THE DISCRETE CLASSICAL OPTIMAL CONTROL PROBLEM of A SEMILINEAR PARABOLIC EQUATION (COCP)

Chryssoverghi, I.[1] **and J. Amir Ali Al-hawasy**[2]

[1] Department of Mathematics, School of Applied Mathematics & Physical Science, National Technical University, Athens-Greece

[2] Department of Mathematics, College of Science, Al-Mustansiriya University.

hawasy20@yahoo.com

ABSTRACT

In this paper, the continuous classical optimal control for systems of a semilinear parabolic partial differential equations is described with a equality and inequality state constraints. First, the considered continuous classical optimal control problem is discretized into a discrete classical optimal control problem by using the Galerkin finite element method in space and the theta finite difference scheme (θ-method)in time. The classical continuous controls are approximated by picewise constants. Second the existence of a unique solution of the discrete state equations for fixed discrete classical control is studied. Third, the existence theory for optimality of the discrete classical problem is proved, and the discrete adjoint equations are developed corresponding to the discrete state equations. Finally the necessary conditions and a picewise minimum principle are derived for optimality of the discrete classical problem so as the sufficient conditions.

Introduction: During the last decades, many researchers ([1],[2],[3] and many others), interested to study the discretization for the continuous relaxed optimal control problems for systems defined by ordinary and partial differential equations. Since many applications in different fields of natural sciences lead to mathematical modules represent a classical optimal control governed by semilinear partial differential equations as a heat equation [4]. Solving such problems numerically need to discretize the continuous optimal control problems to a discrete classical optimal control problems, so we interest in this paper to study the discretization of the classical optimal control problem for systems defined by semilinear parabolic partial differential equations with al equality and inequality state constraints.

In this paper in order to give a complete idea about this work, we start to give a description for the continuous classical optimal control problem which is studied in [5], then we discretize this continuous classical optimal control problem to a discrete classical optimal control problem. First we discretize the weak form of state equations in the continuous problem by using the Galerkin finite element methods in space and the finite difference scheme (θ-method) in time, while the continuous controls are approximated by piecewise constants with respect to an independent partition of the space-time domain. Then the existence of a unique solution of the discrete state equations for fixed discrete classical control is proved. Also, we prove the existence theory of optimal control for the discrete classical problem, and we derive the discrete adjoint-state equations corresponding to the discrete state equations. Finally the necessary conditions and a piecewise minimum principle for optimality of the discrete classical optimal control problem are derived as well as the sufficient conditions.

1.Description of The Continuous Classical Optimal Control Problem(CCOCP):- In this section we describe the continuous classical optimal control problem of a semilinear parabolic partial differential equations which is studied by [5], we begin with let $\Omega \subset \mathbb{R}^d$ be an open and bounded region with Lipschitz boundary $\Gamma = \partial\Omega$, and let $I = (0,T)$, $0 < T < \infty$, $Q = \Omega \times I$. We denote the Euclidean norm in $\mathbb{R}^n$ by $|.|$, the norm in $L^\infty(\Omega)$ by $\|.\|_\infty$, the inner product and norm in $L^2(\Omega)$ by $(.,.)$ and $\|.\|_0$ respectively, the inner product and norm in Sobolev space $V = H_0^1(\Omega)$ by $(.,.)_1$ and $\|.\|_1$ respectively, the duality between V and V^*(its dual) by $<.,.>$, and finally we denote by $\|.\|_Q$ the norm in $L^2(Q)$.

The weak form of the semilinear parabolic state equations is:-

$$< y_t, v > + a(t,y,v) = (f(x,t,y,u), v),$$
$\forall v \in V$, a.e. on I, (1)

$y(.,t) \in V$,a.e.on I, $y(0) = y^0$, (2)

where $y = y_u$ is the *state*, u is the *classical control*, $y^0 \in L^2(\Omega)$ and

$$a(t,y,v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij}(x,t) \frac{\partial y}{\partial x_j} \frac{\partial v}{\partial x_i} dx$$

The *control constraints (The controls set) are*

$$u \in W,\ W \subset L^2(Q)$$

where W is defined by one of the following forms: -

$W = W_U = \{w \in L^2(Q) \mid w(x,t) \in U$, a.e. in $Q\}$, with $U \subset \mathbb{R}$

the *cost functional is*

$$G_0(u) = \int_Q g_0(x,t,y,u)dxdt,$$

the *constraints* on the state and the control variables are

$$G_1(u) = \int_Q g_1(x,t,y,u)dxdt = 0,$$
$$G_2(u) = \int_Q g_2(x,t,y,u)dxdt \le 0,$$

The set of *admissible controls* is

$$W_A = \{u \in W \mid G_1(u) = 0,\ G_2(u) \le 0\}.$$

We suppose that $a(t,v,w)$ is symmetric and for some α_1, α_2 , for each $v, w \in V$, and $t \in \bar{I}$, we obtain $|a(t,v,w)| \le \alpha_1 \|v\|_1 \|w\|_1$, and $a(t,v,v) \ge \alpha_2 \|v\|_1^2$.

We suppose also that f is of Caratheodory type on $Q \times (\mathbb{R} \times \mathbb{R})$ (e.g. continuous), it satisfies the following sublinearity condition w.r.t. y & u, and Lipschitz w.r.t. y

$$|f(x,t,y,u)| \le \eta(x,t) + c_1|y| + c_1'|u|,$$
$$|f(x,t,y_1,u) - f(x,t,y_2,u)| \le L|y_1 - y_2|,$$

where $(x,t) \in Q$, $y, y_1, y_2, u \in \mathbb{R}$ and $\eta \in L^2(Q, \mathbb{R})$.

The ***Continuous Classical Optimal Control Problem (CCOCP)*** is to minimize the cost functional subject to the above constraints, i.e., to find u such that

$u \in W_A$ and $G_0(u) = \underset{w \in W_A}{Min}\, G_0(w)$,

2. Descritization , Description and Solution of the Discrete Classical Optimal Control Problem :- Suppose for simplicity the operator $a(t,.,.)$ is independent of t, the domain Ω is a polyhedron . For every integer n, let $\{S_i^n\}_{i=1}^{M(n)}$ be an admissible regular triangulation of $\bar{\Omega}$ into closed d-simplices [6] , $\{I_j^n\}_{j=0}^{N(n)-1}$ be a subdivision of the interval $\bar{I}$ into $N(n)$ intervals, where $I_j^n := [t_j^n, t_{j+1}^n]$ of equal lengths $\Delta t = T / N$.

Set $Q_{ij} := S_i^n \times I_j^n$. Let $V_n \subset V = H_0^1(\Omega)$ be the space of continuous piecewise affine mapping in Ω. Let W^n be the set of discrete (blockwise constants) classical controls (piecewise constants classical controls), i.e.

$$W^n = \{w = w^n \in W \,|\, w(x,t) = w_{ij} \text{ in} Q_{ij}\}$$

The discrete state equation (for $\theta \in [0,1]$), for each $v \in V_n$, is written in the form:

$$\frac{1}{\Delta t}\left(y_{j+1}^n - y_j^n, v\right) + a(y_{\theta j}^n, v)$$
$$= \left(f(t_{\theta j}^n, y_{\theta j}^n, u_j^n), v\right), y_j^n \in V_n,$$

(3a)

Where $t_{\theta j}^n = (1-\theta)t_j^n + \theta t_{j+1}^n$, $y_{\theta j}^n = (1-\theta)y_j^n + \theta y_{j+1}^n$, and $j = 0,1,...,N-1$

And the initial condition associated with the above discrete form($\forall v \in V_n$ with $y^0 \in V$) is

$$\left(y_0^n, v\right)_1 = \left(y^0, v\right)_1,$$

(3b)

The discrete constraint on the control is $w^n \in W^n$. And the discrete constraints on the state and the control are: (i) $\left|G_1^n(u^n)\right| \le \varepsilon_1^n$, or (ii) $G_1^n(u^n) = \eta_1^n$, and $G_2^n(u^n) \le \varepsilon_2^n$, where ε_1^n, ε_2^n, & η_1^n are given numbers, tend to zero as n goes to infinity.

