On (m,n)-Semi Paracompactness In Tritopological Spaces

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Abstract

We study the concept of (m,n)-semi paracompactness for spaces with 2 and 3 topologies and obtain several results concerning (m,n)- semi paracompact in tritopological spaces . Throughout the present paper m and n will denote infinite cardinal numbers . $|A|$ will stand for the cardinality of A .

الخلاصة

قدمنا في هذا البحث تعديل مفهوم (m,n) – شبه فوق المرصوصة في الفضاءات الثلاثية التبولوجيا وحصلنا على عدة نتائج تتعلق بهذا المفهوم . وفي كافة البحث الحالي فأن m و n هي أعداد كاردينالتية وأن $n \leq m$ وكذلك أن الرمز $|A|$ سيساند الكاردينال بالنسبة إلى المجموعة A .

1- Introduction

The concept of paracompactness is due to Dieudonne [Diedonne] , that of (m,n)-compactness due to Gal [Gal] and that of (m,n)- paracompactness due to Singal [Singal] . The concept of paracompact with respect to three topologies is due to Martin [Martin] and that of (m,n)-paracompactness due to AL-Swidi [Al-swidi] .

The term space (X,τ,σ,ρ) is referred as a set X with three , generally nonidentical topologies τ,σ and ρ . A family of subsets of X is called $\sigma - locally$ finite if it is a union of a countably many locally finite subfamilies (This notion has nothing common with the topology also denoted by σ) . A family is called locally-n if each point in the space has a neighborhood which intersection at most n members of the family. We say that the space (X,τ,σ,ρ) is $(\tau,\sigma)-$ paracompact with respect to ρ if every $\tau - open$ cover of X has a $\sigma - open$ refinement which is locally finite with respect to ρ [Martin], and is said to be $(\tau,m)-$compact with respect to ρ if each $\tau - open$ cover of X with cardinal at most m has ρ-open finite sub cover . A tritopological space (X,τ,σ,ρ) is said to be $\sigma -$countable paracompact with respect to ρ if for each countable $\sigma - open$ cover of X , has a $\sigma - open$ refinement which is locally finite with respect to ρ . Let (X,τ) be a topological space and $A \subset X$, A is said to be semi open set in X iff there exist $O \subset \tau$ such that $O \subset A \subset cl(O)$[Levin] , and A is said to be semi closed set in X iff X-A is semi open in X [[Das] .

2 - On (m,n)-semi paracompactness in tritopological spaces

Definition 2-1:

We say that the tritopological space (X,τ,σ,ρ) is (τ,σ,m,n)-semi paracompact with respect to ρ if every τ-semi open cover of X with cardinal at most m , has a σ-semi open refinement which is locally –n with respect to ρ .

Theorem 2-2 :

Let (X,τ,σ,ρ) be a tritopological space . If $\{F_\alpha : \alpha \in I\}$ be a locally finite family with respect to ρ of τ-semi closed subsets of X such that $\bigcup\{\text{int}_\sigma(F_\alpha) : \alpha \in I\} = X$, then X is (τ,σ,m,n)-semi paracompact with respect to ρ iff each F_α is (τ,σ,m,n)-semi paracompact with respect to ρ .

Proof :

Let each F_α be (τ,σ,m,n) -semi paracompact with respect to ρ and $\Re = \{V_\beta : \beta \in J\}$ be any τ-semi open covering of (X,τ,σ,ρ) such that $|\Re| \le m$. For each $\alpha \in I, \Re_\alpha = \{F_\alpha \bigcap V_\beta : \beta \in J\}$ is an τ-semi open covering of F_α such that $|\Re_\alpha| \le m$. Since F_α is (τ,σ,m,n)-semi paracompact with respect to ρ , therefore $\Re_\alpha$ has a σ-semi open refinement which is locally-n with respect to ρ $\coprod_\alpha = \{W_\lambda \bigcap F_\alpha : \lambda \in \Lambda^\alpha, W_\lambda \in \tau\}$. Then $\coprod = \{W_\lambda \bigcap \text{int}_\sigma(F_\alpha) : \lambda \in \Lambda^\alpha, \alpha \in I\}$ is obviously an σ-semi open refinement of $\Re$. To show that $\coprod$ is locally –n with respect to ρ in X , let x be any point of X . Since $\{F_\alpha : \alpha \in I\}$ is locally finite with respect to ρ , x has a ρ-neighbourhood N , which intersects at most a finite number of F_α 's, say $F_{\alpha_1}, F_{\alpha_2}, ..., F_{\alpha_k}$. Also therefore if $i \ne 1,2,......,k$, then $N \bigcap \text{int}_\sigma(F_{\alpha_i}) = \Phi$. For each $i \in \{1,2,......,k\}$, x has a ρ-neighbourhood a N_i which intersect at most n members of $\coprod_{\alpha_i}$ and also therefore at most n members of the family $\{W_\lambda \bigcap \text{int}_\sigma(F_{\alpha_i}) : \lambda \in \Lambda^{\alpha_i}\}$. The ρ-neighbourhood $N \bigcap N_1 \bigcap .... \bigcap N_k$ of x intersect at most n members of $\coprod$. Hence $\coprod$ is locally –n with respect to ρ in X . Thus $\coprod$ is σ-semi open refinement which is locally –n with respect to ρ of X and X is consequently (τ,σ,m,n)-semi paracompact with respect to ρ .

