

Numerical Derivative Using the Piecewise Uniform Mesh

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Abstract

In this paper, a technique of a piecewise-uniform meshes formed on an improvement finite difference algorithm for finding derivatives of functions. The purpose was to overcome difficulties which face numerical derivatives of functions with stiff formula, the main idea is that the formula includes some terms that can lead to rapid variation in the graph of the functions, which have recently been named singular layers in numerical analysis. The fundamental numerical difficulty is related to non-physical oscillations of the solution (instability) when the formula of the function dominates over the formula of its derivatives, this is a characteristic of many fluid flow problems. The use of Shishkin mesh to find derivatives of arbitrary degree and order is the novelty of this paper. The method was applied to find derivatives of some examples until third order and the results were compared with a previous study, mentioned in the paper, to derive the functions numerically.

Keywords: Piecewise Uniform Mesh, Boundary and Interior Singular Layers, Shishkin Mesh, Finite Difference Method, Numerical Differentiation.

الاشتقاق العددي باستخدام الشبكات الاعتيادية جزئياً

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الملخص

قُدِّمَتْ في هذا البحث تشكيل شبكة الاعتيادية جزئياً وتركيبها على خوارزمية الفروقات المنتهية مطورة لإيجاد المشتقات العددية للدوال. كان الهدف هو التغلب على الصعوبات التي تواجه المشتقات العددية للدوال ذوات قواعد اقتران صلدة، والفكرة الرئيسية هي أن قواعد الاقتران تتضمن بعض الصيغ التي يمكن أن تؤدي إلى اختلاف سريع في الرسم البياني للدوال، والتي تم تسميته مؤخراً في التحليل العددي بالطبقات المنفردة (الشاذة). ترتبط الصعوبة العددية الأساسية بالتذبذبات غير الطبيعية للحل (عدم الاستقرار) إذا كانت صيغة قاعدة اقتران الدالة تهيمن على صيغ مشتقاتها، والتي هي خواص كثير من مسائل تدفق الموائع. ان اصاله فكرة البحث تتمثل في تركيب شبكة ششكن في الطرق العددية المستخدمة في ايجاد المشتقات بدرجات ورتب اختيارية. تم تطبيق الطريقة لإيجاد مشتقات بعض الدوال حتى الدرجة الثالثة وتمت مقارنة النتائج مع دراسة سابقة ، مذكورة في البحث، لاشتقاق الدوال عددياً.

الكلمات الدالة: الشبكة الاعتيادية جزئياً، الطبقات الحدودية والداخلية الشاذة، شبكة ششكن، طريقة الفروقات المنتهية، الاشتقاق العددي.

1. Introduction

Functions are known to be Stiff $f_{\Psi}(x)$, if $1 \leq \Psi < \infty$, Its derivatives reach a certain order of ρ that depends on the softness of the data can be limited by:

$$|f^{(k)}(x)| \leq C[1 + \Psi^k e^{-\Psi\mu x}], k = 0, 1, \dots, \rho, x \in [0,1] \quad (1)$$

Where C and μ indicates a general positive constant independent of Ψ and the number of mesh points used. Kellogg and Tsan proved these estimates [1].

Recently, Shishkin showed that the regular and singular components of f can be separated: f have a representation of $f = E + S$, where

$$|E^{(k)}(x)| \leq C \text{ and } |S^{(k)}(x)| \leq C\Psi^k e^{-\Psi\mu x}, k = 0, 1, \dots, \rho, x \in [0,1], \quad (2)$$

This Shishkin decomposition has played a major role in finite difference analysis and finite component methods on the meshes of Shishkin and other adaptive layer meshes in recent years. It was generally thought that decomposition (2) was necessary to demonstrate a uniform convergence of standard numerical methods on the meshes of the adapted layer. It should be noted that (1) and (2) are equivalent[2]. Prandtl originally introduced the term boundary layer in 1904 [3]. It is often easier to locate the boundary layer, it will be found analytically or by plotting the graph of the function, to illustrate this we consider a typical boundary layer function $e^{-\Psi t}$ obviously this boundary layer is the one that fall at the boundary layer $x = 0$ as in Prob.4- Fig. 2. We define the arithmetic representation of the boundary layer as follows: because of the function of the boundary layer $S(x)$ and the separate measurement function $g(N)$ so that $g(N) \rightarrow 0$ as $N \rightarrow \infty$, then the arithmetic width ω of the boundary layer corresponding to the boundary function $S(x)$ for $g(N)$ is the smallest ω value that we have

$$\sup_{x \geq W} |f(x)| \leq g(N). \quad (3)$$

If the domain of the exact solution of a problem involving stiff parameter indicated by $\bar{\Omega}$ then the domain of numerical solutions is denoted by $\bar{\Omega}^N$ and satisfies $\bar{\Omega}^N \subseteq \bar{\Omega}$ where N is the discretization parameter of the numerical method [4]. In the one dimensions, the simplest

meshes are a uniform mesh $\bar{\Omega}^N = \{x_i\}_0^N$, with $N + 1$ spaced grid points $x_{i+1} - x_i = \frac{1}{N}$ for all i . But with this mesh none of the mesh points will be inside the boundary layer, unless if $N \geq \Psi$ (i.e. when $N < \Psi$ layer width $\omega < h$). Piecewise is a unified mesh installed on the boundary layer, consisting of two uniform meshes: a denser fine mesh is for the boundary layer and a coarse mesh is outside the boundary layer. The location of the shift point between the fine and the coarse mesh is the function of the stiffness parameter Ψ and a parameter of the discretization N . The correct distribution of the grid points is to obtain an equal or comparable number of mesh points in the fine and the coarse meshes [4]. A numerical comparison between Shishkin Mesh (**S-Mesh**) and classic Uniform Mesh (**U-Mesh**) will be presented, on same certain finite difference operator, in section (8).

