On the growth of iterated special monogenic functions

Mushtaq Shakir A.Hussein
College of science, Mustansiriha University
Aseel Hameed Abed Sadaa
College of Basic Education, Mustansiriha University

Abstract

In this paper we consider the relative iteration of entire special monogenic functions and we studied the comparative growth of the maximum term of iterated entire monogenic functions with that of the maximum term of the related functions.

1-Introduction

Firstly, following Constales, Almeida and Kraussha r (see [1] and [2]), we give some definitions and associated properties .Let m= $(m_1,m_2,\dots,m_n) \in N_0^n$ be the n-dimensional multi-index and $x \in R^n$ then we define

$x^m=x_1^{m1}\dots x_n^{mn}$, $m! = m_1! \dots, m_n!$, $|m| = m_1+\dots+m_n$ … (1)

By $\{e_1, e_2,\dots,e_n\}$ we denote the canonical basis of the Euclidean vector space R^n. The associated real Clifford algebra CI_n is free algebra generated by R^n modulo $x^2=-\|x\|^2e_0$, where e_0 is the neutral element with respect to multiplication of the Clifford algebra CI_n .In the Clifford algebra CI_n following multiplication rule holds :

$e_ie_j+e_je_i=-2\,\delta_{i,j}$, i,j =1,2,…,n. Where δ_{ij} is kronecker symbol. … (2)

A basis for Clifford algebra CI_n is given by the set $\{e_A : A\subseteq \{1,2,\dots,n\}\}$, with $e_A=e_{l_1}e_{l_2}\dots,e_{l_r}$.

Where $1\le l_1 \le l_2 \dots \le l_r \le n$,$e_\phi=e_0=1$.Each $a \in CI_n$ can be written in the form $a=\sum_{A\subseteq(1,2,,,n)} a_A\, e_A$,

With $a_A\in R$.The conjugation in Clifford algebra CI_n is defined by $\bar{a}=\sum_{A\subseteq(1,2,,,n)} a_A\, \bar{e}_A$ Where $\bar{e}_A=\bar{e}_{l_r}\,\bar{e}_{l_{r-1}} \dots \bar{e}_{l_1}$, and $\bar{e_j}=-\,e_j$ for j=1, 2…,n , $\bar{e}_0=e_0=1$. The linear subspace span $_R\{1,e_1,e_2,\dots,e_n\} \subseteq CI_n$ is the so called space of Para vectors $z = x_0 + x_1\, e_1 + x_2\, e_2 +\dots\dots+x_n\, e_n$ which we simply identify with R^{n+1}: Here $x_0=Sc(z)$ is scalar part and $x=x_1e_1+x_2e_2+\dots+x_ne_n=Vec(z)$ is vector part of Paravector z: The Clifford norm of an arbitrary $a=\sum_{A\subseteq(1,2,..,n)} a_A e_A$ is given by $\|a\| = (\sum_{A\subseteq(1,2,..,n)} |a_A|^2)^{1/2}$. The generalized Cauchy–Riemann operator in R^{n+1} is given by $D= \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$. If $U\subseteq R^{n+1}$ is an open set then the function g: $U \rightarrow CI_n$ is called left (right) monogenic at a point $z \in U$ if $Dg(z)=0$ ($gD(z)=0$). The functions which are left (right) monogenic in the whole space called left (right) entire monogenic functions.

Following Abul-Ez and Constales [3], we consider the class of monogenic polynomials p_m of degree $|\mathbf{m}|$, defined as

$$P_m(z)=\sum_{i+j=|m|}^{\infty} \frac{((n-1/2))_i}{i!}\frac{((n+1)/2))_j}{j!} (\bar{z})^i (z)^j \quad \ldots (3)$$

Let ω_n be n-dimensional surface area of $n+1$-dimensional unit ball and let S^n be n- dimensional sphere. Then, the class of monogenic polynomials described in (3) satisfies (see [3], p. 1259)

$$\frac{1}{\omega_m}\int_{s}^{n} \overline{p_m}\, p_l(z)\, dS_z = k_m\, \delta_{|m||l|} \quad \ldots (4)$$

Also following Abul-Ez and Constales [3], we have

$$\max_{\|z\|=r} \|P_m(z)\| = k_m\, r^{m} \ldots (5)$$

Also we have some definitions that are needed

Definition 1.1. [6]: The order ρ_g and lower order λ_g of monogenic function g is defined as:

$$\rho_g = \lim_{r\to\infty} \sup \frac{\log\log M(r,g)}{\log r} \ldots (6)$$

$$\lambda_g = \lim_{r\to\infty} \inf \frac{\log\log M(r,g)}{\log r} \ldots (7)$$

Notation 1.2. [6]:- $log^{[0]} x = x$, $exp^{[0]} x = x$ and for positive integer k, $log^{[k]} x = \log(\log^{[k-1]} x)$,
$\exp^{[k]} x = \exp(\exp^{[k-1]} x)$.

