda^2 x} E_{\alpha,0}(-\lambda^2 x^\alpha)$.

Now, $y(x) = c_1 x^2 E_{\alpha,3}(-\lambda^2 x^\alpha) - \frac{1}{\lambda^2 x} E_{\alpha,0}(-\lambda^2 x^\alpha) * f(x)$.

Using the convolution property, we can write the general solution in the form

$$y(x) = c_1 x^2 E_{\alpha,3}(-\lambda^2 x^\alpha) - \int_0^x \frac{E_{\alpha,0}(-\lambda^2 (x-s)^\alpha)}{\lambda^2 (x-s)} * f(s) ds,$$

$$y(x) = c_1 x^2 E_{\alpha,3}(-\lambda^2 x^\alpha) - \int_0^x \frac{E_{\alpha,0}(-\lambda^2 (x-s)^\alpha)}{\lambda^2 (x-s)} * q(s) y(s) ds.$$

The symbol $*$ refers to the Volterra integral equation equivalent of 1.1 and 1.2.

Remark 4.1. If $\alpha \to 2$, the integral equation $*$ becomes

$$y(x) = c_1 x^2 E_{2,3}(-\lambda^2 x^2) - \int_0^x \frac{E_{2,0}(-\lambda^2 (x-s)^2)}{\lambda^2 (x-s)} * q(s) y(s) ds.$$

Then, $y(x) = \frac{c_1}{i|\lambda|} cosh(i|\lambda| x) - \int_0^x \frac{sin(i|\lambda|(x-s))}{i|\lambda|} * q(s) y(s) ds$.

Note [35], [38]: In Case 2 (variable coefficients), the only difficulty that we have is LT of two multiplication functions. We can use the following theorems and properties.

We can classify LT for the two multiplication functions through the following cases ($L\{g(x) * f(x)\}$).

9. If $g(x) = e^{ax}$ from the shifting theorem, LT is $L\{e^{ax} f(x)\} = F(s-a)$.
10. If $g(x)$ trigonometric or hyperbolic functions exist, we can represent these functions by using the exponential function and the note above.

$$\left\{ sinx = \frac{e^{ix} - e^{-ix}}{2i},\ cosx = \frac{e^{ix} + e^{-ix}}{2}, sinhx = \frac{e^x - e^{-x}}{2}, coshx = \frac{e^x + e^{-x}}{2} \right\}$$

11. If $g(x)$ polynomial functions exist, we can use the property $L\{x^n * f(x)\} = (-1)^n \frac{d}{ds} F(s)$.
12. If $L\{g(x) * f(x)\}$ is of the time delay type, we can use the property $L\{g(x-a) f(x-a)\} = e^{-as} F(s)$.
13. If $L\{k(x) * f(x)\}$ is of the convolution type, we can use the convolution theorem $L\{g(t) * y(t)\} = G(s) Y(s)$, where $G(s) = L\{g(t)\}$ and $Y(s) = L\{y(x)\}$.
14. If $g(x)$ is a differentiable function, we can use the Taylor series and reduce it to a polynomial function; then, we can employ Note 3. The Taylor series for a differentiable function is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n.$$

5. Illustrative Examples

This section contains four analytical problems that show how to derive the precise (analytic) solution to high-order linear fractional differential equations of the Liouville–Caputo function by using LT.

Example 5.1 Consider the fractional boundary value problem

$$-{}_0^C D_x^{\frac{7}{3}} y(x) + \frac{1}{10} y(x) = \lambda^2 \frac{1}{10} y(x), \qquad x \in [0,1],$$
$$y(0) = y'(0) = 0, \qquad y(1) = y'(1).$$

Solution: We have $M = \frac{1}{10}$.

Now, $-{}_0^C D_x^{\frac{7}{3}} y(x) + \frac{1}{10} y(x) = \frac{1}{10} \lambda^2 y(x)$

$\to {}_0^C D_x^{\frac{7}{3}} y(x) = \frac{1}{10}(1-\lambda^2) y(x)$.

We demonstrate the use of LT to obtain Laplace properties on both sides. $L\left\{ {}_0^C D_x^{\frac{7}{3}} y(x) \right\} = L\{\frac{1}{10}(1-\lambda^2) y(x)\}$.