The *discrete cost* is $G_0^n(u^n)$, where the discrete functional $G_l^n(u^n)$ is defined by

$$G_l^n(u^n) = \Delta t \sum_{j=0}^{N-1} \int_\Omega g_l(t_{\theta j}^n, x, y_{\theta j}^n, u_j^n)dx, \quad l = 0,1,2 .$$

The set of *admissible controls* is given by

$$W_A^n = \left\{u^n \in W^n \,\middle|\, \left|G_1^n(u^n)\right| \le \varepsilon_1^n, G_2^n(u^n) \le \varepsilon_2^n\right\}$$

(for the optimal problem)

$$W_A^n = \left\{u^n \in W^n \,\middle|\, G_1^n(u^n) = \eta_1^n, G_2^n(u^n) \le \varepsilon_2^n\right\}$$

(for the extremal problem)

The ***Discrete Cclassical Optimal Control Problem (DCOCP)*** is to find u such that $u^n \in W_A^n$, and $G_0^n(u^n) = \min_{u^n \in W_A^n} G_0^n(w^n)$.

Now, suppose the function f is defined on $S_i^n \times I_j^n \times \mathrm{R} \times W^n$ ($i = 1,2,...,M$) continuous w.r.t. (with respect to) y_j^n, and u_j^n and satisfies:-

$$\left|f(x, t_{\theta j}^n, y_{\theta j}^n, u_j^n)\right| \le \eta_j(t_{\theta j}^n) + c_1\left|y_{\theta j}^n\right| + c_1'\left|u_j^n\right|,$$

for $0 \le j \le N-1$, $x \in \Omega$, and

$$\left|f(x, t_j^n, y_{j+1}^n, u_j^n) - f(x, t_j^n, y_j^n, u_j^n)\right|$$
$$\le L\left|y_{j+1}^n - y_j^n\right|, \ \forall 0 \le j \le N-1, \ x \in \Omega$$

where, $\eta_j(x) = \eta(x, t_j^n) \in L^2(\Omega)$, and L represents the Lipschtiz constant for any j .

From here and up and for brevity we will drop sometimes the arguments t_j^n of the dependents variable y_j^n, z_j^n, u_j^n and any other terms which contained this independent variable, and the terms y_j^n, z_j^n, u_j^n of the functions which contained their.

Theorem 2.1: For any fixed j $(0 \le j \le N-1)$, and for each control $u^n \in W^n$, the discrete state equations (3a,b) for sufficiently small Δt, has a unique solution $y_{u^n}^n = y^n = (y_0^n, y_1^n, ..., y_N^n)$.

Proof: For fixed any j $(0 \le j \le N-1)$, let $\{v_i, i = 1,2,...,M(n)\}$ be a finite basis of V_n(where $v_i(x)$,for $i = 1,2,...,n$ are continuous piecewise affine mapping in Ω ,with $v_i(x) = 0$ on the boundary Γ), then equations (3a,3b) for any $i = 1,2,...,M$ and $y_j^n, y_{j+1}^n \in V_n$, can be written in the form

$$\left(y_{j+1}^n - y_j^n, v_i\right) + \Delta t a(y_{\theta j}^n, v_i) =$$
$$\Delta t\left(f(t_{\theta j}^n, \theta y_{j+1}^n + (1-\theta)y_j^n, u_j^n), v_i\right)$$

(4)

$\left(y_0^n, v_i\right)_1 = \left(y^0, v_i\right)_1$, $v_i \in V_n$, with $y^0 \in V \subset L^2(\Omega)$.
(5)

where v_i & $y_j^n \in V_n$, $i = 1, 2, ..., M$

Now, form the basis of V_n, using the Galerkin method [6], we write

$y_0^n = \sum_{k=1}^{M} c_k^0 v_k$, $y_j^n = \sum_{k=1}^{M} c_k^j v_k$, and $y_{j+1}^n = \sum_{k=1}^{M} c_k^{j+1} v_k$,

where c_k^0, c_k^j & c_k^{j+1}, $\forall k = 1, 2, ..., M$, are unknown constants.

By substituting y_j^n & y_{j+1}^n as defined above in (4), and y_0^n in (5) we get system (4-5) is equivalent to the following algebraic system

$$(A + \theta \Delta t D) C^{j+1} = (A - (1-\theta)\Delta t D) C^j + b \quad (6)$$

$$AC^0 = b^0$$
(7)

where

$A = (a_{ik})_{M \times M}, a_{ik} = (v_k, v_i)$,

$D = (d_{ik})_{M \times M}, d_{ik} = a\left(v_k, v_i\right)$,

$\vec{V}_{M \times 1} = (v_1, v_2, ..., v_M)^T$,

$C^j_{M \times 1} = (c_1^j, c_2^j, ..., c_M^j)^T, \forall j$,

$b = (b_i)_{M \times 1}, b_i = \Delta t (f(\theta \vec{V}^T C^{j+1} + (1-\theta)\vec{V}^T C^j, u_j^n), v_i)$,

and $b^0 = (b_i^0)_{M \times 1}$, $b_i^0 = (y^0, v_i)_1$.

To solve system (6-7), we use the following **predictor and corrector method** [7]:

Predictor method: we set $\theta = 0$, in the R.H.S. (Right Hand Side) of (4), system (6-7) becomes a linear system. Since $\theta \Delta t > 0$, and from the assumptions on $a(.,.)$ the matrices A, and D are positive definite, then $A + \theta \Delta t D$ is also positive definite, hence it is regular, and then the above system has a unique solution. Solving this system w.r.t. $j+1$, for fixed j, we get C^{j+1}, i.e. we get the predictor solution $y_{j+1}^n = \sum_{k=1}^{M} c_k^{j+1} v_k$ of system (6-7). We use the above procedure for each j, and we get the predictor solution y_{j+1}^n, $j = 0, 1, ..., N-1$. Finally we set $\bar{C}^{j+1} = C^{j+1}$, and $\bar{y}_{j+1}^n = y_{j+1}^n$, $\forall j = 0, 1, ..., N-1$.

Corrector method (first iteration): we set $\theta \neq 0$, and for fixed any j ($j = 0, 1, ..., N-1$), we substitute the predictor solution $y_{j+1}^n = \bar{y}_{j+1}^n$ in the R.H.S. of (4), hence $\forall i = 1, 2, ..., M$ system (4-5) becomes

$$\left(\frac{y_{j+1}^n - y_j^n}{\Delta t}, v_i\right) + a\left(\theta y_{j+1}^n + (1-\theta) y_j^n, v_i\right) =$$
$$(f(\theta \bar{y}_{j+1}^n + (1-\theta) y_j^n, u_j^n), v_i)$$
(8)

&

$$\left(y_0^n, v_i\right)_1 = \left(y^0, v_i\right)_1, v_i \in V_n, \text{ with } y^0 \in V \quad (9)$$

Then system (8-9) is equivalent again with the linear system (6-7), with $C^{j+1} = \bar{C}^{j+1}$ in the components of b in the R.H.S. of (6), and by the above way, this system has a unique solution. Solving this system w.r.t. $j+1$, for fixed j, we get C^{j+1}, i.e. we get the solution (corrector solution) $y_{j+1}^n = \sum_{k=1}^{M} c_k^{j+1} v_k$ of the equations (6,7). We Repeat this procedure by substitute the corrector solution $\bar{C}^{j+1} = C^{j+1}$ in the components of b in the linear system (6), and we solve again system (6-7) w.r.t. $j+1$, for fixed j, to get a new corrector solution C^{j+1},and hence we get a sequence of corrector solutions $C^{j+1(\ell)}$ which can be proved is converge in R by using a certain theorem in [8].

