The Composition Operator C_{ϕ_λ} Induced By The Function ϕ_λ

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Abstract

The main aim of this search to study the composition operator C_{ϕ_λ} induced by the automorphism ϕ_λ and discuss the adjoint of the composition operator, also we discuss some properties of composition operator and tried to see the analogue properties in order to show how the results are changed by changing the function ψ of U. In order to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties.

Introduction //

This search consists of two sections. In section one, we gave the concept of the automorphism ϕ_λ and discussed some of it's. In section two, we are going to the Composition Operator we discussed the adjoint of Composition Operator C_{ϕ_λ}, discussed C_{ϕ_λ} as an invertible operator, C_{ϕ_λ} is normal operator, C_{ϕ_λ} is not idempotent operator, C_{ϕ_λ} is algebraic operator and define eigenvalue of C_{ϕ_λ}.

1. Section One

This section will give concept of the automorphism ϕ_λ and discussed some of it's.

Definition (1.1) : [4]

Let $U = \{z \in C : |z| \leq 1\}$ then U is called unit disk in complex C and $\partial U = \{z \in C : |z| = 1\}$ is called boundary of U.

Example(1.2):

For $\lambda \in U$, define $\phi_\lambda(z) = \dfrac{z}{\lambda z - 1}$ $(z \in U)$. Since the denominator equal zero only at $z = \dfrac{1}{\lambda}$, the function ϕ_λ is holomorphic on the disk $\{|z| \leq \dfrac{1}{|\lambda|}\}$. Since $\lambda \in U$, then this disk contains U. Hence ϕ_λ take U into U and holomorphic on U .

Definition(1.3) : [10]

Let $\psi : U \to U$ be holomorphic on U. then ψ is conformal automorphism or automorphism of U if and only if ψ is bijective.

Proposition (1.4) :

Let $\lambda \in U$, define $\phi_\lambda(z) = \dfrac{z}{\lambda z - 1}$ $(z \in U)$ then ϕ_λ is conformal automorphism or automorphism of U .

Proof :

Since $\phi_\lambda(z) = \dfrac{z}{\lambda z - 1}$ $(z, \lambda \in U)$

Suppose $\phi_\lambda(z_1) = \phi_\lambda(z_2)$ that is $\dfrac{z_1}{\lambda z_1 - 1} = \dfrac{z_2}{\lambda z_2 - 1}$, therefore $\lambda z_1 z_2 - z_1 = \lambda z_1 z_2 - z_2$, hence $z_1 = z_2$. Thus ϕ_λ is injective. Let $y = \phi_\lambda(z)$, that is $y = \dfrac{z}{\lambda z - 1}$, therefore $\lambda z\, y - y = z$, then $\lambda z\, y - z = y$, hence $z = \dfrac{y}{\lambda y - 1}$,

$$\phi_\lambda(z) = \phi_\lambda\left(\frac{y}{\lambda y - 1}\right) = \frac{\dfrac{y}{\lambda y - 1}}{\dfrac{\lambda y}{\lambda y - 1} - 1} = \frac{\dfrac{y}{\lambda y - 1}}{\dfrac{\lambda y - \lambda y + 1}{\lambda y - 1}} = y,$$

for every $y \in U$ there exists $z \in U$ such that $\phi_\lambda(z) = y$ .Thus ϕ_λ is surjective . Hence ϕ_λ is automorphism .

Definition(1.5) : [10]

A point $p \in C$ is a fixed point for the function ψ , if $\psi(p) = p$.

Proposition (1.6) :

Let $\lambda \in U$, then 0 , $\dfrac{2}{\lambda}$ are fixed points of ϕ_λ , where $\phi_\lambda(z) = \dfrac{z}{\lambda z - 1}$.

Proof :

Let $\phi_\lambda(z) = z$ that is $\dfrac{z}{\lambda z - 1} = z$, therefore $\lambda z^2 - 2z = 0$. Hence ϕ_λ has two fixed points $z_1 = 0$, $z_2 = \dfrac{2}{\lambda}$.