By using the properties of LT, we have

$$L\left\{ {}_0^C D_x^{\frac{7}{3}} y(x) \right\} = \left\{ s^{\frac{7}{3}} Y(s) - s^{\frac{4}{3}} y(0) - s^{\frac{1}{3}} y'(0) - s^{\frac{-2}{3}} y''(0) \right\}.$$

Now, $s^{\frac{7}{3}} Y(s) - s^{\frac{4}{3}} y(0) - s^{\frac{1}{3}} y'(0) - s^{\frac{-2}{3}} y''(0) = \frac{1}{10}(1-\lambda^2) Y(s)$.

From the boundary conditions, we have $y(0) = y'(0) = 0$

$\to s^{\frac{7}{3}} Y(s) - s^{\frac{-2}{3}} y''(0) = \frac{1}{10}(1-\lambda^2) Y(s)$

$\to s^{\frac{7}{3}} Y(s) + \frac{1}{10}(\lambda^2 - 1) Y(s) = s^{\frac{-2}{3}} y''(0)$

$\left(s^{\frac{7}{3}} + \frac{1}{10}(\lambda^2 - 1) \right) Y(s) = s^{\frac{-2}{3}} y''(0)$,

such that $y''(0) \neq 0$,

$Y(s) = \frac{y''(0) s^{\frac{-2}{3}}}{s^{\frac{7}{3}} + \frac{1}{10}(\lambda^2 - 1)} \to Y(s) = \frac{A s^{\frac{7}{3}-3}}{s^{\frac{7}{3}} + \frac{1}{10}(\lambda^2 - 1)}$, where $A = y''(0)$.

For both sides, we take the Laplace inverse and derive

$$L^{-1}(Y(s)) = L^{-1}\left(\frac{A s^{\frac{7}{3}-3}}{s^{\frac{7}{3}} + \frac{1}{10}(\lambda^2 - 1)} \right).$$

For the inverse, readers may refer to [29]. With LT and the Mittag–Leffler derivative, we obtain $y(x) = A x^2 E_{\frac{7}{3},3}\left(\frac{1}{10}(1-\lambda^2) x^{\frac{7}{3}} \right)$, *where* $A = y''(0)$ and g is a constant. In accordance with the second condition $y(a) = y'(a) = 0$, the solution is derived as

$y(x) = y''(0)x^2 E_{\frac{7}{3},3}\left(\frac{1}{10}(1-\lambda^2)x^{\frac{7}{3}}\right)$.

Now, $y(1) = y''(0)E_{\frac{7}{3},3}\left(\frac{1}{10}(1-\lambda^2)\right)$, and from the properties of the Mittag–Leffler derivative, we have

$y'(1) = y''(0)E_{\frac{7}{3},2}\left(\frac{1}{10}(1-\lambda^2)\right)$.

Hence, $y(1) = y'(1) \rightarrow \quad y''(0)E_{\frac{7}{3},3}\left(\frac{1}{10}(1-\lambda^2)\right) = y''(0)E_{\frac{7}{3},2}\left(\frac{1}{10}(1-\lambda^2)\right)$.

$y''(0) \neq 0$ because when $y''(0) = 0$, the solution is trivial. Thus,

$$E_{\frac{7}{3},3}\left(\frac{1}{10}(1-\lambda^2)\right) - E_{\frac{7}{3},2}\left(\frac{1}{10}(1-\lambda^2)\right) = 0.$$

λ can be obtained using certain Mittag–Leffler properties.

A solution to the fractional boundary value problem above is

$y(x) = y''(0)x^2 E_{\frac{7}{3},3}\left(\frac{1}{10}(1-\lambda^2)x^{\frac{7}{3}}\right)$.

Example 5.2 Examine the fractional boundary value problem

$-{}_0^C D_x^{\frac{5}{2}} y(t) + y(t) = \lambda^2 y(t); \quad t \in [0,1], \alpha \in (1,2]$

$y(0) = y'(0) = 0, \qquad y'(1) = y(1).$

Solution: If $\lambda = i$, for any M, we have

$-{}_0^C D_x^{\frac{5}{2}} y(t) + y(t) = -y(t) \rightarrow {}_0^C D_x^{\frac{5}{2}} y(t) = 2y(t)$.

