On a sub branch of element in a BE-algebra

حول الفرع الجزئي للعنصر في جبر-BE

Assist. Lec. Hayder Abbas Dahham
University of Kufa /College of Dentistry
Department of Basic sciences
haidera.janabi@uokufa.edu.iq

Abstract

In this paper, we introduce a new notion that we call a sub branch of element R(a) in a BE-algebra, and we link this notion with notions filter and ideal of BE- algebra, We give some properties of R(a) and we study properties of R(a) in implication algebra, self distributive BE-algebra and transitive BE-algebra.

Keywords: Sub branch of element, BE-algebra, self distributive BE-algebra, transitive BE-algebra.

المستخلص:

قدمنا في هذا البحث مفهوم الفرع الجزئي للعنصر R(a) في جبر- BE , كما درسنا علاقة هذا المفهوم في مع المرشح والمثالية في جبر- BE . كما أعطينا بعض خواصه و درسنا خواصه في الجبور الاستنتاجية وجبور- BE التوزيعية الذاتية وجبور- BE المتعدية.

INTRODUCTION

Y. Imai and K. Iseki introduced two classes of abstract algebras: BCK-algebra and BCI-algebra, the notion of BCI-algebra was a generalization of a BCK-algebra[5,6]. In 1999, W. A . Dudek, connect between deductive system of Hilbert algebra and ideals which introduced by I .Chajda and R. Halas[12]. In 1999, J. Neggers. and H. S. Kim introduced the notion of d-algebra which is another generalization of BCK-algebra[4]. In 2006, H. S. Kim. and Y. H. Kim introduced the notion of BE-algebra and they used the notion of upper sets and give an equivalent condition of the filter in BE-algebra[7]. In 2012, A. Walendziak, introduced the notion of a normal filter in BE-algebra[2]. In 2013, A. Rezaei, A. B. Saeid. and R. A. Borzooei introduced relationship between Hilbert algebra and BE-algebra, and they show that a commutative implicative BE-algebra is equivalent to the commutative self distributive BE-algebra, therefore Hilbert algebra and commutative self distributive BE-algebra are equivalent to the notion of a normal filter in BE-algebra[1].

1.Preliminaries

In this section, we review some basic definitions and notations of BE-algebras, implication algebras, filter and ideals, that we need in our work.

Definition (1.1) [7] :

An algebra $(X,*;1)$ of type (2,0) is called a BE-algebra if

1- $x * x = 1$ for all $x \in X$;
2- $x * 1 = 1$ for all $x \in X$;
3- $1 * x = x$ for all $x \in X$;
4- $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (exchange).

We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$.
A nonempty subset S of X is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$.

Example (1.2) [7] :
Let X = {1, a, b, c, d} be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $(X; * , 1)$ is a BE-algebra.

Proposition (1.3) [1]:
Let X be a BE- algebra and $x, y \in X$. Then
1- $x * (y * x) = 1$.
2- $y*((y*x) *x)=1$.

Definition (1.4) [8] :
An algebras $(X, * , 1)$ of type (2, 0) is called an implication algebra if for all $x,y,z \in X$ the following identities hold:
1- $(x * y) * x = x$;
2- $(x * y) * y = (y * x) * x$;
3- $x * (y * z) = y * (x * z)$.

Proposition (1.5) [13] :
Any implication algebra is a BE -algebra.

Definition (1.6) [7] :
Let $(X; * , 1)$ be a BE-algebra and F be a non-empty subset of X. Then F is said to be a filter of X if
1- $1 \in F$;
2- $x * y \in F$ and $x \in F$ imply $y \in F$.

Example (1.7) [7] :
Let X = {1, a, b, c, d,0} be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; * , 1)$ is a BE-algebra, and $F_1 = \{1, a, b\}$ is a filter of X, but $F_2= \{1, a\}$ is not a filter of X, since $a * b \in F_2$ and $a \in F_2$, but $b \notin F_2$.

