On Nil-Injective Rings

Raida D. Mahammod , Husam. Q. Mohammad

Department of mathematics,College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq

(**Received:** 8 / 6 / 2009 ---- **Accepted:** 27 / 4 / 2010)

Abstract :

In this paper, we continue the studies of several other authors, on nil-injective rings. In particular, we investigate some characterizations and several basic properties of these rings and the relationship between them and n-egular rings, SF-rings, the IN-rings and Kasch rings, respectively.

Keywords: right nil-injective, right Wnil-injective , IN-rings,Kasch rings, n-regular rings, SF-rings

1. Introduction :

Throughout this paper R denoted an associative ring with identity, and R-module is unital. For $a\in R$, r(a)and l(a) denote the right annihilator and the left annihilator of a, respectively. We write J(R), Y(R) (Z(R)), N(R) and Soc $(R_R)(Soc(_RR))$ for the Jacobson radical, the right (left) singular ideal, the set of nilpotent elements and right (left) socle of R, respectively.

A right R-module M is called right principally injective(briefly right P-injective) [2] if, for every principal right ideal P of R, any right R-homomorphism of P into M extends to one of R into M. R is called a right P-injective if R_R is P-injective. A ring R is called right mininjective if every homomorphism from a minimal right ideal of R to R_R can be extended from R to R_R[5].

Recall that a ring R is right minsymmetric if kR minimal, $k\in R$ implies that Rk is minimal [5]. A ring R is a left minannihilator ring if every minimal left ideal K of R is an annihilator, equivalently if lr(K)=K [5].A ring R is (Von Neumann) regular provided that for every $a\in R$ there exists $b\in R$ such that a=aba. R is called right Kasch ring if for every maximal right ideal is right annihilator of R [2]. Call A ring R IN-ring if the left annihilator of intersection of any two right ideals is the sum of the two left annihilators, that is $l(T\cap T')=l(T)+l(T)$ for all right ideals T and T' [1]. A ring R is called reduced if contains no non-zero elements of R. A ring R is said to be reversible if ba=0 implies ab=0 for $a,b\in R$ [3].

2. Nil Injective Rings

Following [7] a right R-module M is called nil-injective if for any $a\in N(R)$, any R- homomorphism $f:aR\to M$ can be extended to $R_R\to M$, or equivalently, there exists $m\in M$ such that f(x)=mx for all $x\in aR$. The ring R is called right nil-injective if R_R is nil-injective . Note that right p-injective ring and reduced ring are right nil-injective, but the converse is not true by [7]

We starts with the following lemma

Lemma 2.1 :[7]

let R be a right nil-injective ring. Then

(1) R is a right mininjective ring.

(2) R is right minsymmetric ring.

(3) R is left minannihilator ring.

The following two results are given in [5].

Lemma 2.2 :

Let R be a right mininjective ring and let $k\in R$.

(1) If kR is a minimal right ideals, then Rk is minimal left ideal.

(2) $Soc(R_R)\subseteq Soc(R_R)$.

Lemma 2.3 :

Let R be a right mininjective, right Kasch ring and consider the map

$\theta:T\to l(T)$

From the set of maximal right ideals T of R to the set of minimal left ideals of R. Then the following conditions hold:

(1) θ is one-to –one.

(2) θ is bijection if and only if lr(k)=k for all minimal left ideals k of R. In this case the inverse map is given by $K\to r(k)$.

The following theorem which extends lemma 2.3

Proposition 2.4 :

If R is right nil-injective and right kasch ring, then

(1) lr(K)=K for every minimal left ideal K of R.

(2) The map $\theta:T\to l(T)$ from the set of maximal right ideals T of R to the set of minimal left ideals of R is a bijection . And the inverse map is given by $K\to r(K)$, where K is a minimal left ideal of R.

(3) For $k\in R$, Rk is minimal if and only if kR is minimal in particular $Soc(R_R)=Soc(_RR)$.

Proof :

(1) From Lemma 2.1.

(2) It informed by Lemma 2.1 and Lemma 2.3.

(3) If Rk is minimal then r(k) is maximal by (2) which shows kR is also minimal .

Conversely; if kR is minimal, then Rk is minimal by Lemma 2.2.

We introduced the following lemma which contains several statements, which used frequently in sequel.

Lemma 2.5:

The following conditions are equivalent for a ring R.

(1) R is a right nil-injective ring.

(2) lr(a)=Ra for every $a\in N(R)$.

(3) $b\in Ra$ for every $a\in N(R)$, $b\in R$ with $r(a)\subseteq r(b)$.

(4) $l(r(a)\cap bR) = l(b)+Ra$ for all $a,b\in R$ with $ab\in N(R)$.