Given that $2 < \alpha \leq 3$, by using T, we have

$L[D^\alpha y(t)] = 2L[y(t)] \rightarrow s^\alpha Y - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) - s^{\alpha-3}y''(0) = 2Y$,

$Y(s^\alpha - 2) = cs^{\alpha-3} \rightarrow Y = \frac{cs^{\alpha-3}}{s^\alpha - 2}$,

$Y = \frac{c}{s^\alpha}\frac{1}{1-\frac{2}{s^\alpha}}$,

where $c = y''(0)$.

Using the identity

$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots \qquad , |u| < 1,$

we derive $Y = \frac{c}{s^\alpha}\left(1 + \frac{2}{s^\alpha} + \frac{4}{s^{2\alpha}} + \cdots\right), |\frac{2}{s^\alpha}| < 1$.

Then, $Y = c\left(\frac{1}{s^\alpha} + \frac{2}{s^{2\alpha}} + \frac{4}{s^{3\alpha}} + \cdots\right)$.

The solution is $y(t) = c\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{t^{3\alpha-1}}{\Gamma(3\alpha)} + \cdots\right)$.

We note that $\lim_{\alpha\to2} y(t) = c\left(\frac{t}{1!} + \frac{t^3}{3!} + \frac{t^5}{5!} \ldots\right) = c\, sinh(t)$ and $\lim_{\alpha\to3} y(t) = c\left(\frac{t^2}{2!} + \frac{t^5}{5!} + \frac{t^8}{8!} \ldots \ldots\right)$.

Example 5.3 Consider the fractional boundary value problem

$-{}_0^C D_x^{2.3} y(x) + 0.4y(x) = 0.4\lambda^2 y(x), \qquad x \in [0,1]$

$y(0) = y'(0) = 0, \qquad y'(1) = y(1).$

Solution: We have $M = 0.4$.

Now, $-{}_0^C D_x^{2.3} y(x) + 0.4y(x) = 0.4\lambda^2 y(x)$

$\rightarrow {}_0^C D_x^{2.3} y(x) = 0.4(1-\lambda^2)y(x)$.

We demonstrate the use of LT to obtain Laplace properties on both sides. $L\{{}_0^C D_x^{2.3} y(x)\} = L\{0.4(1-\lambda^2)y(x)\}$. By employing Laplace characteristics, we obtain $L\{{}_0^C D_x^{2.3} y(x)\} = \{s^{2.3}Y(s) - s^{1.3}y(0) - s^{0.3}y'(0) - s^{-0.7}y''(0)\}$.

Now, $s^{2.3}Y(s) - s^{1.3}y(0) - s^{0.3}y'(0) - s^{-0.7}y''(0) = 0.4(1-\lambda^2)Y(s)$.From the boundary conditions, we have $y(0) = y'(0) = 0$

$\rightarrow s^{2.3}Y(s) - s^{-0.7}y''(0) = 0.4(1-\lambda^2)Y(s)$

$\rightarrow s^{2.3}Y(s) + 0.4(\lambda^2 - 1)Y(s) = s^{-0.7}y''(0)$

$\left(s^{2.3} + 0.4(\lambda^2-1)\right)Y(s) = s^{-0.7}y''(0)$,

such that $y''(0) \neq 0$ and

$Y(s) = \frac{y''(0)s^{-0.7}}{s^{2.3}+0.4(\lambda^2-1)} \rightarrow \quad Y(s) = \frac{As^{2.3-3}}{s^{2.3}+0.4(\lambda^2-1)}$, where $A = y''(0)$.

For both sides, we take the Laplace inverse and obtain $L^{-1}(Y(s)) = L^{-1}\left(\frac{As^{2.3-3}}{s^{2.3}+0.4(\lambda^2-1)}\right)$.

For the inverse, from related inverse LT to the Mittag–Leffler function in [36], we derive

$$y(x) = Ax^2 E_{2.3,3}(0.4(1-\lambda^2)x^{2.3}),$$

where g is a constant and $A = y''(0)$.

We obtain $y(1) = y'(1)$ from the second condition.