Lemma 2.1:- The operator $u^n \mapsto y^n = y_{u^n}^n$ is continuous.

Proof: Let $u^n = (u_0^n, u_1^n, ..., u_j^n, ..., u_{N-1}^n)$,
$u^{nk} = (u_0^{nk}, u_1^{nk}, ..., u_j^{nk}, ..., u_{N-1}^{nk})$,
$y^n = (y_0^n, y_1^n, ..., y_j^n, ..., y_{N-1}^n)$, and
$y^{nk} = (y_0^{nk}, y_1^{nk}, ..., y_j^{nk}, ..., y_{N-1}^{nk})$.
Assume that $u_j^{nk} \to u_j^n$ as $k \to \infty$, $\forall j$; we will prove this by induction, i.e.
First from the initial condition (9), and the projection theory that $y_0^{nk} \to y_0^n$ as $k \to \infty$, i.e. the relation is true for $j = 0$.
Second: assume for any fixed j, that $y_j^{nk} \to y_j^n$ as $k \to \infty$, and we shall prove that $y_{j+1}^{nk} \to y_{j+1}^n$, as $k \to \infty$.
Let $y_{j+1}^{nk} = g(y_j^{nk}, u_j^{nk}, y_{j+1}^{nk})$, and $y_{j+1}^n = g(y_j^n, u_j^n, y_{j+1}^n)$, then

$$\left\| y_{j+1}^{nk} - y_{j+1}^n \right\|_0 \le$$

$$\left\| g(y_j^{nk}, u_j^{nk}, y_{j+1}^{nk}) - g(y_j^{nk}, u_j^{nk}, y_{j+1}^n) \right\|_0$$

$$+ \left\| g(y_j^{nk}, u_j^{nk}, y_{j+1}^n) - g(y_j^n, u_j^n, y_{j+1}^n) \right\|_0,$$

since $u_j^{nk} \to u_j^n$, and $y_j^{nk} \to y_j^n$, as $k \to \infty$ then

$$\left\| g(y_j^{nk}, u_j^{nk}, y_{j+1}^n) - g(y_j^n, u_j^n, y_{j+1}^n) \right\|_0 \le \varepsilon_k$$

with $\varepsilon_k \to 0$ as $k \to \infty$

Hence the above inequality becomes

$$\left\| y_{j+1}^{nk} - y_{j+1}^n \right\|_0 \le \alpha \left\| y_{j+1}^{nk} - y_{j+1}^n \right\|_0 + \varepsilon_k,$$

with $\varepsilon_k \to 0$ as $k \to \infty$, and $\alpha < 1$

$$\Rightarrow \left\| y_{j+1}^{nk} - y_{j+1}^n \right\|_0 \le \frac{\varepsilon_k}{1-\alpha} \to 0 \quad \text{as} \quad k \to \infty$$

$\Rightarrow y_{j+1}^{nk} \to y_{j+1}^n$ as $k \to \infty$.
$\Rightarrow$ $y_j^{nk} \to y_j^n$, $\forall j$, i.e. the operator $u^n \mapsto y^n = y_{u^n}^n$ is continuous**.**

Lemma 2.2: (Stability) with the assumptions on the operator $a(.,.)$, on the function f, and for $\theta \ge 1/2$, if $\|u\|_Q \le c$, then
$\left\| y_j^n \right\|_0 \le b(c), \forall j = 0,1,...,N$
$(\Rightarrow \Delta t \sum_{j=0}^{N-1} \left\| y_j^n \right\|_0^2 \le b(c))$
$\sum_{j=0}^{N-1} \left\| y_{j+1}^n - y_j^n \right\|_0^2 \le b(c)$, $\Delta t \sum_{j=0}^{N-1} \left\| y_{\theta j}^n \right\|_1^2 \le b(c)$,
And
$\Delta t \sum_{j=0}^{N} \left\| y_j^n \right\|_1^2 \le b(c)$ ($\Delta t \le h^2 / c$, if $\theta = 1/2$).

Proof: From (3b) we have $\left\| y_0^n - y^0 \right\|_1 \le \left\| y^0 - v \right\|_1$, where $y_0^n \in V_n$ is the projection of $y^0 \in V$, $\forall v \in V_n$, $V_n \subset V$. Let $\{V_n\}$ be a sequence of a subspaces of V, so by Galerkin method[6], and for $y^0 \in V$ there exists a sequence $\{v_n\}$, $v_n \in V_n, \forall n$, such that
$\left\| v_n - y^0 \right\|_1 \to 0, \forall v_n \in V_n \subset V$.
Since y_0^n is a sequence in V_n (i.e. $y_0^n \in V_n, \forall n = 12,...$), hence

$$\left\| y_0^n - y^0 \right\|_1 \le \left\| y^0 - v_n \right\|_1 \to 0$$

$$\Rightarrow y_0^n \xrightarrow[V]{S} y^0 \Rightarrow \left\| y_0^n \right\|_1 \le \bar{c} \Rightarrow \left\| y_0^n \right\|_0 \le \bar{c}$$

For each n, with $V_n \subset V$, $\forall v \in V_n$, $j = 0,1...,N-1$ consider the state equations (3a&b), i.e.

$$\frac{1}{\Delta t}(y_{j+1}^n - y_j^n, v) + a\left(y_{\theta j}^n, v\right)$$
$$= (f(t_{\theta j}^n, y_{\theta j}^n, u_j^n), v),$$
(10a)

$$(y_0^n, v)_1 = (y^0, v)_1,$$
(10b)

By substituting $v = 2\theta \Delta t y_{j+1}^n$ in (10a), rewriting the terms in the L.H.S. (Left Hand

Side) of the obtained equation we get
$\theta\left[\left\|y_{j+1}^{n}\right\|_0^2-\left\|y_j^n\right\|_0^2+\left\|y_{j+1}^n-y_j^n\right\|_0^2\right]+$
$2\theta\Delta t a\left(y_{\theta j}^n,y_{j+1}^n\right)=2\theta\Delta t(f\ (y_{\theta j}^n,u_j^n),y_{j+1}^n)$

(11)

In the following steps we will use c_1,c_2,c_1' and c_2' denoting for various nonnegative constants.

Now, from the assumptions on f , we can write the term in R.H.S. of (11) by

$$2\theta\Delta t(f\ (y_{\theta j}^n,u_j^n),y_{j+1}^n)\le\theta\Delta t\int_\Omega\eta_j^2dx+$$
$$\theta\Delta t\int_\Omega\left|y_{j+1}^n\right|^2dx+$$
$$+\theta\Delta t\left[c_1\int_\Omega\left|y_{\theta j}^n\right|^2dx+c_1\int_\Omega\left|y_{j+1}^n\right|^2dx\right]$$
$$+\theta\Delta t\left[c_1'\int_\Omega\left|y_{j+1}^n\right|^2dx+c_1'\int_\Omega\left|u_j^n\right|^2dx\right]$$
$$\le\theta\Delta t\left[c_1+c_2\left\|y_{j+1}^n\right\|_0^2+c_2'\left\|y_{\theta j}^n\right\|_0^2+c_1'\left\|u_j^n\right\|_0^2\right]$$

(11a)

Since
$\left\|y_{j+1}^n\right\|_0^2\le2\left\|y_{j+1}^n-y_j^n\right\|_0^2+2\left\|y_j^n\right\|_0^2$ and
$\left\|y_{\theta j}^n\right\|_0^2\le2\theta\left\|y_{j+1}^n-y_j^n\right\|_0^2+(1+\theta)\left\|y_j^n\right\|_0^2.$

Substituting these inequalities in (11a), we get that

$$2\theta\Delta t(f\ (y_{\theta j}^n,u_j^n),y_{j+1}^n)\le$$
$$\theta\Delta t\left[c_1+c_2\left\|y_{j+1}^n-y_j^n\right\|_0^2\right]$$
$$+\theta\Delta t\left[c_2'\left\|y_j^n\right\|_0^2+c_1'\left\|u_j^n\right\|_0^2\right]$$

