

THE CYCLIC DECOMPOSITION OF THE FACTOR GROUP $cf(D_{nh}, Z)/R(\overline{D_{nh}})$ WHEN n IS AN ODD NUMBER

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ABSTRACT

For fixed positive integer $n \geq 3$, let D_n be the **dihedral group**, $D_{nh} = D_n \times C_2$ and $cf(D_{nh}, Z)$ be the abelian group of Z -valued class functions of the group D_{nh} . The intersection of $cf(D_{nh}, Z)$ with the group of all generalized characters of D_{nh} , $R(D_{nh})$ is a normal subgroup of $cf(D_{nh}, Z)$ denoted by $\overline{R}(D_{nh})$, then $cf(D_{nh}, Z)/\overline{R}(D_{nh})$ is a finite abelian factor group which is denoted by $K(D_{nh})$.

In this paper, we determine the cyclic decomposition of the finite abelian factor group $cf(D_{nh}, Z)/R(D_{nh})$ when n is an odd number, we find that the cyclic decomposition of $K(D_{nh})$ depends on the elementary divisor of n , if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ where $p_1, p_2, \dots, p_m$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_m$ are positive integers, then ;

$$K(D_{nh}) = \bigoplus_{i=1}^2 K(D_n)^{(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1)-2} \bigoplus_{i=1}^{C_2} \bigoplus K(C_4).$$
 And we find the rational valued characters table of the group D_{nh} .

1. Introduction

Let G be a finite group, two elements of G are said to be Γ -conjugate if the cyclic subgroups they generate are conjugate in G , this defines an equivalence relation on G . Its classes are called Γ -classes. The Z -valued class function on the group G , which is constant on the Γ -classes forms a finitely generated abelian group $cf(G, Z)$ of a rank equal to the number of Γ -classes.

The intersection of $cf(G, Z)$ with the group of all generalized characters of G , $R(G)$ is a normal subgroup of $cf(G, Z)$ denoted by $\overline{R}(G)$, then $cf(G, Z)/\overline{R}(G)$ is a finite abelian factor group which is denoted by $K(G)$.

Each element in $\overline{R}(G)$ can be written as $u_1\theta_1 + u_2\theta_2 + \dots + u_l\theta_l$, where l is the number

of Γ -classes, $u_1, u_2, \dots, u_l \in Z$ and $\theta_i = \sum_{\sigma \in Gal(Q(\chi_i)/Q)} \sigma(\chi_i)$, where χ_i is an

irreducible character of the group G and σ is any element in Galois group $Gal(Q(\chi_i)/Q)$.

Let $\equiv^*(G)$ denotes the $l \times l$ matrix which corresponds to the θ_i 's and columns correspond to the Γ - **classes** of G . The matrix expressing $\overline{R}(G)$ basis in terms of the $\text{cf}(G, Z)$ basis is $\equiv^*(G)$.

We can use the theory of invariant factors to obtain the direct sum of the cyclic Z -module of orders the distinct invariant factors of $\equiv^*(G)$ to find the cyclic decomposition of $K(G)$. In 1982 M.S. Kirdar [13] studied the $K(C_n)$. In 1994 H.H. Abass [3] studied the $K(D_n)$ and found $\equiv^*(D_n)$. In 1995 N.R. Mahamood [15] studied the factor group $\text{cf}(Q_{2m}, Z) / \overline{R}(Q_{2m})$. In 2005 N.S. Jasim [16] studied the factor group $\text{cf}(G, Z) / \overline{R}(G)$ for the special linear group $SL(2, p)$.

In this paper we study $K(D_{nh})$ and find $\equiv^*(D_{nh})$ when n is an odd number .

2. Preliminaries

In this section we review definitions and some results which will be used in later section.

Definition (1.1): [1]

The set of all $l \times l$ non-singular matrices over the field F which forms a group under the operation of the matrix multiplication is called **the general linear group** of the dimension l over the field F , denoted by $GL(l, F)$.