Definition (1.8) [9] :
A non-empty subset I of X is called an ideal of X if it satisfies:
1- $\forall x \in X$ and $\forall a \in I$ imply $x * a \in I$, i.e, $X * I \subseteq I$;
2- $\forall x \in X$, $\forall a, b \in I$ imply $(a * (b * x)) * x \in I$.

Lemma (1.9) [10] :
A nonempty subset I of X is an ideal of X if and only if it satisfies
1- $1 \in I$;
2- $(\forall x, y \in X) (\forall z \in I) (x * (y * z) \in I \Rightarrow x * z \in I)$.

Definition (1.10) [10] :

A BE-algebra X is said to be transitive if for any x, y, z $\in$X, $y * z \leq (x * y) * (x * z)$.

Proposition (1.11) [3] :

If X is a transitive BE-algebra, then it satisfies the following condition:

For all x, y, z $\in$ X, if $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.

Definition (1.12) [7] :

A BE-algebra X is said to be self distributive if $x * (y * z) = (x * y) * (x * z)$, for all x,y,z $\in$ X.

Example (1.13) :

Consider the BE-algebra in example (1.2),Then X is a self distributive but a BE algebra in example (1.7) is not self distributive.

Proposition (1.14) [11] :

If X is a self distributive BE-algebra, then for all x,y,z$\in$X the following statement hold:

1- if $x \leq y$, then $z * x \leq z * y$.
2- $y * z \leq (x * y) * (x * z)$.

Definition (1.15) [13] :

A BE-algebra X is called commutative if $(x * y) * y = (y * x) * x$ for any x,y $\in$ X.

Definition (1.16) [9] :

Let X be a BE-algebra and let x, y $\in$ X. Define $A(x, y) = \{z \in X \mid x * (y * z) = 1\}$

We call A(x, y) an upper set of x and y.

2.The Main Results:

In this section, we define the notion of a sub branch of element of BE-algebra, and link the notion with another notions in BE-algebra.

Definition (2.1):

Let X be a BE-algebra,we define a sub branch of element a $\in$ X is the set $R(a)=\{x\in X: a*x=1\}$.

Example (2.2):

Let X = {1, a, b, c, d} be a set with the following table:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	c	1	c	d
c	1	b	b	1	d
d	1	a	b	c	1

Then X is a BE-algebra, and

R(1)={1}, R(a)={1,a,b,c}, R(b)={1,b}, R(c)={1,c}, R(d)={1,d}.

Proposition (2.3):

Let X be a BE-algebra. Then

1- $1\in R(x)\ \forall x\in X$
2- $R(1)=\{1\}$
3- $x\in R(x)\ \forall x\in X$

Proof :

1- Since $x* 1=1 \quad \forall x\in X$ [definition (1.1)(2)]

$\Rightarrow 1\in R(x) \quad \forall x\in X$.

2- Let $x\in R(1)$ such that $x\neq 1 \Rightarrow 1* x=1$

and this contradiction since $1* x=x, \forall x\in X$. [definition (1.1)(3)]

$\Rightarrow R(1)=\{1\}$.

3- Since $x* x=1 \quad \forall x\in X$. [definition (1.1)(1)]

$\Rightarrow x\in R(x)\ \forall x\in X$.

Theorem (2.4):

Let X be a self distributive BE-algebra, Then $\forall$ $a\in X$, R(a) is a filter.

Proof :

1- $1\in R(a)$ $\forall a\in X$. [proposition (2.3)(1)]

2- Let $x*y\in R(a)$ and $x\in R(a)$

$\Rightarrow a*(x*y)=1$ and $a*x=1$,

Now, $a*y = 1*(a*y)$ [since (1*x=x) definition (1.1)(3)]

$= (a*x)*(a*y)$ [since a*x=1]

$=a*(x*y)$ [since X is a self distributive definition (1.12)]

$=1$

$\Rightarrow y\in R(a)$.

$\Rightarrow$ R(a) is a filter.