Proof :

See [7].

Following [7] a ring R is said to be right NPP if aR is projective for all $a\in N(R)$. A ring R is called n-regular ring if for every $a\in N(R)$, there exists $b\in R$ such that a=aba [7]. Clearly every n-regular rings are semi-prime by [7].

The following two results are given in [7]

Lemma 2.6 :

The following conditions are equivalent for a ring R.

(1) R is an n-regular ring.
(2) every right R-module is nil-injective.
(3) R is right nil-injective right NPP-ring.

Lemma 2.7 :
If R is a right NPP, then R is a right non-singular ring.

The next result is consider a necessary and sufficient condition for nil-injective ring to be n-regular.

Theorem 2.8 :
Let R be IN-ring . Then R is a right non-singular nil-injective ring if and only if R is n-regular.

Proof:
For any $0\neq a\in N(R)$, consider a principal left ideal Ra. Since R is a right nil-injective. By Lemma 2.5 $Ra=lr(a)$ and by hypothesis, R is right non-singular so $r(a)$ is not essential right ideal of R. Hence $r(a)\oplus L$ is an essential right ideal, for some non-zero right ideal L of R. Since R is IN-ring, then $lr(a)+l(L)=l(r(a)\cap L)=R$ while $lr(a)\cap l(L)\subseteq l(r(a)+L)=0$ since $r(a)+L$ is essential. So $Ra=lr(a)$ is direct summand of R. Therefore R is n-regular.

Conversely; since R is n-regular, then R is right nil-injective and NPP-ring. By Lemma 2.6 and by Lemma 2.7, we get R is a right non-singular ring.

Recall that a ring R is called a right (left) self-injective if for any essential right (left) ideal E of R, every right (left) R-homomorphism of E into R extends to one of R into R_R.

Proposition 2.9 :
Let R be a reversible and left self-injective ring. Then every right R-module is p-injective, if every right R-module is nil-injective.

Proof :
By Lemma 2.6, R is n-regular. So R is semi-prime. Thus for any left ideal I, $l(I)\cap I=0$. Let a be a non-zero element in R. Then $r(a)=l(a)$. Thus $lr(a)\cap l(a)=l(l(a))\cap l(a)=0$ and since R is left self injective ring, then aR is a right annihilator and $R=r(l(r(a))+r(l(a))=r(a)+aR$ by[4].

In particular: $1=d+ab$, for some b in R, and $d\in r(a)$. Hence; $a=a^2b$, and $a=aba$ and let $f:aR\to M$ be a right R-homomorphism, defined by $f(ab)=y\in M$. Then for any $r\in R$, $f(ar)=f(abar)=f(ab)ar= yar$. This means that every right R-module is P-injective.

3. Wnil-Injective Rings :

Recall that a right module M is called Wnil-injective if for any $0\neq a\in N(R)$, there exists a positive integer n such that $a^n\neq 0$ and any right R-homomorphism $f:a^nR\to M$ can be extended to $R\to M$, or equivalently, there exists $m\in M$ such that $f(x) = mx$ for all $x\in a^nR$[7]. Clearly every right nil-injective modules is right Wnil-injective. If R_R is Wnil-injective, then we call R is a right Wnil-injective ring.

We start the section with the following theorem which extends Lemma 2.5.

Theorem 3.1 :
A ring R is a right Wnil-injective if and only if for any $a\in N(R)$ there exists a positive integer n such that $an\neq 0$ and $Ran=lr(an)$.

Proof :
Suppose that a ring R is right Wnil-injective. Then for every $0\neq a\in N(R)$, there exists a positive integer n such that $a^n\neq 0$ and any right R-homomorphism of a^nR into R extends to endomorphism of R_R. It is clear that $Ra^n\subseteq lr(a^n)$. Let $d\in lr(a^n)$, since $r(a^n)=r(l(r(a^n)))\subseteq r(d)$, then we may define a right R-homomorphism $f:a^nR\to R$ by $f(a^nb)=db$ for all $b\in R$. Since R is Wnil-injective, there exists $y\in R$ such that $f(a^n)=ya^n$. Then $d=f(a^n)\in Ra^n$, which implies that $lr(a^n)\subseteq Ra^n$ and so that $lr(a^n)=Ra^n$.

Conversely, If $c\in N(R)$, there exists a positive integer n such that $Rc^n=lr(c^n)$. Let $f:c^nR\to R$ be any right R-homomorphism. Then $r(c^n)\subseteq r(f(c^n))$ which implies $Rf(c^n)\subseteq lr(R(f(c^n)))\subseteq lr(c^n) = Rc^n$, and therefore $f(c^n)=dc^n$ for some $d\in R$. This shows that R is a right Wnil-injective ring.