Conversely , let X be a (τ,σ,m,n)-semi paracompact with respect to ρ and let $\Re = \{F_\alpha \bigcap V_\beta : \beta \in J, V_\beta \in \tau\}$ be any τ-semi open covering of F_α such that $|\Re| \le m$. Then $\{X - F_\alpha\} \bigcup \{V_\beta : \beta \in J\}$ is an τ-semi open covering of X of cardinality less than or equal to m . By the (τ,σ,m,n)-semi paracompact with respect to ρ of X , there exists a σ-semi open refinement which is locally –n with respect to ρ of this cover of X . Let this be $\{U_\beta : \beta \in J'\}$. Then $\{U_\beta \bigcap F_\alpha : \beta \in J'\}$ is a σ-semi open refinement which is locally –n with respect to ρ of $\Re$. Hence F_α is (τ,σ,m,n)-semi paracompact with respect to ρ .

Corollary 2-3 :

Every τ-semi closed subset of an (τ,σ,m,n)-semi paracompact with respect to ρ is (τ,σ,m,n)-semi paracompact with respect to ρ .

Corollary 2-4 :

If $\{F_\alpha : \alpha \in I\}$ be a locally finite with respect to ρ , σ-semi open covering of X , then X is (τ,σ,m,n) -semi paracompact with respect to ρ if each $cl_\tau(F_\alpha)$ is (τ,σ,m,n)-semi paracompact with respect to ρ .

Proof :

$F_\alpha \subset cl_\tau(F_\alpha)$, then $F_\alpha \subset \text{int}_\sigma(cl_\tau(F_\alpha))$, as F_α is σ-semi open. $\{cl_\tau(F_\alpha) : \alpha \in I\}$ is a locally finite family with respect to ρ of τ-semi closed subsets of X such that $\bigcup\{\text{int}_\sigma(cl_\tau(F_\alpha)) : \alpha \in I\} = X$.

Definitions 2-5 :

A subset K is said to be (τ-σ)-semi regularly closed in Tritopological space (X,τ,σ,ρ) if there exist σ-semi open subset U of X such that $K = cl_\tau(U)$. A Tritopological space (X,τ,σ,ρ) is said to be weakly (τ-σ)-semi regularly if every non-null σ-semi open subset of the space contains a non-null (τ-σ)-semi regularly closed set .

Theorem 2-6 :

A weakly (τ-σ)-semi regularly space is (τ,σ,m,n)-semi paracompact with respect to ρ iff every proper (τ-σ)-semi regularly closed subset of X is (τ,σ,m,n) -semi paracompact with respect to ρ .

Proof :

If X is (τ,σ,m,n) -semi paracompact with respect to ρ , then every (τ-σ)-semi regularly closed set being a τ-semi closed subset of X , is (τ,σ,m,n)-semi paracompact with respect to ρ .

Conversely , let U be any proper non-null σ-semi open subset of X . Since X is weakly (τ-σ)-semi regularly space , therefore there exists a non-null σ-semi open set V such that $cl_\tau(V) \subset U$. Since V is non-null and σ-semi open , therefore there exists a non-null σ-semi open set W , such that $cl_\tau(W) \subset V$. U being a proper subset of X , $cl_\tau(V)$ is a proper (τ-σ)-semi regularly closed subset of X . Now W is σ-semi open and $W \bigcap (X - cl_\sigma(W)) = \Phi$, so $W \bigcap cl_\tau((X - cl_\sigma(W))) = \Phi$, then we get that $cl_\tau((X - cl_\sigma(W))) \subset (X - W)$, hence $cl_\tau((X - cl_\sigma(W)))$ is a proper subset of X as W nonempty . Obviously $cl_\tau((X - cl_\sigma(W)))$ is (τ-σ)-semi regularly closed . Now V and $(X - cl_\sigma(W))$ are σ-semi open subsets of X and $X = V \bigcup (X - V) \subset V \bigcup (X - cl_\sigma(W)$. Hence $\{V, X - cl_\sigma(W)\}$ is a locally finite with respect to ρ , σ-semi open covering of X , each member of which (τ,σ,m,n)-semi paracompact with respect to ρ . Therefore by corollary 2 of theorem 1-2 , tritopological space (X,τ,σ,ρ) is (τ,σ,m,n) -semi paracompact with respect to ρ .