2. Epsilon-Uniform Convergent

Consider the family of mathematical problems parameterized by a singular perturbation parameter $1/\Psi$, where $1/\Psi$ lies in the semi-open interval $0 < 1/\Psi < 1$. Suppose that each family problem has a unique solution denoted by f_Ψ , and that f_Ψ approaches a sequence of numerical solutions $\{F_\Psi, \bar{\Omega}^N\}_{N=1}^\infty$, where F_Ψ is defined on $\bar{\Omega}^N$ and N is a discretization parameter. It is then said that numerical solutions F_Ψ converge Epsilon- uniformly with the correct solution f_Ψ , if there is a positive integer N_0 , positive numbers C and p where N_0 , C , and p are independent of N and Ψ , so that for all $N \geq N_0$,

$$\sup_{1 \leq \Psi < \infty} \|F_\Psi - f_\Psi\|_{\bar{\Omega}_\tau^N} \leq CN^{-p}. \quad (4)$$

P is called the Epsilon-uniform rate of convergence and C is called Epsilon-uniform constant error. The definition above can be interpreted as follows, If the numerical method is Epsilon-uniform, for all values of $1 \leq \Psi < \infty$, however small, the pointwise error at the mesh points is never greater than CN^{-p} for all $N \geq N_0$ values. Note that since C, p and N_0 are independent of Ψ , it follows that the upper error limit is independent of Ψ [5].

3. Maximum Absolute Value Norm

The norm comes with boundary layer functions that do not involve averages namely L_∞ or the maximum norm defined by $\|f\|_\infty$ and it also follows this arithmetic law:

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)| \quad (5)$$

The maximum absolute value norm is the only appropriate criterion for studying the phenomena of the boundary layer; this is because differences between distinct functions are detected, no matter how much Ψ [6]. The choice of the maximum error measurement rule is due to the need to measure the error in the very small ranges in which the boundary or interior layers occur. Other parameters, such as the root average square, include error means, which slow down rapid changes in solutions and thus fail to capture the local behavior of error in these layers. To be sure, these assertions are valid, for example, the function of the boundary layer $e^{-\psi x}$ and $e^{-\sqrt{\psi}x}$ on $\bar{\Omega}$, where $\Omega = (0,1)$. Also known as L_∞ - Norm, uniform norm, max norm, or infinity norm defined as maximum absolute values of its components [6]:

$$\|\vec{v}\|_\infty = \max\{|v_i|: i = 1, 2, \dots, n\} \quad (6)$$

4. Shishkin Mesh

In particular, this area of numerical analysis has been strengthened by the contributions of Russian mathematician Grigorii Ivanovich Shishkin. We will describe the construction of Shishkin's formations and analyze the Shishkin solution, and we also aim to highlight the mainstream approach of Shishkin, which is evident from the wide range of problems that Shishkin has applied his methodology. The Shishkin mesh is a uniform mesh of piecewise. What distinguishes the Shishkin network from any other uniform piecewise mesh is the selection of so-called mesh transition parameter(s), which is the point at which the mesh scale changes abruptly [7].

5. The modest composition of one dimension piecewise uniform mesh

The resulting piecewise uniform meshes blow, depends only on one parameter τ . We will denote, the Piecewise uniform meshes with N mesh elements and one parameter τ , by $\bar{\Omega}_\tau^N$:

1- A single Boundary Layer Mesh:

A simple example of a piecewise uniform mesh is constructed on the interval $\Omega = (0,1)$, as follows; Select a point τ satisfactory $0 < \tau \leq \frac{1}{2}$ and assume that $N = 2^r$, for some $r \geq 2$. The point τ is called a mesh transition point which divides Ω into two subintervals the fine one is $(0, \tau)$ and the coarse one is $(\tau, 1)$ if the boundary layer is close to zero but the coarse one is $(0, 1 - \tau)$ and the fine one is $(1 - \tau, 1)$ if the boundary layer close to one. The

corresponding piecewise uniform mesh is constructed by dividing both $(0, \tau)$ and $(\tau, 1)$ or $(0, 1 - \tau)$ and $(1 - \tau, 1)$ into $\frac{N}{2}$ equal subintervals and it becomes uniform when $\tau = \frac{1}{2}$, as in Fig. 1 (a and b).

2- Couple Boundary Layers Meshes:

The following piecewise uniform fitted mesh is constructed if the boundary layer located at both boundary points $x = 0$ and $x = 1$, the fine mesh must be in the neighborhood of each of these points. Thus, two transition points exist and three subintervals are needed. The simplest construct is to choose τ achieving $0 < \tau \leq \frac{1}{3}$, and determine the mesh transition points at τ and $1 - \tau$. Assuming that $N = 3^r$, with $r \geq 2$, the intervals $(0, \tau)$, $(\tau, 1 - \tau)$ and $(1 - \tau, 1)$ each is divided into $\frac{N}{3}$ equal mesh elements and it becomes uniform when $\tau = \frac{1}{3}$, as in Fig. 1 (c).

3- Interior Layer:

As well as in the case of the boundary layer settled at the midpoint $x = \frac{1}{2}$ of the interval domain $\Omega = (0, 1)$, the fine mesh must be in the neighborhood of $x = \frac{1}{2}$. Thus, two transition points exist and three uniform pieces are needed. The simplest construct is to choose τ achieving $0 < \tau \leq \frac{1}{2}$, and determine the transition points at $\frac{1-\tau}{2}$ and $\frac{1+\tau}{2}$. Assuming that $N = 4^r$, with $r \geq 1$, the coarse intervals $(0, \frac{1-\tau}{2})$, and $(\frac{1+\tau}{2}, 1)$ are divided into $\frac{N}{4}$ equal mesh elements, and the fine subinterval $(\frac{1-\tau}{2}, \frac{1+\tau}{2})$ divided into $\frac{N}{2}$ equal mesh elements, and it becomes uniform when $\tau = \frac{1}{4}$. The resulting piecewise uniform mesh, illustrated in Fig.(1) (d) [6-9].