Definition (1.2): [1]

A **matrix representation** of a group G is a group homomorphism T of G into $GL(l, F)$, l is called **the degree of matrix representation T** .

Definition (1.3): [1]

The trace of an $l \times l$ matrix is the sum of the main diagonal elements, denoted by $\text{tr}(A)$.

Definition (1.4): [3]

A matrix representation $T: G \rightarrow GL(l, F)$ is said to be **reducible** if there exists a non-singular matrix A over F such that:

$$A^{-1} T(g) A = \begin{bmatrix} T_1(g) & E(g) \\ o & T_2(g) \end{bmatrix}, \text{ for all } g \in G.$$

Where $T_1(g)$, $T_2(g)$ are matrices of representations T_1 and T_2 of a group over F of the dimension $r \times r$, $s \times s$ respectively and $E(g)$ is a matrix of the dimension $r \times s$ such that $0 < r < l$ and $r+s = l$. If no such reducible matrix exists then T is called **an irreducible matrix representation**.

Theorem (1.5):[1]

Let $T_1: G_1 \rightarrow GL(V_1)$ and $T_2: G_2 \rightarrow GL(V_2)$ be two irreducible representations of the groups G_1 and G_2 respectively, then $T_1 \otimes T_2$ is irreducible representations of the group $G_1 \times G_2$.

Definition (1.6): [3]

Let T be a matrix representation of a group G over the field F , **the character** χ of a matrix representation T is the mapping $\chi: G \rightarrow F$ defined by $\chi(g) = \text{Tr}(T(g))$ for all $g \in G$ where $\text{Tr}(T(g))$ refers to the trace of the matrix $T(g)$ and $\chi(1)$ is the degree of χ .

Remark (1.7):

(i) A finite group G has a finite number of conjugacy classes and a finite number of distinct irreducible character, the group character of a group representation is constant on a conjugacy class, the values of irreducible characters can be written as a table whose columns are the conjugacy class and rows the value of irreducible characters on each conjugacy class, this table of the group G , denoted by $\equiv(G)$.

(ii) If $G = C_n = \langle r \rangle$ is the cyclic group of order n generated by r . If $\omega = e^{2\pi i/n}$ is the primitive n -th root of unity, the

CL_α	1	r	r^2	...	r^{n-1}
$ \text{CL}_\alpha $	1	1	1	...	1
$ C_G(C_\alpha) $	n	n	n	...	n
χ_1	1	1	1	...	1
χ_2	1	ω	ω^2	...	ω^{n-1}

$\equiv(C_n) =$

χ_3	1	ω^2	ω^4	...	ω^{n-2}
M	M	M	M	O	M
χ_n	1	ω^{n-1}	ω^{n-2}	...	ω

Definition (1.8):[3]

Let χ and ψ as characters of a group G , then :

1. The sum of characters is defined by:

$$(\chi+\psi)(g) = \chi(g)+\psi(g) \quad , \text{ for all } g \in G$$

2. The product of characters is defined by : $(\chi.\psi)(g) = \chi(g).\psi(g)$, for all $g \in G$.

Theorem (1.9):[3]

Let $T_1: G_1 \rightarrow GL(n, K)$ and $T_2: G_2 \rightarrow GL(m, K)$ are two matrix representations of the groups G_1 and G_2 , χ_1 and χ_2 be two characters of T_1 and T_2 respectively, then the character of $T_1 \otimes T_2$ is $\chi_1 \chi_2$.

Definition (1.10):[1]

A rational valued character θ of G is a character whose values are in the set of integers Z , which is $\theta(g) \in Z$, for all $g \in G$.

Proposition (1.11):[4]

The rational valued characters $\theta_i = \sum_{\sigma \in Gal(Q(\chi_i)/Q)} \sigma(\chi_i)$ form basis for $\bar{R}(G)$, where

χ_i are the irreducible characters of G and their numbers are equal to the number of all distinct Γ - classes of G .

