

ON THE MEASURABLE FAMILIES OF $\hat{S}$ - BANACH LATTICE

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ABSTRACT:

The aim of this paper is to study a measurable families of $\hat{S}$ - Banach lattice and their decomposition of a separable Banach $\hat{S}$ - vector lattice into measurable field of Banach lattice . Moreover there exists a Freudenthal unit in $\hat{X}$ if and only if it exists in X_τ .

المستخلص :-

يهدف هذا البحث الى دراسة الفصائل القابلة للقياس للحزم المتجه البناخية - $\hat{S}$ وتحللها من الحزم البناخية المتجه المنفصلة - $\hat{S}$ الى الحقول القياسية من تلك الحزم . إضافة الى ذلك وجود الوحدة الفريدنتالية في $\hat{X}$ اذا وفقط كانت في X_τ .

1.Introduction :-

In this paper , we introduce the important definitions and some information about the measurable field of metric space and of Boolean algebras that we needed it in this work . We get the main results about the decomposition of separable Banach $\hat{S}$ - vector lattice into a measurable field of Banach lattice .

2.The preliminaries

We recall the basic definitions and information which are needed in our work .

2.1 Definition: [11]

A mapping $\rho : X \times X \rightarrow \hat{S}$ is called a **metric** on a set X with values in $\hat{S}$ if

1. $\rho(x, y) \geq 0$ for any $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$.
2. $\rho(x, y) = \rho(y, x)$ for any $x, y \in X$.
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for any $x, y, z \in X$.

2.2 Remark:[8]

Suppose that T is the interval $[0,1]$, let A be the σ - algebra of the Lebesgue measurable subset of T , and P be the Lebesgue measure on T .

We shall denote by $\hat{S}$ the ring of all real measurable functions on $[0,1]$.

2.3 Definition: [11]

The space (X, ρ) is called **separable** , if there exists a countable subset $M \subset X$ such that for any $x \in X$, there is $\{z_n\}_{n=1}^{\infty} \subset M$ for which $\rho(x, z_n) \xrightarrow{(t)} 0$.

Suppose that (X_τ, ρ_τ) be a complete separable metric space defined for P -almost every $\tau \in T$.

2.4 Definition: [11]

A **measurable field of metric space** is a pair $\{X, \{X_\tau\}_{\tau \in T}\}$ where X is a totality of functions $x: \tau \rightarrow x(\tau) \in X_\tau$ for P -almost every $\tau \in T$ such that :

1. There exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\{x_n(\tau)\}_{n=1}^{\infty}$ is dense in (X_τ, ρ_τ) for P -almost every $\tau \in T$.
2. The function $\tau \rightarrow \rho_\tau(x(\tau), y(\tau))$ is measurable on $(T, \mathcal{A}, P)$ for all $x, y \in X$.
3. If $\{y_n\}_{n=1}^{\infty} \subset X$, $y: \tau \rightarrow y(\tau) \in X_\tau$ for P -almost every $\tau \in T$ and $\rho_\tau(y_n(\tau), y(\tau)) \rightarrow 0$ as $n \rightarrow \infty$ for P -almost every $\tau \in T$, then $y \in X$.

2.5 Remark:[9,10]

Let $\{X, \{X_\tau\}_{\tau \in T}\}$ be a measurable field of metric space and let $\hat{X}$ be the set of all classes for P -almost everywhere coinciding elements from X .

For any $\hat{x}, \hat{y} \in \hat{X}$, we denote by $\hat{\rho}(\hat{x}, \hat{y})$ the class from $\hat{S}$ containing the function $\rho_\tau(x(\tau), y(\tau))$, where $x(\tau)$ and $y(\tau)$ are representatives of the classes $\hat{x}$ and $\hat{y}$, respectively . Then $(\hat{X}, \hat{\rho})$ is a complete separable space with a $\hat{S}$ - valued metric .

2.6 Definition: [11]

A measurable field of metric space is said to be **saturated** if the following two conditions are satisfied :

1. $y: \tau \rightarrow y(\tau) \in X_\tau$ for P -almost every $\tau \in T$.
2. The function $\tau \rightarrow \rho_\tau(x(\tau), y(\tau))$ is measurable on $(T, \mathcal{A}, P)$ for any $x \in X$, imply $y \in X$.

2.7 Remark: [11]

Suppose that in the definition of saturated measurable field of metric space all X_τ coincide and are equal to (Z, ρ_τ) .

We shall denote by $S(T, Z)$ the set of measurable mapping from $(T, \mathcal{A}, P)$ into Z and by $\hat{S}(T, Z)$ the set of the classes of P -almost everywhere coinciding function from $S(T, Z)$.