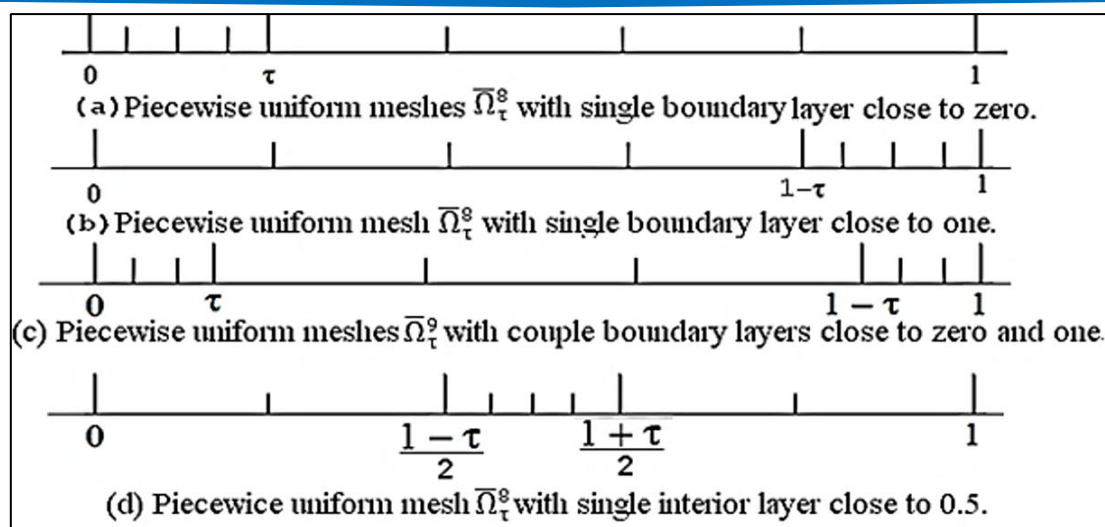


Fig. 1: One Dimension Piecewise Uniform Mesh.

6. On the accuracy of the proposed difference schemes (HMG)

The difference schemes we use are algorithm for numerical differentiation presented by H.Z.Hassan, et al [10] of discrete functions. The algorithm provided is suitable for calculating derivatives of any degree with an arbitrary order of precision on the prior domain of functions. The algorithm provided avoids the work of the initial difference and, in fact, more convenient than the use of specific differential formulas tab defined difference formulas, especially when derivatives are required with high roughness properties. Moreover, special terms schemes. The solution of normal differential equations with restrict interval value and partial differential equations with high numerical precision can be solved by implement the difference schemes. The numerical technique is based on an undetermined coefficient method in conjunction with Taylor's expansion. To avoid the difficulty of solving a system of linear equations, an explicit closed-form equation for the weighting coefficients is derived in terms of the elementary symmetric functions. This is done using an explicit closed formula for inverse of Vandermonde matrix. Furthermore, the difference scheme code is designed to give a uniform approximation order across a given domain. Suppose that $x_1, x_2, \dots, x_n$ are n different real numbers and $x_1 < x_2 < \dots < x_n$. Let f to be a continuous differentiable function in the interval $[x_1, x_n]$ and its values be given at these numbers. Then, for any one x_i the function value will be denoted as f_i . Moreover, for an equal tabular interval for the function f , the uniform mesh step size will be denoted by h where $h = x_{i+1} - x_i$ and $x_i = x_1 + (i - 1)h$ (for $h \geq 0$ and i, l will be denoted as an integer everywhere). Using the

method of undetermined coefficients, it can be derived m th-degree at the base point i approximated to the order of accuracy O by using a stencil with $n = m + O$ points as follows:

$$f_i(m, O) = \sum_{l=0}^n C_{i,l}(m, O) \cdot f_l, \quad i = 1, 2, 3, \dots, n \quad (7)$$

where the n unknown factors, $C_{i,l}(m, O)$, are the weighting coefficients for the derivative. The values of the function f at all other nodes $l \neq i$ can be expressed as Taylor series in terms of the reference point i as given in equation (2):

$$f_l = \sum_{k=0}^{\infty} \frac{(l-i)^k}{k!} h^k f_i^{(k)} = f_i^{(0)} + \frac{(1-i)}{1!} h f_i^{(1)} + \frac{(1-i)^2}{2!} h^2 f_i^{(2)} + \frac{(1-i)^3}{3!} h^3 f_i^{(3)} + \dots$$

$$+ \frac{(1-i)^n}{n!} h^n f_i^{(n)} + \dots \quad (8)$$

Substituting into Eq. (7) we get

$$f_i(m, O) = \sum_{l=1}^{l=n} C_{i,l}(m, O) = f_i + \frac{(l-i)}{1!} h f_i' + \frac{(l-i)^2}{2!} h^2 f_i'' + \frac{(l-i)^3}{3!} h^3 f_i^{(3)} + \dots$$

$$\frac{(l-i)^{(n-1)}}{(n-1)!} h^{n-1} f_i^{(n-1)} + \frac{(l-i)^n}{n!} h^n f_i^{(n)} + \dots \quad (9)$$

This can be rearranged to take the following form:

$$f_i(m, O) = f_i \left[\sum_{l=1}^{l=n} C_{i,l}(m, O) \right] + f_i' \frac{h}{1!} \left[\sum_{l=1}^{l=n} C_{i,l}(m, O) \cdot (1-i) \right]$$

$$+ f_i'' \frac{h^2}{2!} \left[\sum_{l=1}^{l=n} C_{i,l}(m, O) \cdot (1-i)^2 \right] + \dots$$

$$+ f_i^{(n-1)} \frac{h^{(n-1)}}{(n-1)!} \left[\sum_{l=1}^{l=n} C_{i,l}(m, O) \cdot (1-i)^{(n-1)} \right] + E_i(m, O) \quad (10)$$

The term $E_i(m, O)$ in the previous equation represents the remainder or error term in the approximation of $f_i(m, O)$. Eq. (10) can be rewritten in a more compact form as follows:

$$f_i(m, O) = \sum_{g=1}^{g=n} \left[f_i^{(g-1)} \frac{h^{(g-1)}}{(g-1)!} \sum_{l=1}^{l=n} (C_{i,l}(m, O) \cdot (1-i)^{(g-1)}) \right] + E_i(m, O) \quad (11)$$

Eq. (11) represents the exact value of themth derivative of the function at i. An approximated value up to the order of accuracy of O can be obtained by neglecting the remainder term $E_i(m, O)$. Therefore, we can write

$$f_i(m, O) \approx \sum_{g=1}^{g=n} \left[f_i^{(g-1)} \frac{h^{(g-1)}}{(g-1)!} \sum_{l=1}^{l=n} (C_{i,l}(m, O) \cdot (1-i)^{(g-1)}) \right], g = 1, 2, 3, \dots, n \quad (12)$$

We equate the coefficient of any degree derivative from the right-hand side to the corresponding coefficient of the same degree derivative from the left-hand side. Hence, a system of n linear equations in the unknown weighting coefficients can be represented by the following equation:

$$b_g \frac{(g-1)!}{h^{(g-1)}} = \sum_{l=1}^{l=n} [(C_{i,l}(m, O) \cdot (1-i)^{(g-1)})], g = 1, 2, 3, \dots, n \quad (13)$$

Where

$$b_g = \begin{cases} 0 & \text{if } g \neq m+1 \\ 1 & \text{if } g = m+1. \end{cases}$$

Therefore,

$$\sum_{l=1}^{l=n} [(C_{i,l}(m, O) \cdot (1-i)^{(g-1)})] = \begin{cases} 0 & g = 1, 2, 3, \dots, n \text{ and } g \neq m+1 \\ \frac{m!}{h^m} & g = m+1. \end{cases} \quad (14)$$

Then equation (14) represented in matrix form, transpose of the Vandermonde matrix $V_i(m, O)$ used, the Vandermonde matrix inverse $V_i^{-1}(m, O)$ expressed in a closed form, and using El-Mikkawy explicit closed form expression for the inverse matrix of the generalized Vandermonde matrix by using the elementary symmetric functions [10].