Theorem 3.2 :
Let R be a right Wnil-injective ring. Then $Soc(R_R)\subseteq r(J)$, where $J=J(R)$.

Proof :
Let $kR\subseteq R$ be a minimal right ideal. If $kR\not\subset r(J)$, then there exists $j\in r(J)$ such that $jk\neq 0$. Then $r(jk)=r(k)$. Since R is a right Wnil-injective and $(jk)^2=0$, then $lr(kj)=R(jk)$ by Theorem 3.1. Note that $k\in lr(jk)$ and $k=rjk$ for some $r\in R$. Then $(1-rj)k=0$ since $j\in J$,then$(1-rj)$ is an invertible, so that $k=0$ which is a contradiction. Therefore $Soc(R_R)\subseteq r(J)$.

Theorem 3.3:
Let R be a right Wnil-injective and a right non-singular ring. Then every minimal right ideal of R is direct summand.

Proof:
Let kR be a minimal right ideal of R. Since every minimal one-sided ideal of R is either nilpotent or direct summand of R [5]. If $(kR)^2\neq o$, then kR is a direct summand, we are done. If $(kR)^2=0$. Then $k^2=0$ and $k\in N(R)$ so $Rk=lr(k)$ by Theorem 3.1. Since $Y(R)=0$, then $r(k)$ not essential right ideal of R. Hence $r(k)\oplus L$ is essential right ideal for some non-zero right ideal L of R. Let $b\in L$ such that $kb\neq 0$, then $kR\subseteq L$ implies that $r(k)\cap bR=0$ but $(kb)^2\in (kR)^2=0$, therefore $kb\in N(R)$ and we get $l(r(k)\cap bR)=l(b)+Rk$ by Lemma2.5. But $r(k)\cap bR=0$ implies that $l(b)+Rk =R$. While $Rk\cap l(b)=lr(k)\cap l(b)\subseteq l(rk)+bR)\subseteq l(r(k)+L=(0)$.
So that $Rk\cap l(b)=0$ implies that Rk is a direct summand of R and $Rk=Re$ for some $e^2=e\in R$. Write $e=ck$, $c\in R$, then $k=ke=kck$. Set $g=kc$. Then $g^2=g$, $k=gk$ and we get $kR=gR$, so that kR is a direct summand of R.

Corollary 3.4 :

Let R be a right Wnil-injective and right NPP-ring. Then every minimal right ideal of R is a direct summand of R.

Now, we have the following theorem.

Theorem 3.5:

Let R be a right Wnil-injective ring with $Soc(R_R)\cap Y(R)=0$. Then every minimal right ideal is a direct summand of R.

Proof:

Let kR be a minimal right ideal of R if $(kR)^2\neq 0$, then kR is a direct summand, we are done. If $(kR)^2=0$, then $k^2=0$ and if r(k) essential right ideal of R, then $kR\subseteq Soc(R_R)\cap Y(R)=0$ which is a contradiction. Hence r(k) not essential. By a similar method proof is used in Theorem 3.3, kR is direct summand of R.

Theorem 3.6 :

Let R be a reversible ring. Then R is reduced ring if and only if every maximal essential right ideal of R is a right Wnil-injective.

Proof :

Let $0\neq a\in R$ such that $a^2=0$. If there exists a maximal right ideal M of R containing aR+r(a). Then M must be an essential right ideal. Otherwise M=r(e), $0\neq e^2=e\in R$. Hence $a\in r(e)=l(e)$ [since R reversible] and we get $e\in r(e)$ which a contradiction. Hence M is essential and so M is Wnil-injective, and the inclusion map $aR\rightarrow M$ can be extended to $R\rightarrow M$, this implies a = ma for some $m\in M$ since R is reversible a = am so $1-m\in r(a)\subseteq M$, which a contradiction, which shows that R is reduced.

Conversely; Assume that R is reduced. Then R is a right nil-injective. Since every nil-injective is Wnil-injective. So every maximal essential right ideal of R is a right Wnil-injective.

The following proposition extends Lemma 2.6 and Theorem 3.6

Proposition 3.7 :

Let R be a reversible ring. Then the following conditions are equivalent :

(1) every maximal essential right ideal of R is a right Wnil-injective.

(2) R is reduced.

(3) R is n-regular.

(4) R is a right nil-injective and right NPP.

Proof :

From Theorem 3.6, it is follows (1) implies (2)

(2)$\Rightarrow$(3) It is directly verified.

(3)$\Rightarrow$(4). Assume R is n-regular, then by Lemma 2.6 R a right nil-injective and NPP.

(4)$\Rightarrow$(1) It is obvious.