Theorem (2.5):

Let X be a self distributive BE-algebra. Then $\forall$ $a\in X$, R(a) is an ideal.

Proof :

1- Let $x\in X$ and $c\in R(a)$

$\Rightarrow x\in X$ and $a*c=1$,

Now, $a*(x*c)=(a*x)*(a*c)$ [since X is a self distributive definition (1.12)]

$=(a*x)*1=1$ [since a*c=1]

$\Rightarrow x*c\in R(a)$.

2- Let $x\in X$ and $c,d\in R(a)$

$\Rightarrow x\in X$, $a*c=1$, $a*d=1$

Now, $a*((c*(d*x))*x)$

$= a*(c*(d*x))*(a*x)$ [since X is a self distributive definition(1.12)]

$= ((a*c)*(a*(d*x))*(a*x)$ [since X is a self distributive definition (1.12)]

$= (1*(a*(d*x))*(a*x)$ [since a*c=1]

$= ((a*d)*(a*x))*(a*x)$ [since X is a self distributive def.(1.12)]

$= (1*(a*x))*(a*x)$ [since a*d=1]

$=(a*x)*(a*x)$ [since (1*x=x) def.(1.1)(3)]

$=1$ [since x*x=1]

$\Rightarrow ((c*(d*x))*x)\in R(a)$

$\Rightarrow$R(a) is an ideal.

Proposition (2.6):

Let X be a BE-algebra. Then $\forall a\in X$, R(a) is a subalgebra.

Proof :

Let $x,y\in R(a)\Rightarrow a*x=1, a*y=1$

Now,

$a*(x*y)=x*(a*y)$ [definition (1.1)(4)(exchange)]

$= x*1$ [since a*y=1]

$=1$ [definition (1.1)(2)]

$\Rightarrow x*y\in R(a)$.

Proposition (2.7):

Let X be a transitive BE-algebra. Then, if $y\in R(x)$,then $x*z\in R(y*z)$ $\forall z\in X$.

Proof :

Let $y\in R(x)$ and $z\in X$

$\Rightarrow x*y=1\Rightarrow x\leq y\Rightarrow y*z\leq x*z$ [proposition (1.11)]

$\Rightarrow(y*z)*(x*z)=1$

$\Rightarrow x*z\in R(y*z)$.

Proposition (2.8):

Let X be a transitive BE-algebra. If $y\in R(x)$,then $z*y\in R(z*x)$ $\forall z\in X$.

Proof :

Let $y \in R(x)$ and $z \in X$

$\Rightarrow x*y=1 \Rightarrow x \leq y \Rightarrow z*x \leq z*y$ [proposition (1.11)]

$\Rightarrow (z*x)*(z*y)=1$

$\Rightarrow z*y \in R(z*x)$.

Proposition (2.9):

Let X be a BE-algebra. Then

1- $y*x \in R(x)\ \forall\ x,y \in X$.

2- $(y*x)*x \in R(y)\ \forall\ x,y \in X$.

Proof :

1- Since $x*(y*x)=1$ [proposition (1.3)(1)]

$\Rightarrow y*x \in R(x)\ \forall x,y \in X$.

2- Since $y*((y*x)*x)=1$ [proposition (1.3)(2)]

$\Rightarrow (y*x)*x \in R(y)\ \forall x,y \in X$.

Proposition (2.10):

Let X be a self distributive BE-algebra. Then $(x*y)*(x*z) \in R(y*z),\ \forall\ z \in X$.

Proof :

Since X is a self distributive, then

$\forall\ x, y, z \in X,\ y*z \leq (x*y)*(x*z)$, [proposition (1.14)(2)]

$\Rightarrow (y*z)*((x*y)*(x*z))=1$,

$\therefore ((x*y)*(x*z)) \in R(y*z)$.

Proposition (2.11):

Let X be a self distributive BE-algebra. If $x \in R(a)$, then $y \in R(a),\ \forall\ y \in R(x)$.