2. The factor group $K(G)$

In this section , we study the factor $K(G)$ and discuss the cyclic decomposition of the factor groups $K(C_n)$ and $K(D_n)$.

Definition (2.1):[4]

Let M be a matrix with entries in a principal ideal domain R , a k -minor of M is the determinate of $k \times k$ sub matrix preserving rows and columns order.

Definition (2.2):[4]

A k -th determinant divisor of M is the greatest common divisor (g.c.d) of all the k -minors of M . This is denoted by $D_k(M)$.

Lemma (2.3):[4]

Let M , P and W be matrices with entries in a principal ideal domain R , if P and W are invertible matrices, then $D_k(P M W) = D_k(M)$ modulo the group of unites of R .

Theorem (2.4):[4]

Let M be an $l \times l$ matrix entries in a principal ideal domain R , then there exists matrices P and W such that:

- 1- P and W are invertible.
- 2- $P M W = D$.
- 3- D is diagonal matrix.
- 4- if we denote D_{ii} by d_i then there exists a natural number m ;

$0 \leq m \leq l$ such that $j > m$ implies $d_j = 0$ and $j \leq m$ implies $d_j \neq 0$ and $1 \leq j \leq m$ implies $d_j \mid d_{j+1}$.

Definition (2.5):[4]

Let M be a matrix with entries in a principal ideal domain R be equivalent to a matrix $D = \text{diag} \{d_1, d_2, \dots, d_m, 0, 0, \dots, 0\}$ such that $d_j \mid d_{j+1}$ for $1 \leq j < m$.

We call D *the invariant factor matrix of* M and $d_1, d_2, \dots, d_m$ the invariant factors of M .

Theorem (2.6):[4]

Let K be a finitely generated module over a principal ideal domain R , then K is the direct sum of a cyclic sub modules with an annihilating ideal $\langle d_1 \rangle, \langle d_2 \rangle, \dots, \langle d_m \rangle, d_j \mid d_{j+1}$ for $j = 1, 2, \dots, K-1$.

Proposition(2.7):[4]

Let A and B be two non-singular matrices of the rank n and m respectively, over a principal ideal domain R . Then the invariant factor matrices of $A \otimes B$ equals $D(A) \otimes D(B)$, where $D(A)$ and $D(B)$ are the invariant factor matrices of A and B respectively.

Theorem(2.8):[4]

Let H and L be p_1 -group and p_2 -group respectively, where p_1 and p_2 are distinct primes. Then, $\equiv^*(H \times L) = \equiv^*(H) \otimes \equiv^*(L)$.

Remark (2.9):[4]

Suppose $\text{cf}(G, Z)$ is of the rank l , the matrix expressing the $\overline{R}(G)$ basis in terms of the $\text{cf}(G, Z) = Z^l$ basis is $\equiv^*(G)$.

Hence by theorem (2.4), we can find two matrices P and Q with a determinant ± 1 such that $P \cdot \equiv^*(G) \cdot Q = D(\equiv^*(G)) = \text{diag}\{d_1, d_2, \dots, d_l\}$, $d_i = \pm D_i(\equiv^*(G)) / \pm D_{i-1}(\equiv^*(G))$.

this yields a new basis for $\overline{R}(G)$ and $\text{Cf}(G, Z), \{v_1, v_2, \dots, v_l\}$ and $\{u_1, u_2, \dots, u_l\}$ respectively with the property $v_j = d_j u_j$.

Hence by theorem (2.6) the Z -module $K(G)$ is the direct sum of cyclic submodules with annihilating ideals $\langle d_1 \rangle, \langle d_2 \rangle, \dots, \langle d_l \rangle$.