Definition(1.11) : [11]

Let $\psi : U \to U$ and holomorphic on U. A is the non-singular 2×2 complex matrix

Definition(1.7): [4]

Let $\psi : U \to U$ and holomorphic on U with fixed point r, then:

1) r is interior fixed point of ψ if $r \in U$.
2) r is exterior fixed point of ψ if $r \notin U$.

Proposition (1.8):

Let $\phi_\lambda(z) = \dfrac{z}{\lambda z - 1}$, for $\lambda \in U$. Then 0 is interior fixed point and $\dfrac{2}{\lambda}$ is exterior fixed point of ϕ_λ .

Proof :

Since ϕ_λ has two fixed points $z_1 = 0$, $z_2 = \dfrac{2}{\lambda}$, $|z_1| = |0| = 0 < 1$. Thus z_1 is interior fixed point

Since $\lambda \in U$, then $|\lambda| < 1 < 2 = |2|$, therefore $|\lambda| < |2|$, hence $|z_2| = \left|\dfrac{2}{\lambda}\right| > 1$. Thus z_2 is exterior fixed point.

Remark(1.9) :

ϕ_λ is own compositional inverse in fact $\phi_\lambda(\phi_\lambda(z)) = z$ for every z in U This property of ϕ_λ is called the self-inverse property.

Remark(1.10) :

For $\lambda \in U$, then $\phi'_\lambda(0) = -1$,

$$\phi'_\lambda(\lambda) = \frac{-1}{(\lambda^2 - 1)^2} .$$

We say that ψ is a rotation round the origin if there exists $\sigma \in \partial U$ such that $\psi(z) = \sigma z \ (z \in U)$

Proposition (1.12):

If $\lambda = 0$, then $\phi_\lambda(z)$ is a rotation a round the origin, where $\phi_\lambda(z) = \frac{z}{\lambda z - 1}$.

Proof :

Since $\phi_\lambda(z) = \frac{z}{\lambda z - 1}$, and since $\lambda = 0$, then $\phi_\lambda(z) = -z$, let $\sigma = -1$, $\sigma \in \partial U$, hence $\phi_\lambda(z) = \sigma z$. Thus $\phi_\lambda(z)$ is a rotation a round the origin .

Theorem (1.13) : [11]

Let $\psi : U \to U$ and holomorphic on U, then ψ is elliptic if and only if ψ is automorphism and has an interior fixed point.

Proposition (1.14) :

Let $\lambda \in U$, ϕ_λ is elliptic, where $\phi_\lambda(z) = \frac{z}{\lambda z - 1}$.

Proof :

From (1- 4) , ϕ_λ is automorphism, and from (1-8) ϕ_λ has an interior fixed Point , hence ϕ_λ is elliptic.

Definition(1.15): [10]

A linear fractional transformation is a mapping of the form $\tau(z) = \frac{az + b}{cz + d}$, where a , b , c , and d are complex numbers and sometimes we denote $\tau(z)$ by $\tau_A(z)$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Remark (2.2) : [1]

We can define an inner product of the Hardy space functions as follows :

Let $f(z) = \sum_{n=0}^{\infty} f\hat{}(n)\, z^n$ and

Proposition (1.16) :

The ϕ_λ is a linear fractional transformation.

Proof :

Since $\phi_\lambda(z) = \frac{z}{\lambda z - 1} = \frac{az + b}{cz + d}$ such that a =1 , b = 0 , c = λ , d = -1 where a , b , c , and d are complex numbers and $A = \begin{bmatrix} 1 & 0 \\ \lambda & -1 \end{bmatrix}$, hence by (1.15) ϕ_λ is a linear fractional transformation .

2. Section Two

This section will give the Composition Operator we discussed the adjoint of Composition Operator C_{ϕ_λ}, discussed C_{ϕ_λ} as an invertible operator, C_{ϕ_λ} is normal operator, C_{ϕ_λ} is not idempotent operator, C_{ϕ_λ} is algebraic operator and define eigenvalue of C_{ϕ_λ} .