7. Algorithm

- 1- Determine the singular layer width (ω) and find its location analytically or by plotting as in Fig. 2, then the transition point parameter τ is:

$$\tau = \min\{\sigma, \omega\} \quad (15)$$

2- Generate the piecewise uniform meshes of type Shishkin mesh (S-mesh) $\bar{\Omega}_\tau^N$ so that it fits with the locations in step (1), as follows:

Location (a): If the boundary layer lies at the beginning point $x = 0$ of $\Omega = (0,1)$ then the piecewise uniform mesh is as follows:

$$\bar{\Omega}_\tau^N = \{x_i^a\}_{i=1}^N$$

With mesh points:

$$x_i^a = \begin{cases} \frac{2(i-1)}{N} \tau, i = 1, 2, \dots, \frac{N}{2} \\ x_{N/2}^a + \frac{2(i - \frac{N}{2})}{N} (1 - \tau), i = \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N \end{cases} \quad (16)$$

Where τ value from equation (15), such that $\sigma = \frac{1}{2}$, and ω constant value depends on the problem.

Location (b): If the Boundary layer lies at the end point $x = 1$ of $\Omega = (0,1)$ then the piecewise uniform mesh is as follows:

$$\bar{\Omega}_\tau^N = \{x_i^b\}_{i=1}^N \quad (17)$$

With mesh points:

$$x_i^b = \begin{cases} \frac{2(i-1)}{N} (1 - \tau), i = 1, 2, \dots, \frac{N}{2} \\ x_{N/2}^b + \frac{2(i - \frac{N}{2})}{N} \tau, i = \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N \end{cases} \quad (18)$$

Where τ value from equation (15), such that $\sigma = \frac{1}{2}$, and ω constant value depends on the problem.

Location (c): If the boundary layer lies at both the beginning point $x = 0$ and the end point $x = 1$ of $\Omega = (0,1)$ then the piecewise uniform mesh is as follows: $\bar{\Omega}_\tau^N =$

$$\{x_i^c\}_{i=1}^N \quad (19)$$

With mesh points:

$$x_i^c = \begin{cases} \frac{3(i-1)}{N}\tau, & i = 1, 2, \dots, \frac{N}{3} \\ x_{N/3}^c + \frac{3\left(i - \frac{N}{3}\right)}{N}(1-2\tau), & i = \frac{N}{3} + 1, \frac{N}{3} + 2, \dots, \frac{2N}{3} \\ x_{2N/3}^c + \frac{3(i-1)}{N}\tau, & i = \frac{2N}{3} + 1, \frac{2N}{3} + 2, \dots, N \end{cases} \quad (20)$$

where τ value from equation (15), such that $\sigma = \frac{1}{3}$, and ω constant value depends on the problem.

Location (d): If the interior layer lies at the midpoint $x = \frac{1}{2}$ of $\Omega = (0,1)$ then the piecewise uniform mesh is as follows:

$$\bar{\Omega}_\tau^N = \{x_i^d\}_{i=1}^N \quad (21)$$

With mesh points:

$$x_i^d = \begin{cases} \frac{2(i-1)}{N}(1-\tau), & i = 1, 2, \dots, \frac{N}{4} \\ x_{N/4}^d + \frac{2\left(i - \frac{N}{4}\right)}{N}\tau, & i = \frac{N}{4} + 1, \frac{N}{4} + 2, \dots, \frac{3N}{4} \\ x_{3N/4}^d + \frac{2\left(i - \frac{3N}{4}\right)(1-\tau)}{N}, & i = \frac{3N}{4} + 1, \frac{3N}{4} + 2, \dots, N \end{cases} \quad (22)$$

[4], [6], and [11].

8. Numerical results and Conclusion

In this section four numerical examples presented and plotted in fig. 2 to illustrate the theoretical results. The test problems below are Stiff functions in forms $f_\Psi(x)$, $x \in \Omega$, $1 \ll \Psi < \infty$, Ω is the domain, Ψ is the stiff parameter, ω is a layer width, and α ($\alpha \approx 1$) is arbitrary scalar :

$$1- f_\Psi(x) = e^x + e^{\Psi x}, x \in \Omega = [0,1], \omega = \frac{\alpha}{\Psi} \ln \Psi [10]$$

$$2- f_\Psi(x) = \frac{\Psi}{1+\Psi x}, x \in \Omega = [-1,1], \omega = \frac{\alpha}{\sqrt{\Psi}} \ln \Psi$$

$$3- f_\Psi(x) = \cos \pi x + e^{\sqrt{\Psi}(x-1)} + e^{-\sqrt{\Psi}(x+1)},$$

$$x \in \Omega = [-1,1], \omega = \frac{\alpha}{\sqrt{\Psi}} \ln \Psi$$

$$4- f_{\Psi}(x) = e^{-\sqrt{\Psi}x}, x \in \Omega = [0,1], \omega = \frac{\alpha}{\sqrt{\Psi}} \ln \Psi [12], [13]$$

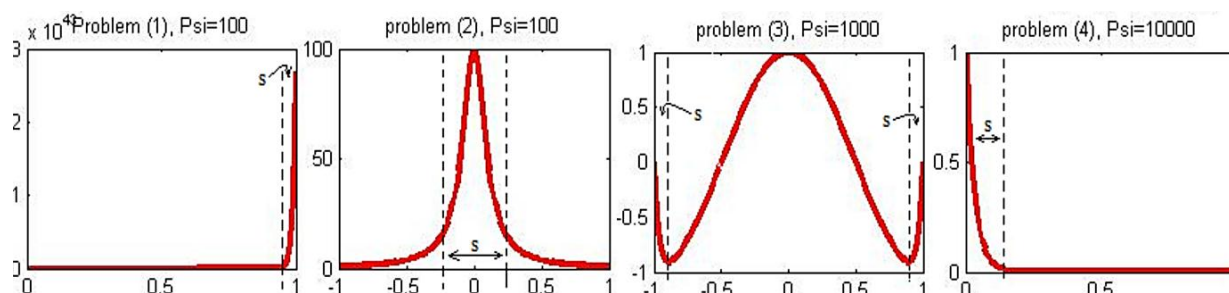


Fig. 2: Matlab plots of Test Problem Functions, these sharper bounds (S) identify both the location and the scale of the layers.