The solution is $y(x) = Ax^2 E_{2.3,3}(0.4(1-\lambda^2)x^{2.3})$.

Now, $y(1) = y''(0)E_{2.3,3}(0.4(1-\lambda^2))$, and from the Mittag–Leffler derivative's attributes, we have

$y'(1) = y''(0)E_{2.3,2}(0.4(1-\lambda^2))$.

Thus, $y(1) = y'(1)$

$\rightarrow y''(0)E_{2.3,3}(0.4(1-\lambda^2)) = y''(0)E_{2.3,2}(0.4(1-\lambda^2))$.

$y''(0) \neq 0$ because when $y''(0) = 0$, the solution is trivial. Hence, $E_{2.3,3}(0.4(1-\lambda^2)) - E_{2.3,2}(0.4(1-\lambda^2)) = 0$.

λ can be obtained using certain Mittag–Leffler properties.

A solution to the fractional boundary value problem above is

$y(x) = y''(0)x^2 E_{2.3,3}(0.4(1-\lambda^2)x^{2.3})$.

Example 5.4 Consider the fractional boundary value problem

$-{}_0^C D_x^{2.9} y(x) + 0.3y(x) = \lambda^2 y(x), \qquad x \in [0,2]$

$y(0) = y'(0) = 0, \qquad y'(2) = y(2).$

Solution: We have $q(x) = 0.3$, $p(x) = 1$.

Now, $-{}_0^C D_x^{2.9} y(x) + 0.3y(x) = \lambda^2 y(x)$

$\rightarrow {}_0^C D_x^{2.3} y(x) = 0.3y(x) - \lambda^2 y(x)$.

We demonstrate the use of LT to obtain Laplace properties on both sides.

$L\{{}_0^C D_x^{2.9} y(x)\} = L\{0.3y(x) - \lambda^2 y(x)\}$. By using Laplace properties and Proposition 5 in Section 3.1, we have $L\{{}_0^C D_x^{2.9} y(x)\} = \{s^{2.9}Y(s) - s^{1.9}y(0) - s^{0.9}y'(0) - s^{-0.1}y''(0)\}$.

Now, $s^{2.9}Y(s) - s^{1.9}y(0) - s^{0.9}y'(0) - s^{-0.1}y''(0) = (0.3 - \lambda^2)Y(s)$.

From the boundary conditions, we have $y(0) = y'(0) = 0$

$\rightarrow s^{2.9}Y(s) - s^{-0.1}y''(0) = (0.3 - \lambda^2)Y(s)$

$\rightarrow s^{2.9}Y(s) + (\lambda^2 - 0.3)Y(s) = s^{-0.1}y''(0)$

$\rightarrow \left(s^{2.3} + 0.4(\lambda^2 - 1)\right)Y(s) = s^{-0.7}y''(0)$,

such that $y''(0) \neq 0$.

$Y(s) = \frac{y''(0)s^{-0.1}}{s^{2.9}+(\lambda^2-0.3)} \rightarrow Y(s) = \frac{As^{2.9-3}}{s^{2.9}+(\lambda^2-0.3)}$, where $A = y''(0)$.

For both sides, we take the Laplace inverse and obtain $L^{-1}\left(Y(s)\right) = L^{-1}\left(\frac{As^{2.9-3}}{s^{2.9}+(\lambda^2-0.3)}\right)$.

For the inverse, readers may refer to [29]. From Section 3.3 and the Mittag–Leffler derivative, we obtain $y(x) = Ax^2E_{2.9,3}\left((0.3-\lambda^2)x^{2.3}\right)$,

where g is a constant and $A = y''(0)$.

From the second condition, we have $y(2) = y'(2)$.

The solution is $y(x) = Ax^2E_{2.9,3}((0.3-\lambda^2)x^{2.9})$.

Now, $y(2) = (2)^{2.9}y''(0)E_{2.9,3}((0.3-\lambda^2)(2)^{2.9})$, and from the Mittag–Leffler derivative's attributes, we derive $y'(2) = (2)^{2.9}y''(0)E_{2.9,2}((0.3-\lambda^2)(2)^{2.9})$.