Definition(2.1): [4]

Let U denote the unit disk in the complex plane, the Hardy space H^2 is the set of functions $f(z) = \sum_{n=0}^{\infty} f\hat{}(n) z^n$ holomorphic on U such that $\sum_{n=0}^{\infty} \left|f\hat{}(n)\right|^2 < \infty$ with $f\hat{}(n)$ denotes the Taylor coefficient of f .

Definition(2.7) :

The composition operator C_{ϕ_λ} induced by ϕ_λ is defined on H^2 by the equation $C_{\phi_\lambda} f = f \circ \phi_\lambda$.

$g(z)=\sum_{n=0}^{\infty} g\hat{}(n)\, z^n$, then inner product of f and g is defined by : $\langle f,g\rangle = \sum_{n=0}^{\infty} f\hat{}(n)\, \overline{g\hat{}(z)}$.

Example (2.3) :[10]

Let $\alpha \in U$ and $k_\alpha(z)=\dfrac{1}{1-\overline{\alpha} z}$ $(z\in U)$. Since $\alpha \in U$ then $|\alpha|\le 1$,hence the geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is convergent and thus $k_\alpha \in H^2$ and $k_\alpha(z)=\overline{\alpha}\, z^n$.

Definition(2.4) : [4]

Let $\psi : U \to U$ and holomorphic on U, the composition operator C_ψ induced by ψ is defined on H^2 by the equation $C_\psi f = f\circ\psi \quad (f\in H^2)$

Definition(2.5) : [2]

Let T be a bounded operator on a Hilbert space H, then the norm of an operator T is defined by :

$$\|T\| = \sup\{\|Tx\| : x\in H, \|x\|=1\}.$$

Littlewood´s Subordination principle (2.6) : [11]

Let $\psi : U \to U$ and holomorphic on U with $\psi(0)=0$, then for each $f\in H^2$, $f\circ\psi\in H^2$ and $\|f\circ\psi\|\le\|f\|$. The goal of this theorem $C_\psi : H^2 \to H^2$.

Proof :

Since ϕ_λ is automorphism of U by (1- 4) and $C_{\phi_\lambda}^{-1} = C_{\phi_\lambda^{-1}} = C_{\phi_\lambda}$ by (1 - 9) , hence C_{ϕ_λ} is an invertible operator on H^2 .

Proposition(2.8) :

If $\lambda \in U$, then for each $f\in H^2$, $f\circ\phi_\lambda \in H^2$ and $\|f\circ\phi_\lambda\|\le\|f\|$

Proof :

Since $\phi_\lambda : U\to U$ and holomorphic on U with $\phi_\lambda(0)=0$, then by (2 - 6) $f\in H^2$, $f\circ\phi_\lambda\in H^2$ and $\|f\circ\phi_\lambda\|\le\|f\|$, hence $C_{\phi_\lambda} : H^2 \to H^2$.

Remark (2.9) : [4]

1) One can easily show that $C_\kappa C_\psi = C_{\psi\circ\kappa}$ and hence $C_\psi^n = C_\psi C_\psi \cdots C_\psi = C_{\psi\circ\psi\circ\cdots\circ\psi} = C_{\psi_n}$

2) C_ψ is the identity operator on H^2 if and only if ψ is identity map from U into U and holomorphic on U.

3) It is simple to prove that $C_\kappa = C_\psi$ if and only if $\kappa = \psi$.

Theorem (2. 10) : [11]

Let $\psi : U\to U$ and holomorphic on U. C_ψ is an invertible operator on H^2 if and only if ψ automorphism of U and $C_\psi^{-1} = C_{\psi^{-1}}$.

Proposition(2.11) :

If $\lambda \in U$, then C_{ϕ_λ} is an invertible operator on H^2.