(11b)

On the other hand, since

$$2\theta a\left(y_{\theta j}^n,y_{j+1}^n\right)=2\theta^2a\left(y_{j+1}^n,y_{j+1}^n\right)+$$
$$2(1-\theta)\theta a\left(y_j^n,y_{j+1}^n\right)$$

and

$$2\theta(1-\theta)a\left(y_{j+1}^n,y_j^n\right)=a\left(y_{\theta j}^n,y_{\theta j}^n\right)-$$
$$\theta^2a\left(y_{j+1}^n,y_{j+1}^n\right)-(1-\theta)^2a\left(y_j^n,y_j^n\right)$$

From these equalities we get that

$$2\theta a\left(y_{\theta j}^n,y_{j+1}^n\right)=\theta^2a\left(y_{j+1}^n,y_{j+1}^n\right)+$$
$$a\left(y_{\theta j}^n,y_{\theta j}^n\right)-(1-\theta)^2a\left(y_j^n,y_j^n\right)$$

(11c)

Now, by substituting (11b) and (11c), in (11), taking the summation for the obtaining equation from $j=0$ to $j=l-1$, drooping the common terms, using the assumptions on $a(.,.)$, we get

$$\theta\left\|y_l^n\right\|_0^2-\theta\left\|y_0^n\right\|_0^2+\theta\sum_{j=0}^{l-1}\left\|y_{j+1}^n-y_j^n\right\|_0^2+$$
$$\alpha_2\Delta t\sum_{j=0}^{l-1}\left\|y_{\theta j}^n\right\|_1^2$$
$$+\Delta t\sum_{j=0}^{l-1}\left[\theta^2a\left(y_{j+1}^n,y_{j+1}^n\right)-(1-\theta)^2a\left(y_j^n,y_j^n\right)\right]$$
$$\le\theta\Delta t\sum_{j=0}^{l-1}\left[c_1+c_2\left\|y_{j+1}^n-y_j^n\right\|_0^2\right]$$
$$+\theta\Delta t\sum_{j=0}^{l-1}\left[c_2'\left\|y_j^n\right\|_0^2+c_1'\left\|u_j^n\right\|_0^2\right]$$

(12)

Since $\theta^2-(1-\theta)^2\le\theta^2$, then

$$\sum_{j=0}^{l}\left(\theta^2-(1-\theta)^2\right)a\left(y_j^n,y_j^n\right)-\theta^2a\left(y_0^n,y_0^n\right)\le$$
$$\sum_{j=0}^{l-1}\left[\theta^2a\left(y_{j+1}^n,y_{j+1}^n\right)-(1-\theta)^2a\left(y_j^n,y_j^n\right)\right]$$

Now, by substituting this inequality in (12), with $c_1=\theta c_1 l\,\Delta t\le\theta c_1 N\,\Delta t=\theta c_1 T\ge0$, adding the common terms, we get that

$$\theta\left\|y_l^n\right\|_0^2+\theta(1-c_2\Delta t)\sum_{j=0}^{l-1}\left\|y_{j+1}^n-y_j^n\right\|_0^2+$$
$$\alpha_2\Delta t\sum_{j=0}^{l-1}\left\|y_{\theta j}^n\right\|_1^2+$$

$$\Delta t\left(\theta^2-(1-\theta)^2\right)\sum_{j=0}^{l} a\left(y_j^n, y_j^n\right)$$
$$-\theta^2\Delta t a\left(y_0^n, y_0^n\right) \le c_1+\theta\Delta t c_2'\sum_{j=0}^{l-1}\left\|y_j^n\right\|_0^2+$$
$$\theta\left\|y_0^n\right\|_0^2+c_1'\theta\Delta t\sum_{j=0}^{l-1}\left\|u_j^n\right\|_0^2$$

(13)

Since $\|u\|_Q \le c \Rightarrow$

$$c_1'\theta\Delta t\sum_{j=0}^{l-1}\left\|u_j^n\right\|_0^2 = c_1'\theta\sum_{j=0}^{l-1}\Delta t\left\|u_j^n\right\|_0^2 \le c_1'\theta\left\|u^n\right\|_Q$$

$\le c_1'\theta c = c_5$,

and since $\|y_0^n\|_1 \le \bar{c}$, & $\|y_0^n\|_0 \le \bar{c}$, then from the assumption on $a(.,.)$, we have $\theta^2\Delta t a(y_0^n, y_0^n) \le \theta^2\Delta t\left|a\left(y_0^n, y_0^n\right)\right|$

$$\le \theta^2\Delta t\alpha_1\left\|y_0^n\right\|_1^2 \le \theta^2\Delta t\alpha_1\bar{c} = c_5'$$

and $\theta\left\|y_0^n\right\|_0^2 \le \theta\bar{c} = c_5''$

by choosing Δt small such that $c_2\Delta t < 1$, we get that $(1-c_2\Delta t) > 0$.

Now, by substituting these results in (13), we get that

$$\theta\left\|y_l^n\right\|_0^2+\theta(1-c_2\Delta t)\sum_{j=0}^{l-1}\left\|y_{j+1}^n-y_j^n\right\|_0^2+$$
$$\alpha_2\Delta t\sum_{j=0}^{l-1}\left\|y_{\theta j}^n\right\|_1^2+$$
$$\left(\theta^2-(1-\theta)^2\right)\Delta t\sum_{j=0}^{l} a\left(y_j^n, y_j^n\right)$$
$$\le c_1+c_2'\Delta t\sum_{j=0}^{l-1}\left\|y_j^n\right\|_0^2$$

(14)

where $c_2' = \theta c_2' \ge 0$, and $c_1 = c_1+c_5+c_5'+c_5'' \ge 0$

Now, for $\theta \ge 1/2$, since the 2nd, 3rd and the 4th terms in L.H.S. of (14) are nonnegative, hence, we can write it in the following form

$$\left\|y_l^n\right\|_0^2 \le c_1+c_2'\Delta t\sum_{j=0}^{l-1}\left\|y_j^n\right\|_0^2,$$

where $c_1 = \dfrac{c_1}{\theta}$, & $c_2' = \dfrac{c_2'}{\theta}$

and by using the discrete Bellman-Gronwall's inequality[2], we get that

$\left\|y_l^n\right\|_0^2 \le b_1^2(c)$, for every l

so it is true for each $j, j = 0,1,\ldots,N-1,N$, i.e.

$$\left\|y_j^n\right\|_0 \le b_1(c),\ j = 0,1,\ldots,N-1,N \Rightarrow$$
$$\Delta t\sum_{j=0}^{N-1}\left\|y_j^n\right\|_0^2 \le b(c)$$

(14a)

Using this result in R.H.S. of (14) with the properties that the 1st, 3rd, and the 4th terms in L.H.S. of (14) are nonnegative, it becomes (for $l = N$)

$$\sum_{j=0}^{N-1}\left\|y_{j+1}^n-y_j^n\right\|_0^2 \le b(c),$$

where $b(c) = (c_1+c_2'b(c))/\theta(1-c_3\Delta t)$.

Also, since the 1st, 2nd, and 4th terms in (14) are nonnegative, and from (14a), (14) gives (for $l = N$)

$$\Delta t\sum_{j=0}^{N-1}\left\|y_{\theta j}^n\right\|_1^2 \le b(c),$$

where $b(c) = (c_1+c_2'b(c))/\alpha_2$

now, by inverse inequality $\Delta t \le h^2/c$, and when $\theta = 1/2$, we have

$$\Delta t\sum_{j=0}^{N-1}\left\|y_{j+1}^n-y_j^n\right\|_1^2 \le b(c)$$

Finally, the 1st, 2nd, and 3rd terms in (14) are nonnegative, and by using (14a), it becomes (for $l = N$)

$$\left(\theta^2-(1-\theta)^2\right)\Delta t\sum_{j=0}^{N} a\left(y_j^n, y_j^n\right) \le c_1+c_2'b(c)$$

Now, we have the following two cases for the parameter θ: -

(i) If $\theta > 1/2$, from the assumption on $a(.,.)$ and the above equation we get

$$\Delta t\sum_{j=0}^{N}\left\|y_j^n\right\|_1^2 \le (c_1 + c_2' b(c)) / \left(\theta^2 - (1-\theta)^2\right)\alpha_2$$
$= \bar{b}_2(c)$.