4. Connection between SF-ring and nil-injective ring

In this section we study the connection between SF-rings and nil-injective rings Recall that A ring R is called a right SF-ring, if every simple right R-module is flat [6].

Proposition 4.1 :

If R is a right SF and Kasch ring, then every maximal right ideal of R is a direct summand.

Proof :

First we have to prove $Y(R)\neq 0$. If not then by [7,Theorem 3.1] there exists $0\neq y\in Y$ such that $y^2=0$. If Y(R)+l(y)=R, then u+v=1 for some $u\in Y(R)$ and $v\in l(y)$. This yields uy=y. Let $x\in yR\cap r(u)$. Then x=yr for some $r\in R$ and ux=0 this implies uyr=0 and hence yr=x=0. Therefore $yR\cap r(u)=0$. On the other hand, since r(u) is an essential right ideal of R, yR=0 and y=0; a contradiction. Suppose that $Y(R)+l(y)\neq R$. Then there exists a maximal right ideal M containing Y(R)+l(y). But R/M is simple flat and $y\in M$. There exists $c\in M$ such that y=cy, whence $1-c\in l(y)\subseteq M$ yielding $1\in M$ and the contradiction $M\neq R$. This proves that Y(R)=0

Now since R is right Kasch ring, then for every maximal right ideal L of R, L=r(a) for some $a\in R$. If L is essential then $a\in Y(R)$, but Y(R)=0 a contradiction, so that L must be a direct summand.

Theorem 4.2 :

Let R be a right Kasch ring, then the following conditions are equivalent:

(1) R is regular ring.

(2) R is right nil-injective and right SF-ring.

Proof :

(1) $\Rightarrow$(2) It is clear.

(2) $\Rightarrow$(1) From Proposition 2.4 $Soc(R_R)=Soc(_RR)=S$ and by proposition 4.1 Y(R)=0 by. But R is nil-injective, therefore every minimal right ideal of R is direct summand of R by Theorem 3.3 so that S is regular. Now since R is right Kasch ring, then R/S is right Kasch and every maximal right ideal of R/S is an image of maximal essential of R under the natural map π : $R\rightarrow R/S$, but by Proposition 4.1 every maximal right ideal of R is direct summand so that R/S is regular. Therefore R is regular.

Wei and Chen [7] introduced the following result.

Lemma 4.3 :

let R be a right nil-injective ring. If N(R) forms an ideal of R, then $N(R)\subseteq Y(R)$.

Theorem 4.4 :

Let N(R) forms an ideal of R, then the following conditions are equivalent:

(1) R is strongly regular ring.

(2) R is right nil-injective and right SF-ring.

Proof :

(1) $\Rightarrow$(2) It is clear.

(2) $\Rightarrow$(1) Since R is right SF-ring, then Y(R)=0 by Proposition 4.1 and since N(R) forms an ideal of R and R right nil-injective then by Lemma 4.3 $N(R)\subseteq Y(R)=0$ so that R is reduced SF-ring. Therefore R is strongly regular ring.

Recall that a ring R is called 2-primal if the set of nilpotent elements of the ring coincides with the prime radical.

Corollary 4.5 :

Let R be 2-primal ring, then the following conditions are equivalent:

(1) R is strongly regular ring.

(2) R is right nil-injective and right SF-ring.

References:

1. V. Camillo; W. K. Nicholson and M. F. Yousif (2000),Ikeda Nakayama rings, J. of Algebra 226(2), pp.1001-1010.
2. J. Chen and N. Ding (1999), On General Principally injective Rings, Comm. In Algebra 27(5) pp. 2097-2116.
3. P. M. Cohn (1999), Reversible Rings, Bull. London Math. Soc., 31, pp. 641-648.
4. R. Y. Ming,(1983) on regular ring and annihilators, Math. Nach. 110, pp.137-142.
5. W.K. Nicholson and M.F. Yosif (2003), Quasi Frobenius Rings, Camberge Tracts in Math.,158, Cambrige univ., Cambrige.
6. M.B. Rege, (1986), On Von Neumann Regular Rings and SF-Rings, Math. Japonica, 31 (6), pp. 927-936.
7. J. Ch. Wei and J. H. Chen (2007), Nil-Injective Rings, Inter. J. of Algebra 2, pp. 1-21

الملخص

في هذا البحث ، نكمل دراسة بعض الباحثين، حول الحلقات الغامرة من النمط – nil . بصورة خاصة ندرس بعض المميزات والخواص الاساسية لهذه الحلقات والعلاقة بينها وبين الخلقات المنظمة من النمط – n ، الحلقات من النمط – SF ، الحلقات من النمط – IN وحلقات كاش.