Proof :

Let $x \in R(a),\ y \in R(x)$

$\Rightarrow a*x=1,\ x*y=1$

Now, $(a*y)=1*(a*y)$ [definition (1.1)(3)]

$= (a*x)*(a*y)$ [$a*x=1$]

$=a*(x*y)$ [since X is a self distributive definition (1.12)]

$= a*1$ [$x*y=1$]

$=1$ [definition (1.1)(2)]

$\Rightarrow a*y=1$

$\Rightarrow y \in R(a)$.

Proposition (2.12):

Let X be a BE-algebra. If $y*z \in R(x)$, then $x*z \in R(y),\ \forall\ x, y, z \in X$.

Proof :

Let $x, y, z \in X,\ y*z \in R(x)$

$\Rightarrow x*(y*z)=1$

$\Rightarrow y*(x*z)=1$ [definition (1.1)(4)]

$\Rightarrow x*z \in R(y)$.

Proposition (2.13):

Let X be a self distributive BE-algebras, if $y*z \in R(x)$, then $\forall\ x,y,z \in X$.

1- $x*z \in R(x*y)$

2- $y*z \in R(y*x)$

3-

Proof :

Let $x, y, z\in X$ and $y * z\in R(x)$,

1- since $y* z\in R(x) \Rightarrow x * (y* z)=1$

$\Rightarrow(x* y)* (x* z)=1$ [since X is a self distributive definition (1.12)]

$\Rightarrow x* z\in R(x* y)$.

2- since $y* z\in R(x)\Rightarrow x * (y* z)=1$

$\Rightarrow y * (x * z)=1$ [definition (1.1)(4)]

$\Rightarrow(y * x) * (y * z)=1$ [since X is a self distributive definition (1.12)]

$\Rightarrow y* z\in R(y* x)$.

Proposition (2.14):

Let X be a commutative BE-algebra. If $y\in R(x* y)$, then $x\in R(y * x)$, $\forall x, y\in X$.

Proof :

Let $y\in R(x * y) \Rightarrow(x * y)* y=1$

$\Rightarrow(y* x)* x=1$ [since X is a commutative definition (1.15)]

$\Rightarrow x\in R(y* x)$.

Proposition (2.15):

Let X be an implication algebra. Then

1- if $x\in R(x* y)$, then $x=1$.

2- if $y\in R(x* y)$, then $x\in R(y* x)$ $\forall x,y\in X$.

Proof :

1- Let $x\in R(x* y)\Rightarrow(x* y)* x=1$

but $(x* y)* x=x$ [definition (1.4)(1)]

$\Rightarrow x=1$.

2- Let $y\in R(x * y)\Rightarrow(x * y)* y=1$

$\Rightarrow(y * x) * x=1$ [definition (1.4)(2)]

$\Rightarrow x\in R(y * x)$.

Theorem (2.16):

Let X be a BE-algebra. Then $R(x)=A(1,x)$.

Proof :

Let $z\in R(x) \Rightarrow x * z = 1$

$\Rightarrow 1* (x* z)=1$ [definition (1.1)(3)]

$\Rightarrow z\in A(1,x)$

$\Rightarrow R(x)\subseteq A(1,x)$.

Now,

Let $z\in A(1,x) \Rightarrow 1* (x* z)=1$

$\Rightarrow x* z=1$ [definition (1.1)(3)]

$\Rightarrow z\in R(x)$

$\Rightarrow A(1,x)\subseteq R(x)$.

$\Rightarrow R(x)=A(1,x)$.

Theorem (2.17):

Let X be a self distributive BE-algebra. Then $R(x)=A(x,y)$ $\forall y\in R(x)$.

Proof :

Let $z \in R(x)$, $y \in R(x)$

$\Rightarrow x * z=1$, $x * y=1$

Now, $x * z=1$

$\Rightarrow 1 * (x * z)=1$ [definition (1.1)(3)]

$\Rightarrow (x * y) * (x * z)=1$ [$x * y=1$]

$\Rightarrow x * (y * z)=1$ [since X is a self distributive definition (1.12)]

$\Rightarrow z \in A(x,y)$

$\therefore R(x) \subseteq A(x,y)$.