(ii) If $\theta = 1/2$, we use here and for each $j = 0,1,...,N-1$, the following inequality

$$\left\|y_j^n\right\|_1 \le \left\|y_{\theta j}^n\right\|_1 + (1/3)\left\|y_{j+1}^n - y_j^n\right\|_1 \Rightarrow$$

$$\left\|y_j^n\right\|_1^2 \le (4/3)\left\|y_{\theta j}^n\right\|_1^2 + (4/9)\left\|y_{j+1}^n - y_j^n\right\|_1^2 \Rightarrow$$

$$\Delta t\sum_{j=0}^{N-1}\left\|y_j^n\right\|_1^2 \le \frac{16}{9} b(c) = b'(c)$$

And for $j = N$, from the inverse inequality we have

$$\Delta t\left\|y_N^n\right\|_1^2 \le \frac{h^2}{c}\frac{c}{h^2}\left\|y_N^n\right\|_1^2 \le b_1^2(c) \Rightarrow$$

$$\Delta t\sum_{j=0}^{N}\left\|y_j^n\right\|_1^2 \le b'(c) + b_1^2(c) = \bar{b}(c)$$

From (i), and (ii), we get for each $\theta \ge 1/2$, that

$$\Delta t\sum_{j=0}^{N}\left\|y_j^n\right\|_1^2 \le b(c)\,, b(c) = \max(\bar{b}_2(c), \bar{b}(c))$$

Lemma 2.3:- If the function f is Lipschitizian w.r.t. y^n and u^n, if $y_{\theta j}^n$ and $y_{\theta j}^n + \Delta y_{\theta j}^n$ are the discrete states solutions correspond to the discrete controls u_j^n and $u_j^n + \Delta u_j^n$, for each $j = 0,1,...,N-1$, where the controls u and $u + \Delta u$ are bounded on $L^2(Q)$, then $\forall j = 0,1,...,N$ the following norms are satisfy:

$$\left\|\Delta y_j^n\right\|_{L^2(\Omega)} \le M\ \left\|\Delta u\right\|_Q\,,$$

$$\Delta t\sum_{j=0}^{N-1}\left\|\Delta y_{\theta j}^n\right\|_1^2 \le M^2\left\|\Delta u\right\|_Q^2, \text{and}$$

$$\Delta t\sum_{j=0}^{N}\left\|\Delta y_j^n\right\|_1^2 \le M^2\left\|\Delta u\right\|_Q^2$$

Proof: The proof is similar to the prove of Lemma 2.2.

3. Existence of a Discrete Optimal Control:- In order to study the existence of a discrete classical optimal control, we suppose in addition to the above assumptions, that the function $g_\ell^n(x, t_{\theta j}^n, y_{\theta j}^n, u_j^n)$, for each ($\ell = 0,1,2$) is defined on $\Omega \times I_j^n \times S_i^n \times W^n$,($\forall i = 1,...,M$, & $\forall j = 0,1,...,N$), continuous w.r.t. $y_{\theta j}^n$ & u_j^n for fixed x and j, measurable w.r.t. x for fixed $y_{\theta j}^n$ & u_j^n, and satisfies

$$\left|g_\ell^n(x, t_{\theta j}^n, u_j^n, y_{\theta j}^n)\right| \le \eta_{2\ell,j}(x) + c_{2\ell,j}(y_{\theta j}^n)^2 + c'_{2\ell,j}(u_{\theta j}^n)^2$$

where $\eta_{2\ell,j}(x) = \eta_{2\ell,j}(x, t_{\theta j}^n) \in L^1(Q)$ $\forall x \in \Omega$, $c_{2\ell,j} \ge 0$, $c'_{2\ell,j} \ge 0$ $j = 0,1...,N$.

Lemma 3. 1: The functional $G_\ell^n(u^n)$, ($\ell = 0,1,2$) is continuous w.r.t. u^n on $L^2(Q)$.

Proof: For each ℓ ($\ell = 0,1,2$), we have

$$G_l^n(u^n) = \Delta t\sum_{j=0}^{N-1}\int_\Omega g_\ell^n(t_{\theta j}^n, y_{\theta j}^n, u_j^n)dx$$

From the assumptions on g_ℓ^n, Lemma 2.1 and Proposition 1.2 in [9], the functional $\int_\Omega g_\ell^n(t_{\theta j}^n, y_{\theta j}^n, u_j^n)dx$ is continuous w.r.t. $y_{\theta j}^n$ & u_j^n, $\forall j$, and $\forall x \in \Omega$.

Then $G_\ell^n(u^n)$ is continuous w.r.t. u^n on $L^2(Q)$.

Theorem 3.1: With the assumptions on f , g_ℓ^n ($\ell = 0,1,2$), if W^n is closed, $W_A^n \ne \phi$, and $G_0^n(u^n)$ is coercive if W^n is unbounded. Then there exists an optimal control for the discrete problem.

Proof: From Lemma 3.1, we get that $G_l^n(u^n)$ is continuous w.r.t. u^n, $\forall x \in \Omega$, $\forall l = 0,1,2$. Since $W_A^n \ne \varnothing$, then there exists

$u^n \in W^n$, such that $\left|G_1^n(u^n)\right| \le \varepsilon_1^n$, & $G_2^n(u^n) \le \varepsilon_2^n$. To prove W_A^n is closed, let $u^{nk} \in W_A^n$ (for each k), such that $u^{nk} \to u^n$ as $k \to \infty$, then for each k, $u^{nk} \in W^n$, but W^n is closed, therefore $u^n \in W^n$, but from Lemma 2.1 the operator $u^n \mapsto y^n = y_{u^n}$ is continuous, and hence

$$\left|G_1^n(u^n)\right| = \lim_{k\to\infty}\left|G_1^n(u^{nk})\right| \le \lim_{k\to\infty}\varepsilon_1^{nk} \to \varepsilon_1^n$$

$$G_2^n(u^n) = \lim_{k\to\infty} G_2^n(u^{nk}) \le \lim_{k\to\infty}\varepsilon_2^{nk} \to \varepsilon_2^n$$

$$\Rightarrow \left|G_1^n(u^n)\right| \le \varepsilon_1^n \text{ and } G_2^n(u) \le \varepsilon_2^n$$

$$\Rightarrow u^n \in W_A^n$$

i.e. W_A^n is closed. Now, we have the following two cases: -

Case1: If W^n is bounded, then W_A^n is bounded, and then is compact. Since $G_0^n(u^n)$ is continuous w.r.t. u^n, for each $u^n \in W_A^n$, then there exists $\bar{u}^n \in W_A^n$, such that

$$G_0^n(\bar{u}^n) = \min_{u^n \in W_A^n} G_0^n(u^n).$$

Case2: If $G_0^n(u^n)$ is coercive, since W_A^n is closed (from Case1), and $G_0^n(u^n)$ is continuous w.r.t. u^n, for each $u^n \in W_A^n$ then there exists $\bar{u}^n \in W_A^n$, such that

$$G_0^n(\bar{u}^n) = \min_{u^n \in W_A^n} G_0^n(u^n).$$

4. The necessary conditions for discrete optimality: In order to state the necessary conditions for discrete classical optimal control, we suppose in addition that the functions f, f_u, f_y, g_ℓ^n, $g_{\ell y}^n$, $g_{\ell u}^n$ are defined on $S_i^n \times I_j^n \times R \times W^n$, (where U' is an open set containing the compact set U), measurable w.r.t. x, and continuous w.r.t. y_j^n & u_j^n, and satisfy(for each $j = 0,1,\dots,N$ with $(x, t_{\theta j}^n, y_{\theta j}^n, u_j^n) \in S_i^n \times I_j^n \times R \times W^n$)