Numerical comparisons between the maximum errors of the HMG-algorithm [10] on Shishkin Mesh (**S-Mesh**) vs the maximum errors of HMG-algorithm on **Uniform Mesh** presented, the MATLAB computer program are used. The data and results are presented through the **Tables (2),(3), (4), and (5)** each table contains maximum error of 1st, 2nd, and 3rd of approximation derivatives of the four test functions with their corresponding exact solutions, i.e. each table contains the numerical data of three of the **Figs..** The results of the work are presented through a set of tables; in addition, it is represented by a set of **Figs..** In order to further clarify the image of the work, **Table 1** was provided for describe each of the test problems, details of the data, the size of the solutions, and to illustrate the figures and other tables.

Table 1: Description of test problems and numerical applications.

Prob . No.	Tab No.	Fig. No.	Layer type	layer location	Ψ	ω term	determined derivatives	# Solved Problem
1	2	3 to 4(a)	boundary	Lower	10	$\alpha \ln \Psi / \Psi$	$f_{\Psi}^{(1)}, f_{\Psi}^{(2)}, f_{\Psi}^{(3)}$	120
2	3	4(b) to 5	interior	center	10^2	$\alpha \ln \Psi / \Psi$	$f_{\Psi}^{(1)}, f_{\Psi}^{(2)}, f_{\Psi}^{(3)}$	120
3	4	6 to 7(a)	boundary	upper	10^2	$\alpha \ln \Psi / \sqrt{\Psi}$	$f_{\Psi}^{(1)}, f_{\Psi}^{(2)}, f_{\Psi}^{(3)}$	120
4	5	7(b) to 8	boundary	Both	10^4	$\alpha \ln \Psi / \sqrt{\Psi}$	$f_{\Psi}^{(1)}, f_{\Psi}^{(2)}, f_{\Psi}^{(3)}$	120

Table 2: Comparison of maximum norm error of problem 1.

N	1 st Derivative		2 nd Derivative		3 rd Derivative	
	U.Mesh	S.Mesh	U.Mesh	S.Mesh	U.Mesh	S.Mesh
90	3.827E-04	5.884E-05	1.852E-02	2.914E-03	5.905E-01	9.361E-02
92	3.237E-04	4.966E-05	1.566E-02	2.449E-03	4.836E-01	7.914E-02
94	2.747E-04	4.204E-05	1.329E-02	2.030E-03	4.008E-01	8.926E-02
96	2.340E-04	3.578E-05	1.136E-02	1.797E-03	3.640E-01	5.762E-02
98	1.999E-04	3.051E-05	9.717E-03	1.500E-03	3.140E-01	1.045E-01
100	1.713E-04	2.610E-05	8.318E-03	1.299E-03	2.563E-01	4.242E-02
102	1.472E-04	2.239E-05	2.914E-03	1.105E-03	2.198E-01	3.644E-02
104	1.269E-04	1.932E-05	6.210E-03	9.891E-04	2.144E-01	3.124E-02
106	1.097E-04	1.661E-05	5.363E-03	8.017E-04	1.809E-01	1.148E-01
108	9.497E-05	1.436E-05	4.612E-03	6.634E-04	1.385E-01	2.377E-02
110	8.254E-05	1.246E-05	4.048E-03	6.338E-04	1.492E-01	2.065E-02
112	7.183E-05	1.081E-05	3.508E-03	4.693E-04	1.216E-01	2.627E-02
114	6.277E-05	9.438E-06	3.118E-03	4.192E-04	1.366E-01	1.938E-02
116	5.487E-05	8.237E-06	2.683E-03	3.739E-04	9.060E-02	2.066E-02
118	4.811E-05	7.153E-06	2.389E-03	2.438E-04	1.032E-01	2.432E-02
120	4.233E-05	6.263E-06	2.148E-03	2.059E-04	1.315E-01	1.029E-01
122	3.726E-05	5.557E-06	1.906E-03	2.409E-04	1.056E-01	8.799E-02
124	3.278E-05	4.916E-06	1.587E-03	2.247E-04	2.310E-02	7.022E-02
126	2.897E-05	4.271E-06	1.406E-03	1.405E-04	4.483E-02	7.259E-03
128	2.570E-05	3.845E-06	1.268E-03	2.256E-04	3.750E-02	2.676E-02

Table 3: Comparisons of maximum norm error of problem 2.

N	1 st Derivative		2 nd Derivative		3 rd Derivative	
	U.M	S.M	U.M	S.M	U.M	S.M
332	7.741E-5	7.462E-7	1.7960E-3	4.43915E-5	1.64875	1.62221E-2
336	7.022E-5	6.771E-7	1.6360E-3	4.13786E-5	1.50293	1.47939E-2
340	6.388E-5	6.150E-7	1.4920E-3	3.85816E-5	1.37104	1.34992E-2
344	5.8640E-5	5.596E-7	1.3620E-3	3.60060E-5	1.25165	1.23301E-2
348	5.385E-5	5.120E-7	1.244E-3	3.36100E-5	1.14351	1.12650E-2
352	4.9490E-5	4.687E-7	1.1380E-3	3.13825E-5	1.04548	1.03021E-2
356	4.550E-5	4.293E-7	1.0420E-3	2.93287E-5	9.5655E-1	9.42747E-3