A $\hat{S}$ - valued metric $\hat{\rho}(\hat{x}, \hat{y}) = \rho_\tau(\widehat{x(\tau)}, y(\tau))$ is naturally defined on $\hat{S}(T, Z)$. In [8] it is shown that $\hat{S}(T, Z)$ is $\hat{S}$ - isometric to $\hat{X}$.

2.8 Remark: [11]

Let (U, d) be a universal metric space of Uryson (one can consider that $U = C[0,1]$ with the uniform metric .

In [9] it is shown that any complete separable space X with an $\hat{S}$ -valued metric is $\hat{S}$ -isometric to some closed subset of $\hat{S}(T, Z)$.

Now , let $\{X, \{X_\tau\}_{\tau \in T}\}$ be a measurable field of metric space , and let A_τ be closed subset of X_τ for P -almost every $\tau \in T$.

2.9 Definition: [9]

A **measurable field of closed sets** is a pair $\{A, \{A_\tau\}_{\tau \in T}\}$ where $A \in X$ and

1. If $x \in A$, then $x(\tau) \in A_\tau$ for P -almost every $\tau \in T$.
2. There exists $\{x_n\}_{n=1}^\infty \subset A$ such that $\{x_n(\tau)\}_{n=1}^\infty$ is dense in A_τ for P -almost every $\tau \in T$.
3. If $\{y_n\}_{n=1}^\infty \subset A$, $y \in X$ and $\rho_\tau(y_n(\tau), y(\tau)) \rightarrow 0$ as $n \rightarrow \infty$ for P -almost every $\tau \in T$, then $y \in A$.

2.10 Remark:[10]

We say that X be a bimodule over $\hat{S} = [0,1]$, i.e. X is abelian group with respect to addition operation (+) and right and left multiplication by element from $\hat{S}$ are defined on X having the properties:

1. $\lambda(x + y) = \lambda x + \lambda y$, $(x + y)\lambda = x\lambda + y\lambda$
2. $(\lambda + \mu)x = \lambda x + \mu x$, $x(\lambda + \mu) = x\lambda + x\mu$
3. $\lambda(\mu x) = (\lambda\mu)x$, $(x\lambda)\mu = x(\lambda\mu)$
4. $\hat{1} \cdot x = x \cdot \hat{1}$, for all $x, y \in X, \lambda, \mu \in \hat{S}$

2.11 Definition: [2, 4, 7]

The collection B of Borel sets of a topological space X is the smallest σ -algebra containing all open sets of X . That is, in addition to containing open sets, B must be closed under complement and countable intersections (and, thus, is also closed under countable unions).

2.12 Definition: [12, 13]

A **Borel mapping** is a mapping such that the inverse image of every Borel set is Borel. It is called Borel function.

2.13 Definition:[3]

A set $C \subseteq R^n$ is called **convex** if for all $x, y \in C$, we have $\lambda x + (1-\lambda)y \in C$ for all $\lambda \in [0, 1]$.

2.14 Definition:[5]

A **Banach space** is a normed linear space in which every Cauchy sequence is convergent.

We cite now example of **S^* -vector lattices**

2.15 Example:[10]

Let P be the Lebesgue measure on $\nabla(S^*)$, $L_1(\nabla(S^*), P)$ be the Banach space of all P -integrable functions on $([0,1], p)$, ∇_1 be a σ -subalgebra of $\nabla(S^*)$ isomorphic to $\nabla(S^*)$, $E: L_1(\nabla(S^*), P) \rightarrow L_1(\nabla_1, m)$ be the conditional expectation.

Then $\mu(e) = E(e)$ is a strictly positive measure on $\nabla(S^*)$ with values in the σ -complete sublattice S_1 of all functions from S^* which are measurable with respect to ∇_1 (it is clear that S_1 can be identified with S^* , since ∇_1 is isomorphic to $\nabla(S^*)$).

In addition, $\mu(eg) = e\mu(g)$ for any $e \in \nabla_1, g \in \nabla(S^*)$.

It is clear that S^* is a normal S_1 - module .

Denote by $\hat{\mu}$ the integral constructed with respect to the measure μ . For every number $P \geq 1$ we set

$$L_P(\nabla(S^*), \mu) = \{x \in S^* : \hat{\mu}(|x|^P) \in S\}$$

And

$$\|x\|_P = (\hat{\mu}|x|^P)^{\frac{1}{P}}$$

It is shown in [3 ,4] that $L_P(\nabla(S^*), \mu, \|\cdot\|_P)$ is a Banach S_1 - vector lattice .

3. The Main Results :

This section is devoted to the main results concerning the measurable families of Banach lattice and decomposition of separable Banach $\hat{S}$ -vector lattice of ordinary Banach lattices .

We begin with the useful information that used in our paper .