Theorem(2.10) :[4]

Let p be a prime number, then :

$$K(G) = \bigoplus_{i=1}^s C_{d_i} \text{ such that } d_i = \pm D_i(\equiv^*(G)) / \pm D_{i-1}(\equiv^*(G)).$$

Theorem (2.11):[4]

$$|K(G)| = \det(\equiv^*(G)).$$

Proposition (2.3.1):[14]

The *rational valued characters table of the cyclic group* C_{p^s} of the rank $s+1$ where p is a prime number which is denoted by $(\equiv^*(C_{p^s}))$, is given as follows:

Γ -classes	[1]	$[r^{p^{s-1}}]$	$[r^{p^{s-2}}]$	$[r^{p^{s-3}}]$	...	$[r^{p^2}]$	$[r^p]$	[r]
θ_1	$p^{s-1}(p-1)$	$-p^{s-1}$	0	0	...	0	0	0
θ_2	$p^{s-2}(p-1)$	$p^{s-2}(p-1)$	$-p^{s-2}$	0	...	0	0	0
θ_3	$p^{s-3}(p-1)$	$p^{s-3}(p-1)$	$p^{s-3}(p-1)$	$-p^{s-3}$	...	0	0	0
					0			
θ_{s-1}	$p(p-1)$	$p(p-1)$	$p(p-1)$	$p(p-1)$	...	$p(p-1)$	$-p$	0
θ_s	$p-1$	$p-1$	$p-1$	$p-1$	...	$p-1$	$p-1$	-1
θ_{s+1}	1	1	1	1	...	1	1	1

where its rank $s+1$ represents the number of all distinct Γ -classes.

Example (2.13):

Consider the cyclic group C_{25} by using table (2.3), we can find the rational valued characters table of C_{25} as follows:

$$\equiv^*(C_{25}) = \equiv^*(C_{5^2}) =$$

Γ -classes	[1]	$[r^5]$	$[r]$
θ_1	20	-5	0
θ_2	4	4	-1
θ_3	1	1	1

Remark (2.14):

In general, for $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_m^{\alpha_m}$ where $\text{g.c.d}(p_i, p_j) = 1$, if $i \neq j$, p_i 's are prime numbers and $\alpha_i \in \mathbb{Z}^+$, then we have the following formula :

$$\equiv^*(C_n) = \equiv^*(C_{p_1^{\alpha_1}}) \otimes \equiv^*(C_{p_2^{\alpha_2}}) \otimes \dots \otimes \equiv^*(C_{p_m^{\alpha_m}}).$$

Proposition (2.14):[4]

If p is a prime number, then $D(\equiv^*(C_{p^s})) = \text{diag}\{p^s, p^{s-1}, \dots, p, 1\}$.

Remark (2.15):[4]

For $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_m^{\alpha_m}$ where p_i 's are distinct primes and $\alpha_i \in \mathbb{Z}^+$, then :

$$D(\equiv^*(C_n)) = D(\equiv^*(C_{p_1^{\alpha_1}})) \otimes D(\equiv^*(C_{p_2^{\alpha_2}})) \otimes \dots \otimes D(\equiv^*(C_{p_m^{\alpha_m}})).$$

Theorem(2.16) :[4]

Let p be a prime number, then $K(C_{p^s}) = \bigoplus_{i=1}^s C_{p^i}$.

Example(2.17):-

$$K(C_{25}) = K(C_{5^2}) = C_5 \oplus C_{5^2}.$$

Proposition(2.18):[4]

Let $n = \prod_{i=1}^k p_i^{a_i}$, where p_i 's are distinct primes and $a_i \in \mathbb{Z}^+$, then :

$$K(C_n) = \bigoplus_{i=1}^k \left(\bigoplus_{j=1}^{a_i} K(C_{p_i^{a_i-j+1}}) \right) \left[\prod_{j=1}^k (a_j + 1) \right] \text{ time}.$$

Example(2.19) :