N	1 st Derivative		2 nd Derivative		3 rd Derivative	
	U.M	S.M	U.M	S.M	U.M	S.M
360	4.185E-5	3.936E-7	9.5450E-4	2.74078E-5	8.758E-1	8.63391E-3
364	3.851E-5	3.6104E-7	8.754E-4	2.56116E-5	8.0246E-1	7.91227E-3
368	3.546E-5	3.313E-7	8.0370E-4	2.39668E-5	7.3577E-1	7.25474E-3
372	3.267E-5	3.043E-7	7.384E-4	2.24197E-5	6.751E-1	6.65690E-3
376	3.0120E-5	2.796E-7	6.7910E-4	2.09985E-5	6.1985E-1	6.11087E-3
380	2.778E-5	2.571E-7	6.250E-4	1.96731E-5	5.6952E-1	5.61343E-3
384	2.563E-5	2.366E-7	5.758E-4	1.84159E-5	5.2363E-1	5.16336E-3
388	2.367E-5	2.178E-7	5.308E-4	1.72780E-5	4.8175E-1	4.74861E-3
392	2.186E-5	2.007E-7	4.898E-4	1.61956E-5	4.4352E-1	4.37162E-3
396	2.0210E-5	1.850E-7	4.5226E-4	1.51984E-5	4.0859E-1	4.02842E-3
400	1.868E-5	1.706E-7	4.1794E-4	1.42697E-5	3.7665E-1	3.70992E-3
404	1.729E-5	1.575E-7	3.8651E-4	1.33969E-5	3.4947E-1	3.42169E-3
408	1.600E-5	1.454E-7	3.5772E-4	1.25904E-5	3.2533E-1	3.16647E-3

Table 4: Comparison of maximum norm error of problem 3.

N	1st Derivative		2nd Derivative		3rd Derivative	
	U.M	S.M	U.M	S.M	U.M	S.M
69	2.0058E-05	5.2982E-06	0.00089836	0.00024241	0.02680568	0.00738692
72	1.4869E-05	3.9014E-06	0.00066944	0.00017928	0.02007716	0.0054866
75	1.1142E-05	2.9053E-06	0.00050401	0.00013404	0.01518717	0.00411832
78	8.4323E-06	2.1861E-06	0.00038313	0.00010123	0.01159503	0.0031218
81	6.4413E-06	1.661E-06	0.00029387	7.7182E-05	0.00892966	0.00238823
84	4.9635E-06	1.2736E-06	0.00022731	5.9367E-05	0.00693324	0.00184278
87	3.8561E-06	9.8484E-07	0.00017722	4.6044E-05	0.00542452	0.00143347
90	3.0188E-06	7.6765E-07	0.0001392	3.599E-05	0.00427487	0.00112354
93	2.3805E-06	6.0286E-07	0.00011011	2.8338E-05	0.00339193	0.00088699
96	1.8899E-06	4.768E-07	8.7677E-05	2.2467E-05	0.00270869	0.00070493
99	1.5102E-06	3.7962E-07	7.0253E-05	1.7929E-05	0.00217633	0.00056384
102	1.2141E-06	3.0415E-07	5.6626E-05	1.4396E-05	0.00175863	0.00045365
105	9.8171E-07	2.4514E-07	4.5899E-05	1.1627E-05	0.00142896	0.00036708
108	7.9816E-07	1.987E-07	3.7404E-05	9.4425E-06	0.00116719	0.00029876
111	6.5231E-07	1.6192E-07	3.0637E-05	7.7093E-06	0.00095811	0.00024445
114	5.3576E-07	1.3263E-07	2.5215E-05	6.3252E-06	0.00079019	0.00020085
117	4.421E-07	1.0916E-07	2.0848E-05	5.214E-06	0.00065452	0.0001657

N	1st Derivative		2nd Derivative		3rd Derivative	
	U.M	S.M	U.M	S.M	U.M	S.M
120	3.6646E-07	9.0258E-08	1.7314E-05	4.3182E-06	0.00054458	0.00013742
123	3.0507E-07	7.496E-08	1.4439E-05	3.5919E-06	0.00045499	0.00011469
126	2.5501E-07	6.2517E-08	1.209E-05	3.0003E-06	0.00038165	9.5947E-05

Table 5: Comparison of maximum norm error of problem 4.

N	1 st Derivative		2 nd Derivative		3 rd Derivative	
	U.M	S.M	U.M	S.M	U.M	S.M
552	6.78908E-06	3.74541E-10	3.18711E-03	2.53785E-07	9.96265E-01	3.21388E-04
554	6.61049E-06	3.59876E-10	3.10416E-03	1.45961E-07	9.70665E-01	5.50270E-04
556	6.43716E-06	3.49786E-10	3.02363E-03	2.53320E-07	9.45688E-01	3.82423E-04
558	6.26893E-06	3.35405E-10	2.94544E-03	2.00467E-07	9.21508E-01	7.41005E-04
560	6.10562E-06	3.24974E-10	2.86952E-03	2.48430E-07	8.98040E-01	8.81195E-04
562	5.94708E-06	3.13975E-10	2.79579E-03	1.32364E-07	8.75173E-01	2.64168E-04
564	5.79316E-06	3.12980E-10	2.72417E-03	1.29191E-07	8.52981E-01	4.11034E-04
566	5.64370E-06	2.93198E-10	2.65463E-03	1.26103E-07	8.31436E-01	1.05000E-03
568	5.49856E-06	2.98741E-10	2.58707E-03	1.47615E-07	8.10488E-01	1.41430E-03
570	5.35761E-06	2.87002E-10	2.52144E-03	2.11643E-07	7.90144E-01	1.66893E-04
572	5.22071E-06	2.78249E-10	2.45767E-03	1.17325E-07	7.70368E-01	2.19345E-03
574	5.08773E-06	2.74213E-10	2.39571E-03	1.14554E-07	7.51159E-01	2.13623E-04
576	4.95854E-06	2.64293E-10	2.33550E-03	1.44821E-07	7.32451E-01	6.80923E-04
578	4.83304E-06	2.54886E-10	2.27699E-03	1.36206E-07	7.14293E-01	2.05421E-03
580	4.71109E-06	2.52868E-10	2.22011E-03	1.06670E-07	6.96640E-01	2.63596E-03
582	4.59260E-06	2.38742E-10	2.16484E-03	1.56695E-07	6.79465E-01	3.12805E-04
584	4.47744E-06	2.28056E-10	2.11110E-03	1.01753E-07	6.62760E-01	6.08444E-04
586	4.36552E-06	2.36923E-10	2.05886E-03	1.96975E-07	6.46543E-01	7.24793E-04
588	4.25674E-06	2.21945E-10	2.00807E-03	9.70890E-08	6.30743E-01	1.52397E-03
590	4.15100E-06	2.12765E-10	1.95869E-03	9.48475E-08	6.15410E-01	1.60980E-03