3.1 Definition : [10]

A measurable field X of Banach spaces $(X_\tau, \|\cdot\|_\tau)$, $\tau \in T = ([0,1])$, P) is a set of functions $x: \tau \rightarrow x(\tau) \in X_\tau$ defined p - almost everywhere on T and such that

1. The function $\tau \rightarrow \|x_\tau\|_\tau$ is measurable on T for all $x \in X$;
2. If $x, y \in X$, $\alpha, \beta \in \hat{S}$, then the function $\alpha(\tau)x(\tau) + \beta(\tau)y(\tau)$ also belongs to X .
3. There exists a sequence $\{x_n\} \subset X$ such that the set $\{x_n(\tau)\}$ is dense in X_τ for p - almost every $\tau \in T$.
4. If $\{y_n\} \subset X$ and $y: \tau \rightarrow y(\tau) \in X_\tau$ is a function and $\|y_n(\tau) - y(\tau)\|_\tau \rightarrow 0$ as $n \rightarrow \infty$ for p - almost every $\tau \in T$, then $y \in X$.

It is clear that a measurable field Banach space X is also a measurable field metric space (X_τ, d_τ) , $\tau \in T$, where

$$d_\tau(x(\tau), y(\tau)) = \|x(\tau) - y(\tau)\|_\tau, \quad x(\tau), y(\tau) \in X_\tau .$$

3.2Note:[10]

A measurable field Banach space $(X_\tau, \|\cdot\|_\tau)$ with property (1,2) of definition (2.1) is said to be **saturated**.

In the sequel the record $\{X, \{X_\tau\}_{\tau \in T}\}$ will mean that X is a measurable field Banach space $(X_\tau, \|\cdot\|_\tau)$.

Now , for any measurable field Banach space X denoted by $\hat{X}$ the set of all classes $\hat{x}$ of p - almost everywhere equal functions

$x: \tau \rightarrow x(\tau)$ from X .

The pointwise operations :-

$$x + y: \tau \rightarrow x(\tau) + y(\tau),$$

$$\alpha x: \tau \rightarrow \alpha(\tau)x(\tau),$$

$$x\alpha: \tau \rightarrow x(\tau)\alpha(\tau),$$

$x, y \in X, \alpha \in \hat{S}$ and the function $\|\cdot\|: \tau \rightarrow \|x(\tau)\|_\tau, x \in X$, determine on $\hat{X}$ a structure of a bimodule over $\hat{S}$ with the $\hat{S}$ -valued norm $\|\cdot\|_{\hat{x}}$. In addition, $(\hat{X}, \|\cdot\|_{\hat{x}})$ is a separable bimodule, i.e. there exists a countable set $\{\hat{X}_n\} \subset \hat{X}$ such that, for any

$\hat{x} \in \hat{X}$ there exists a subsequence $\{\hat{X}_{n_k}\}$ for which $\|\hat{x} - \hat{X}_{n_k}\|_{\hat{x}} \xrightarrow{(o)} 0$.

3.3 Remark :[12]

Suppose $\hat{X}$ be a separable Banach $\hat{S}$ -module with partial order in which represents a lattice coincide with the norm in $\hat{X}$.

From $|\hat{x}| \leq |\hat{y}|$ imply that $\|\hat{x}\| \leq \|\hat{y}\|$ for each $\hat{x}, \hat{y} \in \hat{X}$.

3.4 Definition :[6]

A vector lattice X is **Archimedean** whenever the relation $0 \leq nx \leq y, n \in N$, imply that $x = 0$.

3.5 Proposition:

Let $\hat{X}$ be the set of all classes for P- almost everywhere coinciding elements from X , The Archimedean condition is satisfied in $\hat{X}$.

Proof :-

$$\text{Let } \hat{x} \geq 0 \text{ and } n\hat{x} \leq \hat{y}, n = \overline{1, \infty}$$

$$\text{Then } \|n\hat{x}\| \leq \|\hat{y}\|, \text{ or } \|\hat{x}\| \leq \frac{1}{n} \|\hat{y}\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\text{i.e. } \hat{x} = 0$$

Therefore

$$|x \vee z - y \vee z| \leq |x - y| \dots\dots(1)$$

□

3.6 Definition:[14]

Let X be a vector space over the real field R . A nonempty convex subset P of X is called a **cone** if $\lambda P \subset P$ for all $\lambda \geq 0$.

3.7 Theorem:

Let $\{X, \{X_\tau\}_{\tau \in T}\}$ be a measurable field Banach space generating $\hat{X}$, then for almost everywhere $\tau \in T$ we can define a Banach Lattice on X_τ such that, the order in X_τ for almost everywhere $\tau \in T$ induces the order in $\hat{X}$.

Proof :

let $\hat{G}$ be a countable dense subset of $\hat{K}$ such that :

- 1- $\hat{G} + \hat{G} \subseteq \hat{G}$.
- 2- $r\hat{G} \subseteq \hat{G}$ for all rational numbers $r \geq 0$.
- 3- $\hat{G} \cap (-\hat{G}) = 0$.