Find $K(C_{540})$

$$\begin{aligned}
K(C_{540}) = K(C_{2^2 \cdot 3^3 \cdot 5}) &= K(C_{2^2}) \oplus K(C_{2^2}) \oplus K(C_{2^2}) \oplus K(C_{2^2}) \oplus K(C_{2^2}) \oplus K(C_{2^2}) \oplus K(C_{2^2}) \oplus K(C_{2^2}) \\
&\quad (1+3).(1+1) \text{ times} \\
&\oplus K(C_{3^3}) \oplus K(C_{3^3}) \oplus K(C_{3^3}) \oplus K(C_{3^3}) \oplus K(C_{3^3}) \oplus K(C_{3^3}) \oplus K(C_{3^3}) \oplus K(C_{3^3}) \\
&\quad (2+1).(1+1) \text{ times} \qquad (2+1).(3+1) \text{ times}
\end{aligned}$$

By theorem (2.16) we can find :

$$K(C_{2^2}) = C_{2^2} \oplus C_2, K(C_{3^3}) = C_{3^3} \oplus C_{3^2} \oplus C_3 \text{ and } K(C_5) = C_5$$

$$\text{Then } K(C_{540}) = C_{2^2}^{(8)} \oplus C_2^{(8)} \oplus C_{3^3}^{(6)} \oplus C_{3^2}^{(6)} \oplus C_3^{(6)} \oplus C_5^{(12)}.$$

Definition (2.20):[3]

For a fixed positive integer $n \geq 3$, **the dihedral group D_n** is a certain non-abelian group of the order $2n$. In general can write it as: $D_n = \{ S^j r^k : 0 \leq k \leq n-1, 0 \leq j \leq 1 \}$ which has the following properties : $r^n = 1$, $S^2 = 1$, $Sr^k S^{-1} = r^{-k}$.

Definition (2.21):[3]

The group D_{nh} is the direct product group $D_n \times C_2$.

Lemma (2.22):[2]

The rational valued characters table of D_n when n is an odd number is given as follows:

	Γ - classes of C_n	[S]
θ_1	$\equiv^*(C_n)$	0
	1 1 1 ... 1 1	
θ_{l-1}		0
θ_l		1
θ_{l+1}	1 1 1 ... 1 1	-1

Where l is the number of Γ - classes of C_n .

Example (2.23):

To find $\equiv^*(D_{25})$, From example (2.13), we obtain $\equiv^*(C_{25})$ and by using lemma (2.22) , we have

$$\equiv^*(D_{25}) = \equiv^*(D_{5^2})$$

Γ -classes	[1]	$[r^5]$	$[r]$	[S]
θ_1	$\equiv^*(C_{5^2})$			0
θ_2				0
θ_3				1
θ_4	1	1	1	-1

=

Γ -classes	[1]	$[r^5]$	$[r]$	[S]
θ_1	20	-5	0	0
θ_2	4	4	-1	0
θ_3	1	1	1	1
θ_4	1	1	1	-1

Proposition(2.24):[2]

$D(\equiv^*(D_n)) = \left[\begin{array}{c|c} D(\equiv^*(C_n)) & 0 \\ \hline 0 & -2 \end{array} \right]$ Where $D(\equiv^*(D_n))$ and $D(\equiv^*(C_n))$ are the invariant factors matrices of $\equiv^*(D_n)$ and $\equiv^*(C_n)$ respectively .

Theorem(2.25) :- [2]

Let $n = \prod_{i=1}^k P_i^{a_i}$, where P_i are distinct primes , $a_i \in \mathbb{Z}^+$ and $P_i \neq 2$ for all i , then :

$$K(D_n) = K(C_n) \oplus C_2$$

Example(2.26):-

To find $K(D_{540})$

$$K(D_{540}) = K(C_{540}) \oplus C_2$$

From example(2.19) we find $K(C_{540})$, then we have

$$K(C_{540}) = C_{2^2}^{(8)} \oplus C_2^{(8)} \oplus C_{3^3}^{(6)} \oplus C_{3^2}^{(6)} \oplus C_3^{(6)} \oplus C_5^{(12)}.$$

3. The Main Results

This section is devoted to study the rational valued characters table of the group D_{nh} and to find the cyclic decomposition of $K(D_{nh})$ where n is an odd number.