Definition 1.3. [3]: The radius of regularity R_g of special monogenic function g is defined by:

$R_g = 1/\lim_{\|m\|\to\infty} \|c_m\|^{\frac{1}{\|m\|}}$. The function g is called entire monogenic function if $R_g=\infty$.

A simple but useful relation between M(r,g) and μ (r,g) is the following theorem :-

Theorem 1.4. [7]: For $0 \le r < R$, $\mu(r,g) \le M(r,g) \le \frac{R}{R-r}\mu(r,g)$,Taking R=2r ,for all sufficiently large values of r

$$\mu(r,g) \le M(r,g) \le 2\,\mu(2r,g) \quad . \ldots (8)$$

In theorem 1.4 taking two times logarithms in (8) it is easy to verify that:

$$\rho_g = \lim_{r\to\infty} \sup \frac{log^{[2]}\mu(r,g)}{\log r} \quad \text{and}$$

$$\lambda_g = \lim_{r\to\infty} \inf \frac{log^{[2]}\mu(r,g)}{\log r} .$$

In 1997 Lahiri and Banerjee [5] form the iterations of f (z) with respect to g (z) as follows:-

$f_1(z) = f(z)$
$f_2(z) = f(g(z)) = f(g_1(z))$
$f_3(z) = f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))$
… … …
$f_n(z) = f(g(f\ldots\ldots(f(z) \text{ or } g(z))\ldots))$, according as n is odd or even
And so
$g_1(z) = g(z)$
$g_2(z) = g(f(z)) = g(f_1(z))$
$g_3(z) = g(f_2(z)) = g(f(g(z)))$
… …
$g_n(z) = g(f_{n-1}(z)) = g(f(g_{n-2}(z)))$. Clearly all $f_n(z)$ and $g_n(z)$ are entire monogenic functions.
In this paper we study growth properties of the maximum terms of iterated entire monogenic functions as compared to the growth of the maximum term of the related monogenic function to generalize some entire results.

2- Lemmas

The following lemmas will be needed in sequel

Lemma 2.1. [4]: If f and g are any two entire functions, for all sufficiently large values of r, then

$$M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f \circ g) \leq M(M(r, g), f).$$

Lemma 2.2: ρ_f and ρ_g are finite, then for any $\varepsilon > 0$

$$\operatorname{Log}^{[n]} \mu(r, f_n) \leq \begin{cases} (\rho_f + \varepsilon) \log M(r, g) + O(1) & \text{when } n \text{ is even.} \\ (\rho_g + \varepsilon) \log M(r, g) + O(1) & \text{when } n \text{ is odd.} \end{cases}$$

For all sufficiently large values of r, where f and g are entire functions.

Proof

First suppose that n is even, Then in view of (1.1) and by lemma (2.1) it follows that for all sufficiently large values of r,

$\mu(r, f_n) \leq M(r, f_n) \leq M(M(r, g_{n-1}), f)$, implies

$$\log \mu(r, f_n) \leq \log M(M(r, g_{n-1}), f) \leq [M(r, g_{n-1})]^{\rho_f + \varepsilon}$$

So, $\log^{[2]} \mu(r, f_n) \leq (\rho_f + \varepsilon) \log M(r, g(f_{n-2})) \leq (\rho_f + \varepsilon)\, [M(r, f_{n-2})]^{\rho_g + \varepsilon}$

Thus,

$$\log^{[3]} \mu(r, f_n) \leq (\rho_f + \varepsilon) \log M(r, f_{n-2}) + O(1).$$

Therefore $\log^{[n]} \mu(r, f_n) \leq (\rho_f + \varepsilon) \log M(r, g) + O(1)$.

Similarly if n is odd then for all sufficiently large values of r

$\log^{[n]} \mu(r, f_n) \leq (\rho_g + \varepsilon) \log M(r, g) + O(1)$.

This proves the lemma.

Now in ([7], p.113) we have the following lemma:

Lemma 2.3

If λ_f ,λ_g are non-zero finite , Then

$\mathrm{Log}^{[n]}\,\mu(r,f_n) > (\lambda_f - \varepsilon)\log M(r,g)+O(1)$ when n is even.