$\left|f_y(x, t_{\theta j}^n, y_{\theta j}^n, u_j^n)\right| \le L$, and

$\left|f_u(x, t_{\theta j}^n, y_{\theta j}^n, u_j^n)\right| \le L'$,

$$\left|g_{\ell y}^n(x, t_{\theta j}^n, y_{\theta j}^n, u_j^n)\right| \le \eta_{1\ell}^n(x, t_{\theta j}^n) + \gamma_{1\ell}\left|y_{\theta j}^n\right| + \delta_{1\ell}\left|u_j^n\right|,$$

$$\left|g_{\ell u}^n(x, t_{\theta j}^n, y_{\theta j}^n, u_j^n)\right| \le \eta_{2\ell}^n(x, t_{\theta j}^n) + \gamma_{2\ell}\left|y_{\theta j}^n\right| + \delta_{2\ell}\left|u_j^n\right|,$$

where $\eta_{1\ell}^n \in L^2(\Omega)$, $\eta_{2\ell}^n \in L^2(\Omega)$, $\gamma_{1\ell} \ge 0$, $\gamma_{2\ell} \ge 0$, $\delta_{1\ell} \ge 0$ & $\delta_{1\ell} \ge 0$, ($\ell = 0,1,2$)

Lemma 4.1: With the all above assumptions (drooping the index ℓ) , the *Hamiltonian* is denoted by:-

$$H^n(t_{\theta j}^n, y_{\theta j}^n, z_{1-\theta,j}^n, u_j^n) = \Delta t \sum_{j=0}^{N-1}\left(z_{(1-\theta)j}^n f(t_{\theta j}^n, y_{\theta j}^n, u_j^n) + g^n(t_{\theta j}^n, y_{\theta j}^n, u_j^n)\right)$$

where $z^n = z_{u^n} = (z_0^n, z_1^n, \dots, z_N^n)$, is the solution of the discrete adjoint equation

$$-(z_{j+1}^n - z_j^n, v) + \Delta t a\left(v, z_{1-\theta,j}^n\right) = \Delta t\left(z_{1-\theta,j}^n f_y(y_{\theta j}^n, u_j^n) + g_y^n(y_{\theta j}^n, u_j^n), v\right)$$

$\forall v \in V^n, \forall j = N-1, N-2, \dots, 0$,

(15a)

$z_N = 0$

(15b)

Then the *Fréchet derivative* of G^n w.r.t. u^n, is given by:

$$G^{n\prime}(u^n)\Delta u^n = \Delta t \sum_{j=0}^{N-1}\left(H_u^n(t_{\theta j}^n, y_{\theta j}^n, z_{1-\theta,j}^n, u_j^n), \Delta u_j^n\right)$$

Proof: From the above assumptions and for a fixed control $u \in W$ the adjoint-state equations (15a-15b), has a unique solution $z = z_u$ (this can be proved by using the same way which used in Theorem 2.1).

By using equation (12a), with $v = z^n_{1-\theta,j}$, and then using the Frechét derivative of the function f in R.H.S. of the obtained equation (which it exists from the assumptions on f [10]), then multiplying both sides by Δt, summing over j (from $j=0$ to $j=N-1$), using the results of Lemma 3.2, we get

$$(\frac{\Delta y^n_{j+1} - \Delta y^n_j}{\Delta t}, z^n_{1-\theta,j}) + a\left(\Delta y^n_{\theta j}, z^n_{1-\theta,j}\right) =$$
$$\left(f_y(y^n_{\theta j}, u^n_j)\Delta y^n_{\theta j} + f_u(y^n_{\theta j}, u^n_j)\Delta u^n_j, z^n_{1-\theta,j}\right)$$

$$+\varepsilon_1(\Delta u)\|\Delta u\|_Q$$
(16a)
where $\varepsilon_1(\Delta u) \to 0$, as $\Delta u \to 0$, with
$\Delta y_0(x) = 0$
(16b)

Substituting $v = \Delta y^n_{\theta j}$ in (15a), and taking the summation for the obtained equation and for (16a), from $j=0$ to $j=N-1$, then subtracting the obtained equation of (15a) from the obtained equation of (16a), we get that

$$\Delta t \sum_{j=0}^{N-1} \int_\Omega g^n_y(y^n_{\theta j}, u^n_j)\Delta y^n_{\theta j}\, dx =$$
$$\Delta t \sum_{j=0}^{N-1} \int_\Omega f_u(y^n_{\theta j}, u^n_j)\Delta y^n_{\theta j} z^n_{1-\theta,j}\, dx$$
$$+\bar{\varepsilon}_1(\Delta u)\|\Delta u\|_Q$$
(17)
where $\bar{\varepsilon}_1(\Delta u) = T\,\varepsilon_1(\Delta u) \to 0$, as $\Delta u \to 0$

On the other hand, we have that (from the assumption on g)

$$G^n(u^n + \Delta u^n) - G^n(u^n) =$$
$$\Delta t \sum_{j=0}^{N-1} \int_\Omega \left(g^n_y \Delta y^n_{\theta j} + g^n_u \Delta u^n_j\right)dx$$
$$+\varepsilon_2(\Delta u)\|\Delta u\|_Q$$
(18)
where $\varepsilon_2(\Delta u) \to 0$, as $\Delta u \to 0$.

From (17) and (18), we get that

$$G^n(u^n + \Delta u^n) - G^n(u^n) =$$
$$\Delta t \sum_{j=0}^{N-1} \int_\Omega f_u(y^n_{\theta j}, u^n_j) z^n_{1-\theta,j} \Delta u^n_j dx$$
$$+\Delta t \sum_{j=0}^{N-1} \int_\Omega g^n_u(y^n_{\theta j}, u^n_j)\Delta u^n_j dx + \varepsilon(\Delta u)\|\Delta u\|_Q$$

(19)
where $\varepsilon(\Delta u) = \bar{\varepsilon}_1(\Delta u) + \varepsilon_2(\Delta u)$ &
$\varepsilon(\Delta u) \to 0$, as $\Delta u \to 0$

but

$$G^n(u^n + \Delta u^n) - G^n(u^n) =$$
$$G^{n\prime}(u^n)\Delta u^n + \varepsilon(\Delta u)\|\Delta u\|_Q$$

Then

$$G^{n\prime}(u^n)\Delta u^n =$$
$$\Delta t \sum_{j=0}^{N-1} (H^n_u(t^n_{\theta j}, y^n_{\theta j}, z^n_{1-\theta,j}, u^n_j), \Delta u^n_j) \qquad (20)$$

Lemma 4.2: The operator $u^n \mapsto z_{u^n}$ is continuous.
Proof: The proof is similar to Lemma 2.1.

Lemma 4.3: The operator $u^n \mapsto G^{n'}(u^n)$, is continuous w.r.t. u^n, for each $\ell = 0,1,2$.
Proof:- The proof is similar to Lemma 3.1.