Theorem(3.1) :-

The rational valued characters table of the group D_{nh} when n is an odd number is given as follows: $\equiv^*(D_{nh}) = \equiv^*(D_n) \otimes \equiv^*(C_2)$

Proof :-

since

$$\equiv C_2 =$$

	g'_1	g'_2
χ'_1	1	1
χ'_2	1	-1

and by proposition (2.12),

$$\equiv^*(C_2) =$$

	g'_1	g'_2
θ'_1	1	1
θ'_2	1	-1

Table (3.5)

then, $\chi'_1(g'_1) = \chi'_1(g'_2) = \theta'_1(g'_1) = \theta'_1(g'_2) = 1$, $\chi'_2(g'_1) = \theta'_2(g'_1) = 1$, $\chi'_2(g'_2) = \theta'_2(g'_2) = -1$.

from the definition of D_{nh} , (Theorems (1.5) and (1.9)), $\equiv D_{nh} = \equiv D_n \otimes \equiv C_2$, each element in D_{nh} $g_{kh} = g_k \cdot g'_h \quad \forall \quad g_k \in D_n, g'_h \in C_2, k = 1, 2, 3, \dots, 2n, h = 1, 2$ and each irreducible character of D_{nh} is $\chi_{ij} = \chi_i \cdot \chi'_j$ where χ_i is an irreducible character of D_n and χ'_j is the irreducible character of C_2 , then

$$\chi_{ij}(g_{kh}) = \begin{cases} \chi_i(g_k) & \text{if } j=1 \text{ and } h=1,2 \\ \chi_i(g_k) & \text{if } j=2 \text{ and } h=1 \\ -\chi_i(g_k) & \text{if } j=2 \text{ and } h=2 \end{cases}$$

from proposition (1.11) $\theta_{ij} = \sum_{\sigma \in \text{Gal}(Q(\chi_{ij})/Q)} \sigma(\chi_{ij})$ where θ_{ij} is the rational valued character of D_{nh}

$$\text{Then, } \theta_{ij}(g_{kh}) = \sum_{\sigma \in \text{Gal}(Q(\chi_{ij}(g_{kh}))/Q)} \sigma(\chi_{ij}(g_{kh}))$$

(I) If $j=1$ and $h=1,2$ $\theta_{ij}(g_{kh}) = \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_k))/Q)} \sigma(\chi_i(g_k)) = \theta_i(g_k) \cdot 1 = \theta_i(g_k) \cdot \theta'_j(g'_h)$ where θ_i is

the rational valued character of D_n .

(II) (a) If $j=2$ and $h=1$

$$\theta_{ij}(g_{kh}) = \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_k))/Q)} \sigma(\chi_i(g_k)) = \theta_i(g_k) \cdot 1 = \theta_i(g_k) \cdot \theta'_j(g'_h)$$

(b) If $j=2$ and $h=2$

$$\begin{aligned} \theta_{ij}(g_{kh}) &= \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_k))/Q)} \sigma(-\chi_i(g_k)) = - \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_k))/Q)} \sigma(\chi_i(g_k)) \\ &= \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_k))/Q)} \sigma(\chi_i(g_k)) \cdot -1 = \theta_i(g_k) \cdot \theta'_j(g'_h). \end{aligned}$$

From [I] and [II] we have $\theta_{ij} = \theta_i \cdot \theta'_j$. Then $\equiv^*(D_{nh}) = \equiv^*(D_n) \otimes \equiv^*(C_2)$.

Example(3.2):-

To find the rational valued characters table of D_{25h} , we can use theorem (3.1).