Thus, $y(2) = y'(2) \rightarrow$ $y''(0)E_{2.9,3}((0.3 - \lambda^2)(2)^{2.9}) = y''(0)E_{2.9,2}((0.3-\lambda^2)(2)^{2.9})$.

$y''(0) \neq 0$ because when $y''(0) = 0$, the solution is trivial. Hence,

$$E_{2.9,3}((1-\lambda^2)(2)^{2.9}) - E_{2.9,2}\left((0.3-\lambda^2)\right) = 0.$$

λ can be obtained using certain Mittag–Leffler properties.

A solution to the fractional boundary value problem above is $y(x) = y''(0)x^2E_{2.9,3}((0.3-\lambda^2)x^{2.3})$.

6. Results and Discussion

In this study, the Schrodinger equation for a fractional order with special condition boundaries was solved by LT, and the exact solution was obtained accurately and technically.

7. Conclusions

In summary, the Schrödinger fractional-order boundary value problem can be effectively solved in the field of mathematical physics by using the LT method. The fractional-order differential equation can be transformed effectively into a straightforward algebraic form via LT, which accelerates the analytical solution procedure. The effective use of the LT approach in this study emphasizes the approach's usefulness in solving challenging mathematical issues in physics and shows promise for future developments in fractional-order differential equation research. The findings show that the solution to the fractional boundary value problem in Equations (1.1) and (1.2) is connected to the Mittag–Leffler function. The condition is attained using the operator we established for the problem. Examples are also provided to highlight the key findings.

Author Contributions: All of the authors read and approved the final manuscript.

Acknowledgments

The authors would like to thank the editor and the referees for their valuable suggestions to improve this work.

Conflict of Interest

The authors declare that they have no conflict of interest.

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حل مسألة القيمة الحدودية ذات الرتبة الكسرية لشرودنجر بطريقة تحويلات لابلاس

شوان سواره فتاح[١]، هوزان دلشاد محمد حلمي[٢*]، كاروان حمه فرج جوامێر[٢]

[١] قسم الفیزیاء ، كلية العلوم ، جامعة جرمو، جمجمال ، العراق
[٢] قسم الرياضيات ، كلية العلوم ، جامعة السليمانية ، السليمانية ، العراق
hozan.mhamadhilmi@univsul.edu.iq

الخلاصة:

يلعب عدم التماثل دورًا مهمًا في ديناميكيات النقل لحساب التفاضل والتكامل الكسري الجديد. قامت دراسات قليلة بتصميم خصائص عدم التماثل هذه رياضيًا، ولم تقم أي منها بتطوير نماذج شرودنغر التي تتضمن مراحل تطور التماثل المختلفة. تم اقتراح تحويل لابلاس، والذي يمكن استخدامه لاكتشاف الحل التحليلي (الدقيق) للمعادلات التفاضلية الكسرية الخطية، تم اقتراحه في هذا العمل كتقنية لحل معادلة شرودنغر ذات الترتيب الكسري مع الشروط الحدودية، حيث المشتقات الكسرية لكابوتو وريمان –يتم تطبيق ليوفيل. يمكن استخدام هذا لحل للمعادلات التفاضلية الكسرية والعادية. بعد ذلك، اكتشف المؤلفون الحل الدقيق لمثال محدد لمعادلة تفاضلية كسرية. توضح النتائج أن التحويل المقترح، "تحويل لابلاس"، ينتج حلولًا صحيحة لمسألة قيمة حدود المعادلات التفاضلية الكسرية المحددة دون الحاجة إلى حسابات شاقة. في هذه الدراسة، قمنا بالتحقيق في فئة من مسائل القيمة الحدودية الكسرية مع شرطين من القيمة الحدودية التي تتضمن أوامر كسرية مشتقة كسرية $\alpha \in (2,3]$ و$x \in (0,a]$ تم توضيح استنتاجاتنا الرئيسية بالأمثلة. تم توضيح الحل باستخدام تحويل لابلاس، وحصلنا على الوجود على أساس طريقة التكامل الكسري وعامل التكامل.

الكلمات المفتاحية: عامل شرودنجر، التفاضل الكسري، التكامل كسري، المسألة الحدودىة الكسرية، تحويل لابلاس الكسري، التفاضل كابوتو، المعادلات التفاضلية الكسرية.