Theorem 4.1: Necessary Conditions for Optimality : - In addition to the all above assumptions, if W^n is convex, and $u^n \in W_A^n$ is optimal. Then there exist multipliers $\lambda_\ell^n \in R$, $\ell = 0,1,2$, with $\lambda_0^n \geq 0$ $\lambda_2^n \geq 0$ and $\sum_{\ell=0}^{2} |\lambda_\ell^n| = 1$,such that the following **K.T.L (Kuhan, Tanger and Lagrange)** conditions are satisfied: -

$$\Delta t \sum_{j=0}^{N-1} (H_u^n(t_{\theta j}^n, y_{\theta j}^n, z_{1-\theta,j}^n, u_j^n), w_j^n - u_j^n) \geq 0, \quad \forall w_j^n \in W^n \tag{21a}$$

&

$$\lambda_2^n \left[G_2^n(u^n) - \varepsilon_2^n \right] = 0. \tag{21b}$$

If W^n has the form $W^n = \{ w | w_j^n \in U, j = 0,1,...,N-1 \}$, with $U \subset R$. Then the inequality (21) is equivalent to the following minimum principle in blockwise form

$$(H_u^n(t_{\theta j}^n, y_{\theta j}^n, z_{1-\theta,j}^n, u_{ij}^n), u_{ij}^n)_{T_i} = \min_{w \in U} (H_u^n(t_{\theta j}^n, y_{\theta j}^n, z_{1-\theta,j}^n, u_{ij}^n), w^n)_{T_i}$$

$$\forall j = 0,1,...,N-1, \ i = 1,2,,...,M \tag{22}$$

Proof: From Lemma 3.1, the functional $G_\ell^n(u^n)$ (for each $\ell = 0,1,2$), is continuous for each $u^n \in W^n$. From Lemma 4.3 the functional $G_\ell^n(u^n)$ has a continuous Fréchet derivative at each $u^n \in W^n$, for each n and since $W^n \subset L^2(\Omega)$, and $L^2(\Omega)$ is open, then by [10] (for each $\ell = 01,2$) we have

$$DG_\ell(u^n, w^n - u^n) = G_\ell^{n\prime}(u^n)(w^n - u^n).$$

Then the conditions of the **K.T.L.** theorm in [10] are satisfy here, then by applied this theorem there exist multipliers $\lambda_\ell^n \in R$, $\ell = 0,1,2$, with $\lambda_0^n \geq 0$, $\lambda_2^n \geq 0$, & $\sum_{\ell=0}^{2} |\lambda_\ell^n| = 1$, such that $\forall w^n \in W^n$

$$(\lambda_0^n G_0^{n\prime}(u^n) + \lambda_1^n G_1^{n\prime}(u^n) + \lambda_2^n G_2^{n\prime}(u^n), w^n - u^n) \geq 0 \tag{23a}$$

and

$$\lambda_2^n \left[G_2^n(u^n) - \varepsilon_2^n \right] = 0 \tag{23b}$$

Now by using Lemma 4.1, setting $\Delta u^n = w^n - u^n$, and substituting the Fréchet derivative of $G_\ell^n(u^n)$, for $\ell = 0,1,2$ in (23a), we get that

$$\Delta t \sum_{j=0}^{N-1} \int_\Omega \sum_{l=0}^{2} \lambda_\ell^n f_u(y_{\theta j}^n, u_j^n) z_{\ell_{1-\theta,j}}^n \Delta u_j^n dx + \Delta t \sum_{j=0}^{N-1} \int_\Omega \sum_{l=0}^{2} \lambda_\ell^n g_{\ell_u}^n(y_{\theta j}^n, u_j^n) \Delta u_j^n dx \geq 0 \Rightarrow$$

$$\Delta t \sum_{j=0}^{N-1} \int_\Omega z_{1-\theta,j}^n f_u(y_{\theta j}^n, u_j^n) \Delta u_j^n dx + \Delta t \sum_{j=0}^{N-1} \int_\Omega g_u(y_{\theta j}^n, u_j^n) \Delta u_j^n dx \geq 0, \forall w_j^n \in W^n$$

where $g_u^n(y_{\theta j}^n, u_j^n) = \sum_{\ell=0}^{2} \lambda_\ell^n g_{\ell_u}^n(y_{\theta j}^n, u_j^n)$, and

$$z_{1-\theta,j}^n = \sum_{\ell=0}^{2} \lambda_\ell^n z_{\ell_{1-\theta,j}}^n \Rightarrow$$

$$\Delta t \sum_{j=0}^{N-1} (H_u^n(t_{\theta j}^n, y_{\theta j}^n, z_{1-\theta,j}^n, u_j^n), w_j^n - u_j^n) \geq 0, \quad \forall w_j^n \in W^n$$

First, we prove that (21a) is equivalent with the minimum principle in blockswise form (22). Let $\bar{W} = W_U^n = \{ w_j^n : w_j^n \in U, j = 0,1,...,N-1 \}$, and let $w_j^n = u_j^n, \forall j = 0,1,...,N-1$, except one, say $w_k^n \neq u_k^n$, hence (21a) becomes

$$(H_u^n(y_{\theta k}^n, z_{1-\theta,k}^n, u_k^n), w_k^n - u_k^n) \geq 0 \Rightarrow$$

$$(H_u^n(y_{\theta k}^n, z_{1-\theta,k}^n, u_k^n), u_k^n) =$$

$\min_{w^n \in \bar{W}} (H_u^n(y_{\theta k}^n, z_{1-\theta,k}^n, u_k^n), w^n)$

The above procedure is true for any k, hence it is true for any $j, j = 0,1,...,N-1$, and hence, we get

$(H_u^n(y_{\theta j}^n, z_{1-\theta,j}^n, u_j^n), u_j^n) =$

$\min_{w^n \in \bar{W}} (H_u^n(y_{\theta j}^n, z_{1-\theta,j}^n, u_j^n), w^n)$,

i.e.

$(H_u^n(y_{\theta j}^n, z_{1-\theta,j}^n, u_{ij}^n), u_{ij}^n)_{T_i} =$

$\min_{w^n \in \bar{W}} (H_u^n(y_{\theta j}^n, z_{1-\theta,j}^n, u_{ij}^n), w^n)_{T_i}$.

The converse is clear .

Theorem 4.2: (Sufficient Conditions For Optimality): In addition to the all above assumptions, we assume that W^n is convex (e.g. $W^n = W_U^n$, with U is convex), f & g_1 are affine w.r.t. (y,u), for each $(x, t_{\theta j}^n)$, and $\forall j = 0,1,...,N-1$, and g_0 & g_2 are convex w.r.t. (y,u), for each $(x, t_{\theta j}^n)$, and $\forall j = 0,1,...,N-1$. Then the necessary conditions of Theorem 4.1, with $\lambda_0 > 0$, are also sufficient.

Proof: From the proof of Theorem 4.1 we get

$DG_\ell(u^n, w^n - u^n) = G_\ell'^n(u^n)(w^n - u^n)$,

for each $\ell = 01,2$, and $u^n \in W^n$

Assume that $u^n \in W_A^n$, and as in Theorem 4.1, we get that the conditions of the **K.T.L.** theorem (see [10]) are satisfy here, then by applied this theorem again, i.e. there exist multipliers $\lambda_\ell^n \in \mathrm{R}$, $\ell = 0,1,2$, with $\lambda_0^n > 0$, $\lambda_2^n \geq 0$, & $\sum_{\ell=0}^{2} |\lambda_\ell^n| = 1$, such that (21a &b) are satisfy .

Now, let $G^n(u^n) = \sum_{\ell=0}^{2} \lambda_\ell^n \widehat{G}_\ell^n(u^n)$, with $\widehat{G}_0^n(u^n) = G_0^n(u^n)$, $\widehat{G}_1^n(u^n) = G_1^n(u^n) - \eta_1^n$, $\widehat{G}_2^n(u^n) = G_2^n(u^n) - \varepsilon_2^n$.