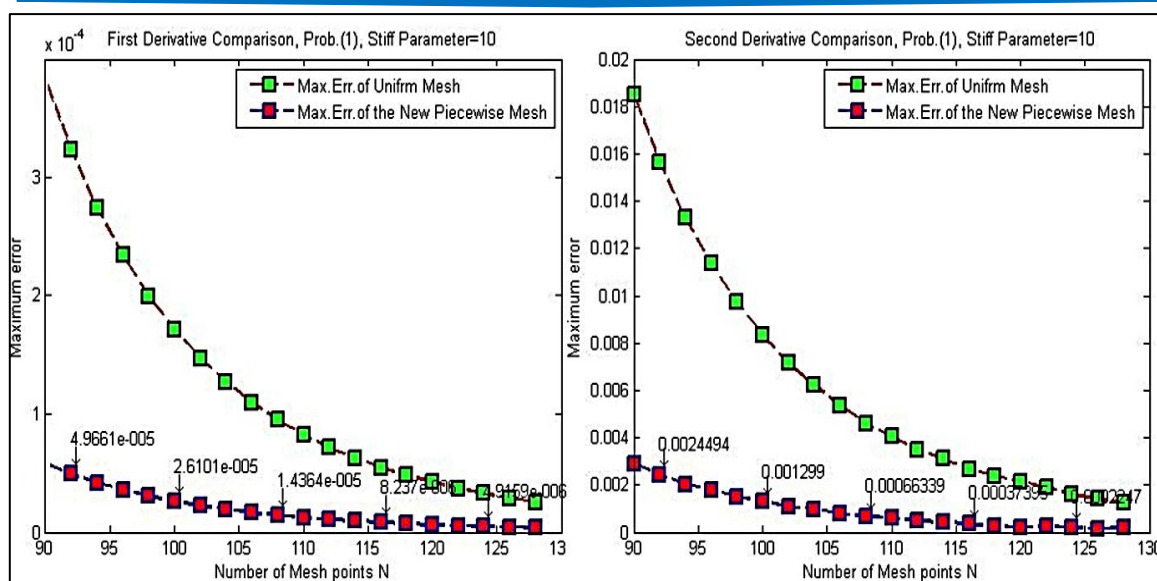


Fig. 3: (a) 1st derivative comparison of prob. 1 (b) 2nd derivative comparison of prob.1.

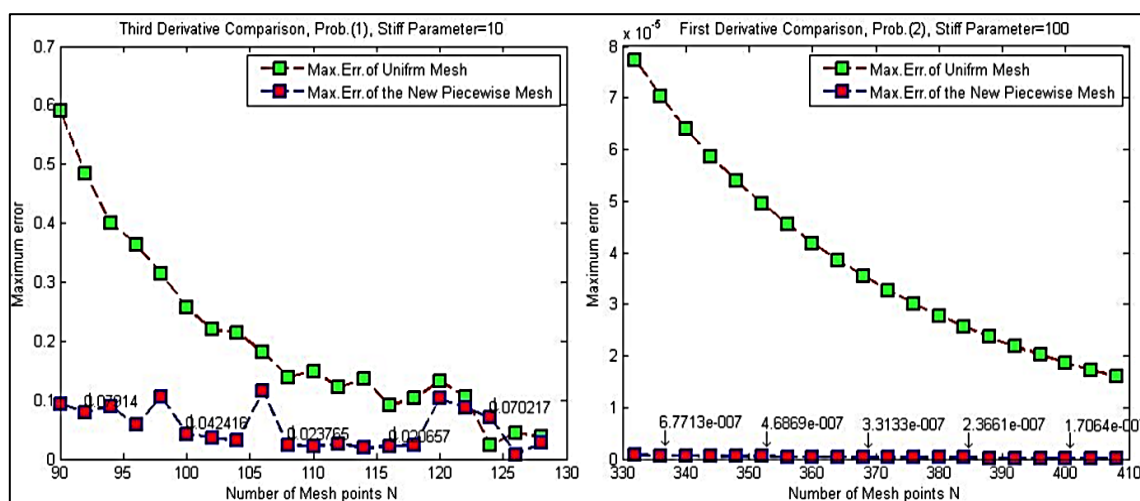


Fig. 4: (a) 3rd derivative comparison of prob.1 (b) 1st derivative comparison of prob.2.

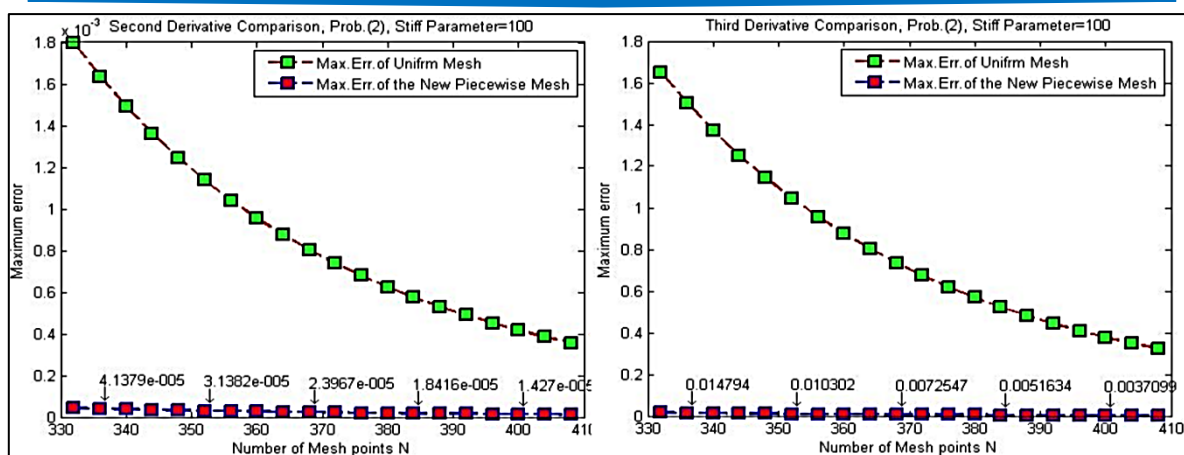


Fig. 5: (a) 2nd derivative comparison of prob.2 (b) 3rd derivative comparison of prob.2.