Proof :

By (2-16), $T_h^* f = T_{\overline{h}} f$ for each $f\in H^2$. Hence for all $\alpha\in U$,

$$\langle T_h^* f, k_\alpha\rangle = \langle T_{\overline{h}} f, k_\alpha\rangle = \langle f, T_{\overline{h}}^* k_\alpha\rangle \quad \cdots (2\text{-}1)$$

On the other hand ,

Definition(2.12): [3]

Let T be an operator on a Hilbert space

H , the operator T^* is the adjoint of T if $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for each $x, y \in H$.

Theorem (2.13) : [5]

$V_{\alpha\in U}\{K_\alpha\}$ forms a dense subset of H^2.

Theorem (2.14) : [10]

Let $\psi : U \to U$ and holomorphic on U, then for all $\alpha \in U$
$C^*_\psi K_\alpha = K_{\psi(\alpha)}$

Definition(2.15): [6]

Let $g \in H^\infty$, the Toeplits operator T_g is the operator on H^2 given by :
$$(T_g f)(z) = g(z)\, f(z) \quad (f \in H^2,\ z \in U)$$

Remark (2.16) : [7]

For each $f \in H^2$, we have $T^*_h f = T_{\bar h} f$, such that $h \in H^\infty$.

Proposition(2.17) :

Let $C^*_{\phi_\lambda} = T_g\, C_\gamma\, T_h$, where $h(z) = (\lambda z - 1)$, $g(z) = -1$, $\gamma(z) = \bar\lambda - z$.

Theorem (2.19) : [9]

Let $\psi : U \to U$ and holomorphic on U, then C_ψ is normal if and only if $\psi(z) = \beta z$ for some $\beta \in \partial U$, $|\beta| = 1, z \in U$.

Proposition(2.20) :

If $\lambda = 0$, then C_{ϕ_λ} is normal composition operator .

Proof :

Since $\phi_\lambda(z) = \dfrac{z}{\lambda z - 1}$, and since $\lambda = 0$, then $\phi_\lambda(z) = -z = \beta z$

$$\langle T^*_h f, k_\alpha\rangle = \langle f, T_h f\rangle = \langle f, h(\alpha) k_\alpha\rangle \quad \cdots (2\text{-}2)$$

From (2-1)and (2-2) one can see that

$T^*_{\bar h}\, k_\alpha = h(\alpha)\, k_\alpha$. Hence $T^*_h\, k_\alpha = \overline{h(\alpha)}\, k_\alpha$.

Calculation give $C^*_{\phi_\lambda} k_\alpha(z) = k_{\phi_\lambda(\alpha)}(z)$

$$= \frac{1}{1 - \overline{\phi_\lambda(\alpha)}\, z} = \frac{1}{1 - \dfrac{\bar\alpha z}{\bar\lambda\bar\alpha - 1}}$$
$$= \frac{1}{\dfrac{\bar\lambda\bar\alpha - 1 - \bar\alpha z}{\bar\lambda\bar\alpha - 1}} = \frac{\bar\lambda\bar\alpha - 1}{\bar\lambda\bar\alpha - 1 - \bar\alpha z} = \frac{\overline{(\lambda\alpha - 1)}}{-(1 - \bar\alpha(\bar\lambda - z))}$$
$$= \overline{(\lambda\alpha - 1)} \cdot (-1) \cdot \frac{1}{1 - \bar\alpha(\bar\lambda - z)}$$
$$= \overline{h(\alpha)} \cdot T_g\, k_\alpha(\gamma(z)) = T_g\, \overline{h(\alpha)}\, k_\alpha(\gamma(z))$$
$$= T_g\, \overline{h(\alpha)}\, C_\gamma\, k_\alpha(z) = T_g\, C_\gamma\, \overline{h(\alpha)}\, k_\alpha(z)$$
$= T_g\, C_\gamma\, T^*_h\, k_\alpha(z)$, therefore

$$C^*_{\phi_\lambda} k_\alpha(z) = T_g\, C_\gamma\, T^*_h\, k_\alpha(z) \quad (z \in U).$$

But $\overline{V_{\alpha\in U}\{K_\alpha\}} = H^2$, then $C^*_{\phi_\lambda} = T_g\, C_\gamma\, T^*_h$.