Then

$$G^{n\prime}(u^n)\Delta u^n = \sum_{\ell=0}^{2} \lambda_\ell^n \widehat{G}_\ell'^n(u^n)\Delta u^n$$

$$G^{n\prime}(u^n)\Delta u^n = \Delta t \sum_{j=0}^{N-1} (z_{1-\theta,j}^n f_u(y_{\theta j}^n, u_j^n), \Delta u_j^n) + \Delta t \sum_{j=0}^{N-1} (g_u^n(y_{\theta j}^n, u_j^n), \Delta u_j^n)$$

where $g_u^n(y_{\theta j}^n, u_j^n) = \sum_{\ell=0}^{2} \lambda_\ell^n g_{\ell_u}^n(y_{\theta j}^n, u_j^n)$, and

$z_{1-\theta,j}^n = \sum_{\ell=0}^{2} \lambda_\ell^n z_{\ell_{1-\theta,j}}^n \Rightarrow$

$G^{n\prime}(u^n)\Delta u^n =$

$\Delta t \sum_{j=0}^{N-1} (H_u^n(t, y_{\theta j}^n, z_{1-\theta,j}^n, u_j^n), \Delta u_j^n) \geq 0 \Rightarrow$

$G^{n\prime}(u^n)\Delta u^n \geq 0$.

(24)

From the assumptions on f and Lemma 2.1 we have the operator $u^n \mapsto y^n = y_{u^n}^n$ is convex-linear, from the hypotheses on $g_1^n(t_{\theta j}, x, y_{\theta j}^n, u_j^n)$, we have that $g_1^n(t_{\theta j}^n, x, y_{\theta j}^n, u_j^n)$ is convex-linear w.r.t. $(y_{\theta j}^n, u_j^n)$, for each $(x, t_{\theta j}^n)$, $j = 0,1,...,N-1$, also from the hypotheses we have that $g_0^n(t_{\theta j}^n, x, y_{\theta j}^n, u_j^n)$ and $g_2^n(t_{\theta j}^n, x, y_{\theta j}^n, u_j^n)$ are convex w.r.t. $(y_{\theta j}^n, u_j^n)$, for each $(x, t_{\theta j}^n)$, $j = 0,1,...,N-1$, therefore

$G^n(u^n) = \sum_{\ell=0}^{2} \lambda_\ell^n \widehat{G}_\ell^n(u^n)$,

is convex w.r.t. $(y_{\theta j}^n, u_j^n)$, for each $(x, t_{\theta j}^n)$, $j = 0,1,...,N-1$, in the convex set W^n, and satisfy (24), at $u^n \in W^n$, hence the functional $G^n(u^n)$ has a minimum at the point u^n (sufficient condition) , i.e.

$$G^n(u^n) \le G^n(w^n), \forall w^n \in W^n \quad \Rightarrow$$

$$\lambda_0^n G_0^n(u^n) + \lambda_1^n \widehat{G}_1^n(u^n) + \lambda_2^n \widehat{G}_2^n(u^n)$$

$$\le \lambda_0^n G_0^n(w^n) + \lambda_1^n \widehat{G}_1^n(w^n)$$

$$+\lambda_2^n \widehat{G}_2^n(w^n), \forall w^n \in W^n$$

(25)

Let w^n be an admissible control, since $u^n \in W_A^n$, and satisfies (23b) with $\lambda_2^n \ge 0$, i.e.

$$\widehat{G}_1^n(u^n) = G_1^n(u^n) - \eta_1^n = 0 \;\Rightarrow\; \widehat{G}_1^n(u^n) = 0$$

and

$$\lambda_2^n \left[G_2^n(u^n) - \varepsilon_2^n \right] = 0 \Rightarrow \lambda_2^n \widehat{G}_2^n(u^n) = 0$$

On the hand, since $w^n \in W_A^n$, then

$$\widehat{G}_1^n(w^n) = G_1^n(w^n) - \eta_1^n = 0 \Rightarrow \widehat{G}_1^n(w^n) = 0$$

and

$$w^n \in W_A^n \Rightarrow \lambda_2^n \widehat{G}_2^n(w^n) \le 0$$

Substituting these results in (25), we get that

$$G_0^n(u^n) \le G_0^n(w^n), \; \forall w^n \in W^n$$

i.e. u^n is an optimal control for the considered problem.

REFERENCES

[1] Borzi, A., **(2003)**, "Multigrid Methods for Parabolic Distributed Optimal Control Problems", *Journal of Computational and Applied Mathematics*, Vol. 157, pp. 365-382, .

[2] Jaddu, H., **(2002)**, "Direct Solution of Nonlinear Optimal Control Problems Using Quasilinearization and Chebyshev Polynomials", *Journal of The Fraklin Institute*, Vol.339, pp. 479-498,.

[3] Subasi, M., **(2002)** "An Optimal Control Problem Governed by the Potential of a Linear Schrodinger equation", *Applied Mathematics and Computation*, Vol. 131, pp.95-106, .

[4] Al-Hawasy Jamil, **(2004),**"Optimization and Approximation of Parabolic Boundary Value Problem ", *Ph.D. thesis, School of App. Math. & Phy. Sci., Nathional* , Technical Univerity,Athens-Greece,.

[5] Al-Hawasy Jamil, **(2010),** "The Continuous Classical Optimal Control Problem of a Semilinear parabolic Equations(CCOCP)", *The 6th scientific conference, Karbala* ,1-2 may.

[6] Thomee, V., **(1997)** ," Galerkin Finite Element Methods for Parabolic Problems, Springer, Berlin,.

[7] Bacopoulos, A., and Chryssoverghi, I. , **(2003),**" Numerical Solutions of Partial Differential Equations by Finite Elements Methods", Symeon Publishing Co., Athens,.

[8] Burden, R.L., and Faires,J.D., **(2001),**" Numerical analysis",3rd eidition,Pws.

[9] Chryssoverghi, I. , Coletsos, J., and Kokkinis, B., **(1999),** "Discrete Relaxed Methods for Semilinear Parabolic Optimal Control Problems", *Control and Cybernetics*, Vol. 28, N0.2, pp. 157-176,.

[10] Warga, J., **(1972)**, " Optimal Control of Differential and Functional Equations", Academic Press, New York and London.

مسألة السيطرة الامثلية التقليدية من النمط المقسم لمعادلة تفاضلية جزئية شبه خطية من النمط المكافيء

إيون خريسوفيرغي[1] و جميل أمير علي الهوَّاسي[2]

[1] **قسم الرياضيات /كلية العلوم للرياضيات التطبيقية والفيزياء الطبيعية/الجامعة التكنلوجية الوطنية في اثينا**
[2] **قسم الرياضيات/كلية العلوم/ الجامعة المستنصرية**

المستخلص:-

يتناول هذا البحث دراسة مسألة السيطرة الامثلية التقليدية من النمط المقسم(discrete) لنظام من المعادلات التفاضلية الجزئية شبه الخطية من النوع المكافيء بوجود قيدي المساوات و اللامساوات. أولا" قسمت (discretized) مسألة السيطرة الامثلية التقليدية من النمط المستمر الى مسألة سيطرة امثلية تقليدية من النمط المقسم وذلك بتجزئة مجال تعريف المتغيرات المستقلة إلى مناطق صغيرة مستقلة حيث استخدمت طريقة العناصر الثابته بالنسبة للفضاء وطريقة الفروقات المنتهية بالنسبة لمتغير الزمن . ثانيا" تم ايجاد حلول معادلات تفاضلية جزئية مقسمة discrete (solutions) شبه خطية من النوع المكافىء لسيطرة تقليدية مقسمة (discrete control)وثابتة باستخدام طريقة كاليركن لتقريب الحلول المضبوطة.

ثالثا" تم اشتقاق نظرية الوجود للحصول على سيطرة أمثلية تقليدية مقسمة تحت قيدي المساوات و اللامساوات. رابعا" تم ايجاد الصيغة المتقطعة للمعادلات المرافقة (adjoint equations) التفاضلية الجزئية المصاحبة للمعادلات التفاضلية الجزئية المقسمة شبه الخطية من النوع المكافيء. اخيرا" تم اشتقاق الشروط الضرورية والكافية وصيغة مبدأ الحزم الاصغري لمناطق ثابته (Minimum principle blockwise form) لمسالة السيطرة الامثلية التقليدية المقسمة.