By lemma (2.23) and proposition (2.12), we have

$$\equiv^*(D_{5^2}) =$$

Γ - classes	[1]	$[r^5]$	$[r]$	[S]
θ_1	20	-5	0	0
θ_2	4	4	-1	0
θ_3	1	1	1	1
θ_4	1	1	1	-1

$$\equiv^*(C_2) =$$

	1	r^*
θ'_1	1	1
θ'_2	1	-1

Then, by theorem (3.1) $\equiv^*(D_{25h}) = \equiv^*(D_{25}) \otimes \equiv^*(C_2) \equiv^*(D_{25h})$

Γ -Classes	$[1,1]$	$[1,r^*]$	$[r^5,1]$	$[r^5,r^*]$	$[r,1]$	$[r,r^*]$	$[S,1]$	$[S,r^*]$
$ CL_\alpha $	1	1	2	2	2	2	5	5
$ C_G(CL_\alpha) $	20	20	10	10	10	10	4	4
θ_{11}	20	20	-5	-5	0	0	0	0
θ_{12}	20	-20	-5	5	0	0	0	0
θ_{21}	4	4	4	4	-1	-1	0	0
θ_{22}	4	-4	4	-4	-1	1	0	0
θ_{31}	1	1	1	1	1	1	1	1
θ_{32}	1	-1	1	-1	1	-1	1	-1
θ_{41}	1	1	1	1	1	1	-1	-1
θ_{42}	1	-1	1	-1	1	-1	-1	1

Theorem(3.3) :-

For a fixed positive odd integer n such that $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ where

$p_1, p_2, \dots, p_m$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_m$ are positive integers, then ;

$$K(D_{nh}) = \bigoplus_{i=1}^2 K(D_n) \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1)-2} C_2 \bigoplus K(C_4).$$

Proof:-

By theorem (3.1) and proposition(2.7) $D(\equiv^*(D_{nh})) = D(\equiv^*(D_n)) \otimes D(\equiv^*(C_2))$ By proposition(2.24)

$$D(\equiv^*(D_n)) = \left[\begin{array}{c|c} D(\equiv^*(C_n)) & 0 \\ \hline 0 & -2 \end{array} \right]$$

Then ,

$$D(\equiv^*(D_{nh})) = \left[\begin{array}{cc} D(\equiv^*(C_n)) & 0 \\ 0 & -2 \end{array} \right] \otimes \left[\begin{array}{cc} 2 & 0 \\ 0 & -1 \end{array} \right] = \left[\begin{array}{cc|cc} 2D(\equiv^*(C_n)) & 0 & & \\ 0 & -4 & & 0 \\ \hline & & -D(\equiv^*(C_n)) & 0 \\ 0 & & 0 & 2 \end{array} \right]$$

$$= \text{diag} \{ 2d_1, 2d_2, \dots, 2d_{((\alpha_1+1)(\alpha_2+1)L(\alpha_m+1))}, -4, -d_1, -d_2, \dots, \\ -d_{((\alpha_1+1)(\alpha_2+1)L(\alpha_m+1))}, 2 \}$$

where d_i is the invariant factor of $\equiv^*(C_n)$. Then, by theorem (2.10)

$$\begin{aligned} K(D_{nh}) &= \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)} C_{2d_i} \oplus_{C_4} \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)} C_{d_i} \oplus_{C_2} \\ &= \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)} C_{d_i} \oplus_{C_2} \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)} C_2 \oplus_{C_4} \\ &\quad \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)} C_{d_i} \oplus_{C_2} \\ &= \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)} C_{d_i} \oplus_{C_2} \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)-2} C_2 \oplus_{C_2} \oplus_{C_4} \\ &\quad \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)} C_{d_i} \oplus_{C_2} \\ &= \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)} C_{d_i} \oplus_{C_2} \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)} C_{d_i} \oplus_{C_2} \\ &\quad \bigoplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)-2} C_2 \oplus_{C_4} \oplus_{C_2} \end{aligned}$$

By theorems (2.16) and (2.25), we have :

$$K(D_{nh}) = \bigoplus_{i=1}^2 K(D_n) \oplus_{i=1}^{(\alpha_1+1)(\alpha_2+1)L(\alpha_m+1)-2} C_2 \oplus K(C_4).$$