Definition (2.18) : [3]

Let T be an operator on a Hilbert space H , T is called normal operator if $T\,T^* = T^*\,T$

Proposition(2.23) :

Let $\lambda \in U$, then C_{ϕ_λ} is an algebraic composition operator.

Proof :

Take $P(z) = z^4 - z^2$, note that for all $f \in H^2, z \in U$
$$P(C_{\phi_\lambda})\, f(z) = (C^4_{\phi_\lambda} - C^2_{\phi_\lambda})\, f(z)$$
$$= C^4_{\phi_\lambda} f(z) - C^2_{\phi_\lambda} f(z)$$

such that $\beta = -1, |\beta| = 1$.
Hence by (2.19) C_{ϕ_λ} is normal composition operator.

Definition (2.21) : [8]

Let T be an operator on a Hilbert space H , T is called an algebraic operator on H if there exists non-zero complex polynomial P such that P(T) = 0, and T is called idempotent on H if $T^2 = T$.

Proposition(2.22) :

Let $\lambda \in U$, then C_{ϕ_λ} is not idempotent composition operator on H^2

Proof :

Since $C^2_{\phi_\lambda} = C_{\phi_\lambda} C_{\phi_\lambda} = C_{\phi_\lambda \circ \phi_\lambda}$ by (2.9) , since $\phi_\lambda \circ \phi_\lambda = I$ by (self-inverse property) . Hence $C_{\phi_\lambda \circ \phi_\lambda} = C_I$ and $C_I = I$ by (2.9) , hence $C^2_{\phi_\lambda} = I \neq C_{\phi_\lambda}$. Hence C_{ϕ_λ} is not idempotent composition operator on H^2.

Proof :

Since $\phi_\lambda(z) = \dfrac{z}{\lambda z - 1}, \phi'_\lambda(z) = \dfrac{(\lambda z - 1)(1) - (\lambda z)}{(\lambda z - 1)^2} = \dfrac{-1}{(\lambda z - 1)^2}$, and since ϕ_λ has fixed point $0 \in U$, we have by (2.22)

$\kappa = (\phi'_\lambda(0))^n = (-1)^n$ is an eigenvalue of C_{ϕ_λ} for some n = 0 , 1 , 2 ,..

Then , it is easily to see that $C^4_{\phi_\lambda} = C^2_{\phi_\lambda} = I$ (by the self-inverse property). Hence $P(C_{\phi_\lambda}) = 0$. Since P is non-zero complex polynomial, then C_{ϕ_λ} is an algebraic composition operator.

Definition (2.21) : [12]

Let $\psi : U \to U$ and holomorphic on U, the eigenvalue equation for the composition operator is define by $C_\psi f = \kappa f$ or $f \circ \psi = \kappa f$, where k is eigenvalues of C_ψ.

Theorem (2.22): [11]

Let $\psi : U \to U$ and holomorphic on U, and that fixes the point $p \in U$ and suppose that $C_\psi f = \kappa f$ for some non-constant $f \in H^2$ and some $\kappa \in C$. Then $\kappa = (\psi'(p))^n$ for some n = 0, 1, 2, ...

Proposition(2.23) :

If $\lambda \in U$, then $(-1)^n$ is an eigenvalue of C_{ϕ_λ} for some n = 0 , 1 , 2 , …..

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الملخص

الهدف الرئيسي لهذا البحث هو لدراسة المؤثر التركيبي المحتث C_{ϕ_λ} من الدالة ϕ_λ حيث ناقشنا المؤثر المرافق للمؤثر التركيبي . بالإضافة إلى ذلك ناقشنا بعض خصائص المؤثر التركيبي وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير الدالة التحليلية ψ. ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية مع براهين مفصلة وكذلك برهنا بعض النتائج التي أعطيت بدون برهان.