$(\lambda_g - \varepsilon)\log M(r,g)+O(1)$ when n is odd.

Proof

First suppose that n is even, let (ε>0) be such that ε< {λ_f , λ_g }, now we have for all sufficiently large values of r ,

$$\mu(r, f\circ g) > e^{[M(r,g)]^{\lambda_f-\varepsilon}}$$

So, $\log \mu(r, f\circ g) > [M(r,g)]^{\lambda_f-\varepsilon}$ (2.1)

Now

$\mathrm{Log}\,\mu(r, f_n)=\log \mu(r, f(g_{n-1}))$

$> [M(r,g_{n-1})]^{\lambda_f-\varepsilon}$ using (2.1)

$\geq [\mu(r,g_{n-1})]^{\lambda_f-\varepsilon}$ From (1.1)

$\mathrm{Log}^{[2]}\,\mu(r,f_n) > (\lambda_f - \varepsilon)\log \mu(r, g(f_{n-2}))$

$> (\lambda_f - \varepsilon)[M(r,f_{n-2})]^{\lambda_g-\varepsilon}$ using (2.1)

Then $\mathrm{Log}^{[3]}\,\mu(r,f_n) > (\lambda_g - \varepsilon)\log \mu(r, g(f_{n-2}))+O(1)$

$> (\lambda_g - \varepsilon)[M(r,g_{n-3})]^{\lambda_f-\varepsilon}+O(1)$

Taking repeated logarithms

$\mathrm{Log}^{[n-1]}\,\mu(r,f_n) \geq (\lambda_g - \varepsilon)[M(r,g)]^{\lambda_f-\varepsilon}+O(1)$

Then $\mathrm{Log}^{[n]}\,\mu(r,f_n) \geq (\lambda_f - \varepsilon)\log M(r,g)+O(1)$, Similarly,

$\mathrm{Log}^{[n]}\,\mu(r,f_n) \geq (\lambda_g - \varepsilon)\log M(r,f)+O(1)$ when n is odd .

This proves the lemma.

3-Main results

Now we prove the following:

Theorem 3.1:

Let f and g be two entire monogenic functions , such that $0<\lambda_f\leq\rho_f<\infty$, and $0<\lambda_g\leq\rho_g<\infty$ .Then for any positive number A and every real number α :

(i) $$\lim_{r\to\infty}\frac{\log^{[n]}\mu(r,f_n)}{[\log\log\mu(r^A,f)]^{1+\alpha}}=\infty.$$

(ii) $$\mathrm{Lim}_{r\to\infty}\frac{\log^{[n]}\mu(r,f_n)}{[\log\log\mu(r^A,g)]^{1+\alpha}}=\infty.$$

Proof

If α≤-1 then the theorem is trivial. So we suppose that α>-1 and n is even. Then from lemma 2.3 we get for all sufficiently large values of r and any ε ($0<\varepsilon<\min\{\lambda_f,\lambda_g\}$)

$\mathrm{Log}^{[n]}\,\mu(r,f_n) \geq (\lambda_f - \varepsilon)\log M(r,g)+O(1)$

$\geq (\lambda_f - \varepsilon)r^{\lambda_g-\varepsilon}+O(1)$ (3.1)

Again from definition (1.1) it follows that for any ε>0 and for all large values of r,

$\{loglog\mu(r^A, f)\}^{1+\alpha} < (\rho_f + \varepsilon)^{(1+\alpha)} A^{1+\alpha}$ (log r)$^{1+\alpha}$ (3.2)

From (3.1) and (3.2) we have for all large values of r and any ε (0< ε < min { λ_f , λ_g })

$$\frac{log^{[n]}\mu(r,f_n)}{[loglog\,\mu(r^A,f)]^{1+\alpha}} \geq \frac{(\lambda_f-\varepsilon)r^{\lambda_g-\varepsilon}}{(\rho_f+\varepsilon)^{(1+\alpha)}A^{1+\alpha}(logr)^{1+\alpha}} \geq \frac{(\lambda_f-\varepsilon)r^{\lambda_g-\varepsilon}}{(\rho_f+\varepsilon)^{(1+\alpha)}A^{1+\alpha}(logr)^{1+\alpha}} + O(1)$$

Since ε>0 is arbitrary, Then

$$\lim_{r\to\infty}\frac{log^{[n]}\mu(r,f_n)}{[loglog\mu(r^A,f)]^{1+\alpha}} = \infty \quad (3.3)$$

Similarly for odd n we get

Log$^{[n]}$ μ(r,f_n) ≥ (λ_g -ε) $r^{\lambda_f-\varepsilon}$ +O(1) (3.4)

So from (3.2) and (3.4) we have the equation (3.3) for odd n.