Example(3.4):-

To find the cyclic decomposition $K(D_{25h})$, $K(D_{35h})$, $K(D_{1125h})$, $K(D_{1157625h})$ and $K(D_{15015h})$ by theorem (3.3)

$$K(D_{25h}) = K(D_{5^2h}) = \bigoplus_{i=1}^2 K(D_{5^2}) \bigoplus_{i=1}^{(2+1)-2} C_2 \oplus K(C_4) = \bigoplus_{i=1}^2 K(D_{25}) \oplus C_2 \oplus K(C_4)$$

$$K(D_{35h}) = K(D_{5.7h}) = \bigoplus_{i=1}^2 K(D_{5.7}) \bigoplus_{i=1}^{(1+1).(1+1)-2} C_2 \oplus K(C_4) = \bigoplus_{i=1}^2 K(D_{35}) \bigoplus_{i=1}^2 C_2 \oplus K(C_4)$$

$$K(D_{1125h}) = K(D_{3^2.5^3h}) = \bigoplus_{i=1}^2 K(D_{3^2.5^3}) \bigoplus_{i=1}^{(2+1).(3+1)-2} C_2 \oplus K(C_4)$$

$$= \bigoplus_{i=1}^2 K(D_{1125}) \bigoplus_{i=1}^{10} C_2 \oplus K(C_4)$$

$$K(D_{1157625h}) = K(D_{3^3.5^3.7^3h}) = \bigoplus_{i=1}^2 K(D_{3^3.5^3.7^3}) \bigoplus_{i=1}^{(3+1).(3+1)(3+1)-2} C_2 \oplus K(C_4)$$

$$= \bigoplus_{i=1}^2 K(D_{1157625}) \bigoplus_{i=1}^{62} C_2 \oplus K(C_4)$$

$$K(D_{15015h}) = K(D_{3.5.7.11.13h}) = \bigoplus_{i=1}^2 K(D_{3.5.7.11.13}) \bigoplus_{i=1}^{(1+1).(1+1)(1+1)(1+1)-2} C_2 \oplus K(C_4)$$

$$= \bigoplus_{i=1}^2 K(D_{15015}) \bigoplus_{i=1}^{30} C_2 \oplus K(C_4) .$$

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التجزئة الدائرية للزمرة الكسرية $cf(D_{nh}, Z)/R(D_{nh})$ عندما n عدد فردي

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الخلاصة

لعدد صحيح موجب $n \geq 3$ ، لتكن زمرة ثنائي السطوح $D_n = D_n \bar{\cap} C_2$ و $cf(D_{nh}, Z)$ الزمرة الإبدالية لدوال الصف ذات القيم الصحيحة للزمرة D_{nh} . ان تقاطع $cf(D_{nh}, Z)$ مع زمرة كل الشواخص العمومية للزمرة D_{nh} ، $R(D_{nh})$ تكون زمرة جزئية سوية من $cf(D_{nh}, Z)$ ويرمز لها بالرمز $\bar{R}(D_{nh})$ ، إذن $K(D_{nh})$ تكون زمرة كسرية إبدالية منتهية والتي يرمز لها بالرمز $\bar{R}(D_{nh})$ ، حددنا في هذا البحث ، التجزئة الدائرية للزمرة الإبدالية المنتهية $cf(D_{nh}, Z)/R(D_{nh})$ عندما n عدد فردي، وجدنا إن التجزئة للزمرة $K(D_{nh})$ تعتمد على القواسم الأولية للعدد n ، إذا كان $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ حيث ان $p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_m^{\alpha_m}$ أعداد أولية مختلفة أعداد $\alpha_1, \alpha_2, \dots, \alpha_m$ حجة موجبة فـ ان:

$$K(D_{nh}) = \bigoplus_{i=1}^2 K(D_n) \bigoplus_{i=1}^{(\alpha_1+1) \cdot (\alpha_2+1) \cdot L(\alpha_m+1)-2} C_2 \oplus K(C_4)$$

وكذلك وجدنا الشواخص النسبية للزمرة D_{nh} .