Definition 2-7 :

A tritopological space (X,τ,σ,ρ) is said to be $\sigma-$countable semi paracompact with respect to ρ if for each countable σ-semi open cover of X , has a σ-semi open refinement which is locally finite with respect to ρ .

Theorem 2-8 :

If Tritopological space (X,τ,σ,ρ) is $\sigma-$ countably semi paracompact with respect to ρ and every τ-semi open covering of the space of cardinality less than or equal to m has an σ-semi open refinement which is $\sigma-$ locally –n with respect to ρ, then the space is (τ,σ,m,n)-semi paracompact with respect to ρ .

Proof :

Let $\Re=\{U_\alpha:\alpha\in I\}$ be any τ-semi open covering of (X,τ,σ,ρ) such that $|\Re|\le m$. By hypothesis , $\Re$ has an σ-semi open refinement which is $\sigma-$ locally –n with respect to ρ, Π say . Then Π is a countable union of locally –n with respect to ρ families , so that $\Pi=\bigcup\{\Pi_i:i\in N\}$, each Π_i being locally –n with respect to ρ . Let $\Pi_i=\{V_\lambda:\lambda\in\Lambda^i\}$ and let $V_i=\bigcup\{V_\lambda:\lambda\in\Lambda^i\}$. Since Π covers X , therefore $\{V_i:i\in N\}$ is a countable σ-semi open covering of X , X being $\sigma-$countably semi paracompact with respect to ρ, $\{V_i:i\in N\}$ has σ-semi open refinement which is locally finite with respect to ρ $\{W_i:i\in N\}$ such that $W_i\subset V_i$ for each $i\in N$. Then $W=\{V_\lambda\bigcap W_i:\lambda\in\Lambda^i,i\in N\}$ is a σ-semi open refinement which is locally –n with respect to ρ of $\Re$. Hence X is (τ,σ,m,n)-semi paracompact with respect to ρ .

Definition 2-9 :

A bitopological space (X,τ,σ) is said to be (τ-σ)-semi regular space with respect to σ iff for each $x\in X$ and τ-semi open subset U of X containing x , there exist σ-semi open subset V of X , such that $cl_\sigma(V)\subset U$.

Theorem 2-10 :

Let (X,τ,σ) be a (τ-σ)-semi regular space and x be a point of X having a fundamental system of σ-semi open neighborhoods A with the property that $X-A$ is (τ,σ,m,n)-semi paracompact with respect to ρ for each $A\in$ A . Then tirtopological space (X,τ,σ,ρ) is (τ,σ,m,n)-semi paracompact with respect to ρ .

Proof :

Let $\Re=\{G_\alpha:\alpha\in I\}$ be any τ-semi open covering of X with cardinality less than or equal to m . Then some members of $\Re$ contains x , call it G_{α_x} . Since X is (τ-σ)-semi regular space , therefore there exists an $A\in$ A such that $x\in A\subset cl_\sigma(A)\subset G_{\alpha_x}$. Then A'$=\{(X-A)\bigcap G_\alpha:\alpha\in I\}$ is an τ-semi open cover of $X-A$ of cardinality less than or equal to m . Since $X-A$ is (τ,σ,m,n)-semi paracompact with respect to ρ , therefore there exists a σ-semi open refinement which is locally –n with respect to ρ $\{V_\beta:\beta\in J\}$ of A' . Let $\coprod=\{G_{\alpha_x}\}\bigcup\{V_\beta\bigcap(X-A):\beta\in J\}$. Then

I . Since $\{V_\beta:\beta\in J\}$ is locally –n with respect to ρ in X , therefore $\coprod$ also is locally –n with respect to ρ in X .

II . $X-cl_\sigma(A)$ may be regarded as a subspace of $X-A$. Each V_β being σ-semi open in $X-A$, $V_\beta\bigcap(X-cl_\sigma(A))$ is σ-semi open in $X-cl_\sigma(A)$ which itself is σ-semi open in X . Hence $V_\beta\bigcap(X-cl_\sigma(A))$ is an σ-semi open subset of X for each $\beta\in J$.

III. If $\beta\in J$, then $V_\beta\bigcap(X-cl_\sigma(A))\subset V_\beta\subset(X-A)\bigcap G_\alpha$, for some $\alpha\in I$. Therefore $V_\beta\subset(X-cl_\sigma(A))\subset G_\alpha$ for some $\alpha\in I$. Also since $\{V_\beta:\beta\in J\}$ covers

$X-A$, therefore $\{V_\beta \cap (X-cl_\sigma(A)) : \beta \in J\}$ covers $X-cl_\sigma(A)$. Now $(X-cl_\sigma(A)) \cup G_\alpha$ being X , $\amalg$ covers X . Therefore $\amalg$ refines $\Re$.