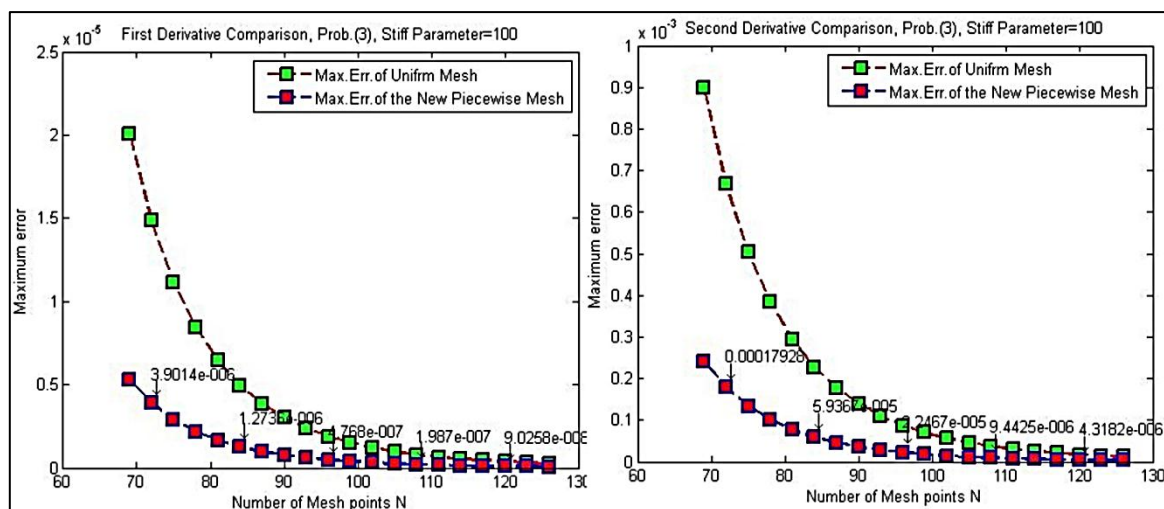


Fig. 6: (a) 1st derivative comparison of prob.3 (b) 2nd derivative comparison of prob.3.

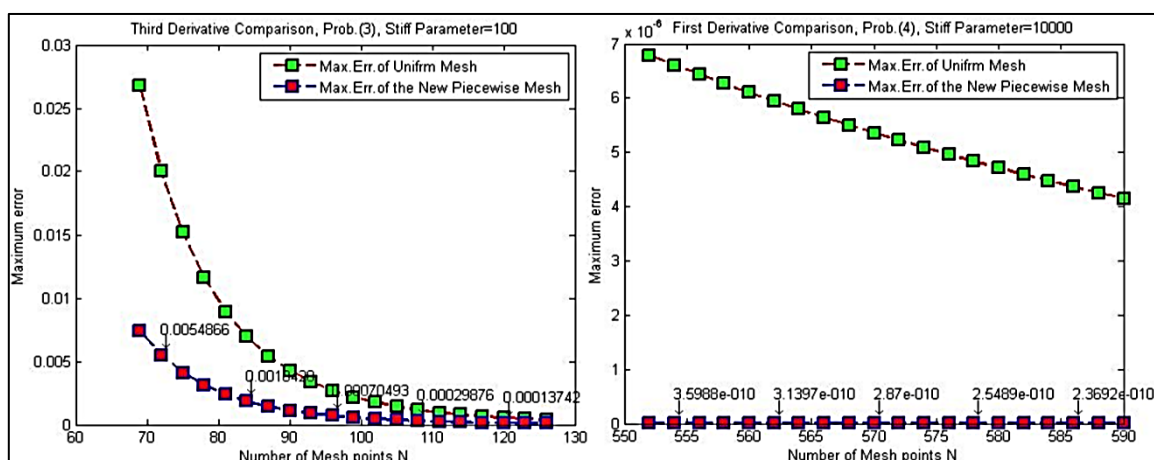


Fig. 7: (a) 3rd derivative comparison of prob.3 (b) 1st derivative comparison of prob.4.

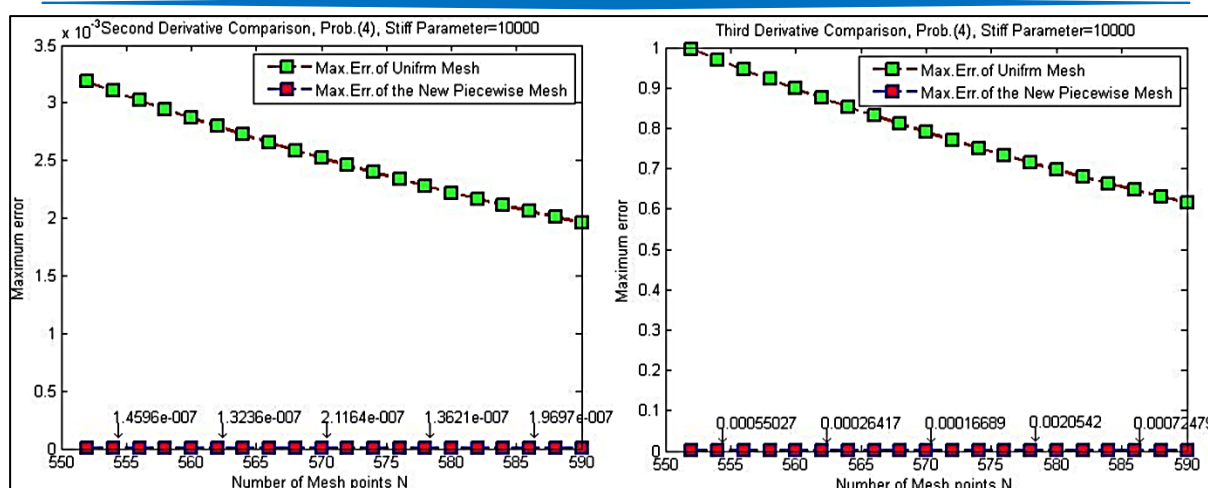


Fig. 8 (a) 2nd derivative comparison of prob.4 (b) 3rd derivative comparison of prob.4.

The numerical results indicate that the new technique has an improvement about (86.5038105%) in Maximum error of the s-mesh method against the uniform mesh method [10], as in Table 6.

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