Now,

Let $z \in A(x,y)$

$\Rightarrow x * (y * z)=1$

$\Rightarrow (x * y) * (x * z)=1$ [since X is a self distributive definition (1.12)]

$\Rightarrow 1 * (x * z)=1$ [$x * y=1$]

$\Rightarrow x * z=1$ [definition (1.1)(3)]

$\Rightarrow z \in R(x)$

$\Rightarrow A(x,y) \subseteq R(x)$.

$\Rightarrow R(x)=A(x,y)$.

Proposition (2.18):

Let X, Y be two BE-algebras, and f be a homomorphism's from X to Y, if $x \in R(y)$ in X, then $f(x) \in R(f(y))$ in Y.

Proof :

Let $x \in R(y) \Rightarrow y \; x=1$,

Now,

$f(y) \; f(x)=f(y \; x)=f(1)=f(x \; x)= f(x) \; f(x)=1$

$\Rightarrow f(x) \in R(f(y))$.

Proposition (2.19):

Let X, Y be two BE-algebras, and f be a monomorphism from X to Y. If $x \in X$, then $R(f(x))=\{f(c) : c \in R(x)\} \; \forall \; x \in X$.

Proof :

If $c \in R(x) \Rightarrow x \; c=1$

$\Rightarrow f(x \; c)=f(1) \Rightarrow f(x) \; f(c)=1$

$\Rightarrow f(c) \in R(f(x))$

Now, suppose $f(c) \in R(f(x))$ such that $c \notin R(x)$,

$\Rightarrow x \; c \neq 1$

$\Rightarrow f(x \; c) \neq f(1)$

$\Rightarrow f(x \; c) \neq 1$

$\Rightarrow f(x) \; f(c) \neq 1$

$\Rightarrow f(c) \notin R(f(x))$

and this contradiction!

$\Rightarrow R(f(x))=\{f(c) : c \in R(x)\} \forall \; x \in X$.

References

1- A. Rezaei, A. Borumand saeid and R. A. Borzooei, Relation between Hilbert algebras and BE-algebras, Application and Applied Mathematics Vol. 8 Iss.2 (2013) 573-584.
2- A. Walendziak, On normal filter and congruence relation in BE- algebras, Commentationes Mathematics Vol. 52 No.2 (2012) 199-205.
3- B. L. Meng, On filters in BE-algebras, Scientiae Mathematicae Japonicae 71 (2010), 201-207.
4- J. Neggers and H. S. Kim, "On d-algebras", Math. Slovaco, 49, 19-26, 1999.
5- K. Is´eki and S. Tanaka, An introduction to theory of BCK-algebras, Math. Japonica 23 (1978),1-26.
6- K. Is´eki, On BCI-algebras, Math. Seminar Notes 8 (1980), 125-130.
7- Kim H. S. and Kim Y. H., On BE-Algebras, Sci. Math. Jpn. Online e-2006, 1299-1302.
8- R. A. Borzooei and S. Khosravi Shoar, Implication algebras are eguivalent to the dual implicative BCK-algebras, Sci. Math. Jpn. 63(2006), 429-431.
9- S. S. Ahn and K. S. So, On generalized upper sets in BE-Algebras, Bull, math. Soc. 46(2009), no. 2, 281-287.
10- S. S. Ahn and K. S. So, On ideals and upper sets in BE-algebras, Sci. Math. Jpn. 68(2008), no. 2, 279-285.
11- Smarandache Florentin, Special Algebrasic Structures, 78-81, in Collected Papers, Vol. III, Abaddaba, Oradea, 2000.
12- W. A. DUDEK, On ideals in Hilbert algebras, Acta Universities Palackianae Olomuciensis 38 (1999), 31-34.
13- Walendziak A., On commutative BE-algebras, Sci. Math. Jpn. online e-(2008), 585-588.