Therefore for all n the statement (i) follows.

Second part of this theorem follows similarly by using the following inequality instead of (3.2)

$\{loglog\mu(r^A, g)\}^{1+\alpha} < (\rho_g + \varepsilon)^{(1+\alpha)} A^{1+\alpha}(\log r)^{1+\alpha}$

For all large values of r and arbitrary ε>0. This proves the theorem.

Theorem 3.2

Let f and g be two entire monogenic functions of finite orders and(λ_f , λ_g)≥ 0 then for p>0 and each real number α ∈ (-∞,∞)

(i) $\lim_{r\to\infty}\frac{\{log^{[n]}\mu(r,f_n)\}^{1+\alpha}}{loglog\mu(\exp(r^p),f)}$ =0 if p> (1+α) ρ_g and n is even,

(ii) $\lim_{r\to\infty}\frac{\{log^{[n]}\mu(r,f_n)\}^{1+\alpha}}{loglog\mu(\exp(r^p),f)}$ =0 if p> (1+α) ρ_f and n is odd.

Proof

If α≤-1 then the theorem is trivial. So we suppose that α >-1 and n is even. Then from lemma 2.2 we get for all sufficiently large values of r and any ε>0.

Log$^{[n]}$ μ(r,f_n) ≤ (ρ_f+ε) log M (r, g) +O(1)

≤ (ρ_f + ε)$r^{\rho_g+\varepsilon}$ +O(1) (3.5)

Again from definition 1.1 it follows that for any 0< ε< λ_f and for all large values of r, (3.6)

So from (3.5) and (3.6) we have for all large values of r and any ε (0< ε< λ_f)

$\frac{\{log^{[n]}\mu\,(r,f_n\,)\}^{1+\alpha}}{loglog\mu(\exp(r^p),f)} \leq \frac{(\rho_{f\,+\varepsilon})^{1+\alpha}\ \ r^{(1+\alpha)(\rho g+\varepsilon)}}{(\lambda_{f-\varepsilon})r^p}$ +O(1) , since ε>0 is arbitrary ,we can choose ε such that 0<ε<min$\{\lambda_f\,,\frac{p}{1+\alpha}-\rho_g\}$,Then

$\lim_{r\to\infty}\frac{\{log^{[n]}\mu\,(r,f_n\,)\}^{1+\alpha}}{loglog\mu(\exp(r^p),f)} = 0$.

Similarly when n is odd then we get the second part of this theorem .This proves theorem.

4- References

[1] D. Constales, R. De Almeida, R. S. Krausshar, On the relation between the growth and the Taylor coefficients of entire solutions to the higher dimensional Cauchy–Riemann system in R^{n+1}; J.b Math. Anal. Appl. 327 (2007), 763–775.

[2] D. Constales, R. De Almeida, R. S. Krausshar, On the growth type of entire monogenic functions, Math.pp.153–163,(2007).

[3] M.A Abul–Ez, Denis Constales, Basic sets of polynomials in Clifford analysis, v o 30(1989),No.4,647-655.

[4] J.Clunie, The composition of entire and meromorphic functions, mathematical essays dedicated to A. J.ohiouniv. press, (1970), 75-92.

[5] B.K.Lahiri and Banerjee, Relative fix points of entire functions, J. Indian Acad.Math,19(1)(1997),87-97.

[6] D.sato ,On the rate of growth of entire functions of fast growth, Bull Amer .Math .Soc. 69(1963), 411-414.

[7] A.P.Singh, On Maximum term of composition of entire functions, Proc.Nat.Acad.Sci.India,59(A),(1989),103- 115.

النمو لدوال تكرارية خاصه أحادية المنشأ عن

م.أسيل حميد عبد السادة — مشتاق شاكر ألشباني. د ا.م
كلية العلوم — كلية التربية الأساسية
الجامعة المستنصرية — الجامعة المستنصرية

الخلاصة:-
في هذه المقاله تدارسنا العلاقة التكراريه لدوال خاصه احاديه المنشأ و درسنا النمو المقارن للحد الأعظم للدوال احاديه المنشأ مع ذللك الحد الأعظم للدوال ذات الصلة.