Now, suppose that $\{G_\tau\}_{\tau \in T}$ generated $\hat{G}$, such that, for almost everywhere countable dense subset G_τ of $\hat{K}_\tau$.

Since $\hat{G}$ and G_τ are countable for almost everywhere $\tau \in T$, then the properties (1 - 3) are satisfied in G_τ for almost everywhere $\tau \in T$.

Thus, K_τ is a cone for each $\tau \in T$.

In fact, we need to prove $\alpha K_\tau \subseteq K_\tau$ for each $\alpha \in R_+$ but, that is clear when we use the property (2) and consider the fact that K_τ is closed.

Thus, for almost everywhere $\tau \in T$, we can define a partial order induced the same order in $\hat{X}$.

Now, we need to prove that X_τ is a Banach lattice for that order, i.e. there exist $x_\tau \vee y_\tau$ for any $x_\tau, y_\tau \in X_\tau$.

From $|x_\tau| \leq |y_\tau|$ we have $\|x_\tau\|_\tau \leq \|y_\tau\|_\tau$.

Let $\hat{\Gamma}$ be a countably densely subset of $\hat{X}$ such that $\hat{\Gamma} \vee \hat{\Gamma} \subseteq \hat{\Gamma}$ and let $\{\Gamma_\tau\}_{\tau \in T}$ be a family of a countable dense set in X_τ , for almost everywhere $\tau \in T$ generating $\hat{\Gamma}$.

Therefore almost everywhere $\tau \in T$ and for all $x_\tau, y_\tau \in \Gamma_\tau$, we can define $x_\tau \vee y_\tau$ such that $\Gamma_\tau \vee \Gamma_\tau \subseteq \Gamma_\tau$ and therefore the relation (1) and the order conditions are satisfied on Γ_τ .

Let $y_\tau \in \Gamma_\tau$, $x_\tau \in X_\tau$ and $\{x_n^\tau\}_{n=1}^\infty \subseteq \Gamma_\tau$, $x_{n=1}^\tau \rightarrow x_\tau$ as $n \rightarrow \infty$

Then

$$|x_n^\tau \vee y_\tau - x_m^\tau \vee y_\tau| \leq |x_n^\tau - x_m^\tau|$$

i.e. $\{x_n^\tau \vee y_\tau\}_{n=1}^\infty$ is a fundamental sequence and it is easy to prove that is convergent to $x_\tau \vee y_\tau$.

Thus , the operations will be defined for all pair x_τ, y_τ such that $x_\tau \in X_\tau, y_\tau \in \Gamma_\tau$,
In the similar way this operation can be extended to all X_τ , and there is no difficult to
show that the order condition can be extended with Γ_τ on X_τ . $\square$

3.8 Definition :[1]

An element from X_+ is called a **Freudenthal** unit and denoted by $\hat{1}$, if it follows from $x \in X, x \wedge \hat{1} = 0$, that $x = 0$.

3.9 Proposition :

There exists a Freudenthal unit in $\hat{X}$ if and only if it exists in X_τ almost everywhere $\tau \in T$.

Proof :

Let $\hat{e} \in \hat{X}$ be a Freudenthal unit . Then from the equality $\hat{e} \wedge \hat{x} = 0$, we deduce that $\hat{x} = 0$. Moreover for all $\hat{x} \in \hat{G}$, we have $e(\tau) \wedge x_\tau = 0$, imply that $x_\tau = 0$ for all $x_\tau \in G_\tau$.

Since G_τ is dense in K_τ , then , we get ,that is true for all $x_\tau \geq 0$. It means that $e(\tau)$ is a Freudenthal unit in X_τ for almost everywhere $\tau \in T$.

Conversely , let $\{X_\tau\}_{\tau \in T} \subseteq S^*(T, U)$, where U is the Uneversal space, $\{X_n(\tau)\}_{n=1}^\infty = G_\tau$ for almost everywhere $\tau \in T$ and $x_n(\tau)$ be a Borel representation of elements in $\hat{G}$, Then the set

$$A = \bigcap_{n=1}^\infty \{ (e, \tau) : \|x_n(\tau) \wedge e\|_\tau = 0 \text{ and } \|x_n(\tau)\|_\tau = 0 \}$$

is a Borelset, so since hypothesis

$$A_\tau = \{e : e \text{ is a freudenthal unit in } X_\tau\} = \{e : (e, \tau) \in A\} \neq \emptyset$$

for almost everywhere $\tau \in T$, then , there exist $e(\tau)$ such that $(e(\tau), \tau) \in A$ for almost everywhere $\tau \in T$. It is clear that , $e = e(\tau)^n$ is a freudenthal unit in $\hat{X}$. $\square$

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