From I , II and III it follows that (X,τ,σ,ρ) is (τ,σ,m,n)-semi paracompact with respect to ρ .

Theorem 2-11 :

A subspace A of tritopological space (X,τ,σ,ρ) is (τ,σ,m,n) -semi paracompact with respect to ρ iff for each τ-semi open set G containing A , there exists an (τ,σ,m,n)-semi paracompact with respect to ρ set B , such that $A \subset B \subset G$.

Proof :

If A is (τ,σ,m,n)-semi paracompact with respect to ρ , then for each τ-semi open set G containing A , we have $A \subset A \subset G$. To prove the converse , for every τ-semi open set G containing A , let there exist an (τ,σ,m,n)-semi paracompact with respect to ρ set B such that $A \subset B \subset G$. Let $\Sigma = \{U_\alpha \cap A; \alpha \in I, U_\alpha \in \tau\}$ be any τ-semi open covering of A , such that $|\Sigma| \le m$. If $U = \bigcup\{U_\alpha; \alpha \in I\}$, then U is an τ-semi open set that containing A . Therefore there exists an (τ,σ,m,n)-semi paracompact with respect to ρ set B such that $A \subset B \subset G$. $\Pi = \{U_\alpha \cap B; \alpha \in I\}$ is an τ-semi open covering of B such that $|\Pi| \le m$. By the (τ,σ,m,n)-semi paracompact with respect to ρ of B , there exists a σ-semi open refinement which is locally-n with respect to ρ $\{V_\beta \cap B; \beta \in J, V_\beta \in \sigma\}$ of Π . Then $\{V_\beta \cap A; \beta \in J\}$ is a σ-semi open refinement which is locally-n with respect to ρ of A , as can easily be verified . Hence A is (τ,σ,m,n)-semi paracompact with respect to ρ .

Corollary 2-12 :

If every τ-semi open subspace of a tritopological space (X,τ,σ,ρ) is (τ,σ,m,n)-semi paracompact with respect to ρ, then every subspace of (X,τ,σ,ρ) is (τ,σ,m,n)-semi paracompact with respect to ρ .

Definition 2-13:

Let (X,τ,σ,ρ) be a tritopological space. A subset A of a space is said to be semi F_σ if it is a union of countably many τ-semi closed sets . A subset A of a space X is said to be a generalized semi F_σ if for every semi open set G containing A , there exists a semi F_σ set B such that $A \subset B \subset G$ [4] .

Theorem 2-14 :

Every countably (σ,ρ)-semi paracompact with respect to ρ generalized semi F_σ subset of an (τ,σ,m,n) -semi paracompact with respect to ρ is (τ,σ,m,n) -semi paracompact with respect to ρ .

Proof :

Let X be an (τ,σ,m,n) -semi paracompact with respect to ρ and let Y be countably (σ,ρ)-semi paracompact with respect to ρ generalized semi F_σ subset of X . Let $\Sigma = \{Y \cap U_\alpha; \alpha \in I, U_\alpha \in \tau\}$ be any τ-semi open covering of Y such that $|\Sigma| \le m$. Let

$U = \bigcup\{U_\alpha ; \alpha \in I\}$. Then U is an τ-semi open set containing Y . Since Y is a generalized semi F_σ , therefore there exist τ-semi closed sets G_i such that $Y \subset \bigcup\{G_i ; i \in N\} \subset U$. For each i , $\Sigma_i = \{X - G_i\} \bigcup \{U_\alpha ; \alpha \in I\}$ is an τ-semi open covering of X such that $|\Sigma_i| \leq m$. Since X is (τ, σ, m, n)-semi paracompact with respect to ρ so there exists a σ-semi open refinement which is locally-n with respect to ρ $\Psi_i = \{X - G_i\} \bigcup \{V_{i\alpha} ; \alpha \in I\}$ such that $V_{i\alpha} \subset U_\alpha$ for each $\alpha \in I$. Then $\Psi = \{Y \bigcap V_{i\alpha} ; \alpha \in I, i \in N\}$ is a σ-semi opem refinement which is $\sigma-$ locally-n with respect to ρ of Π . Thus each τ-semi open covering of Y of cardinality less than or equal to m , has σ-semi open refinement which is $\sigma-$ locally-n with respect to ρ and Y is count ably (σ, ρ)-semi paracompact with respect to ρ . Hence by theorem 1-2 , Y is (τ, σ, m, n)-semi paracompact with respect to ρ .

Corollary 2-15:

Every countably (σ, ρ)-semi paracompact with respect to ρ-semi F_σ subset of an (τ, σ, m, n)-semi paracompact with respect to ρ is (τ, σ, m, n)-semi paracompact with respect to ρ .

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