



The Orthogonality of Martingale in Birkhoff's sense and others

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ABSTRACT

Orthogonality is one of an important the concepts in Mathematics , therefor it will be discussed in this paper, the orthogonality of martingale according to Birkhoff's, Roberts's, Singer's, Carlsson's sense for orthogonality and the conditions that are needed to have orthogonality.

1. Introduction

Let $\{X_1, X_2, \dots\}$ be a sequence of integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $F_n \subseteq F_{n+1}$ an increasing sequence of sub σ -field of $\mathcal{F}$, X_n is F_n -measurable that is $X_n: (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. $\{X_n, F_n\}$ is said to be martingale iff $\forall n, E[X_{n+1} | F_n] = X_n$ a.e.[1]

If $E(|Y|^p) < \infty$ It's said $Y \in L^p(\Omega, \mathcal{F}, P)$ for $P \in [1, \infty)$ and the norm defined as :

$$\|Y\|_p = \left\{ E(|Y|^p) \right\}^{\frac{1}{p}}. [2]$$

James was the first one Who studied Birkhoff's properties of orthogonality, therefor this orthogonality is called Birkhoff- James.[3]

Ash proved that the martingale difference is orthogonal in a Hilbert space $L^2(\Omega, \mathcal{F}, P)$ by usual orthogonality[1].

In 1934 Roberts introduced his orthogonality as Roberts's orthogonality and in 1935 Birkhoff introduced his orthogonality as Birkhoff's orthogonality, which was one of the most important orthogonality senses in normed space. [4]

Singer's orthogonality was introduced by singer in 1957. Therefor, the orthogonality of martingale will

be discussed in normed space according to these senses.[5]

A functional is a mapping g of an element in a normed space $L^p(\Omega, \mathcal{F}, P)$ to an element in its scalar field.

A functional is called linear when it satisfies,

$$g(aX + bY) = ag(X) + bg(Y)$$

for every scalars and $X, Y \in L^p(\Omega, \mathcal{F}, P)$. [6]

We say that functional g is bounded functional if

there exist scalar $k \geq 0$ such that

$$|g(X)| \leq K \|X\|_p$$

for every $X \in L^p(\Omega, \mathcal{F}, P)$

The space of all bounded linear functionals on $L^p(\Omega, \mathcal{F}, P)$ is called the dual space of $L^p(\Omega, \mathcal{F}, P)$ and denoted by L^* . [6]

2- Main results

Definition(2.1)[7]

In a vector normed space $L^p(\Omega, \mathcal{F}, P)$, Z is called Birkhoff – James orthogonal to W and denoted by $Z \perp_B W$ if

$$\|Z + aW\|_p \geq \|Z\|_p$$

for any real number a .

Theorem (2.2)[7]

Let X and Y belongs to a normed space $L^p(\Omega, \mathcal{F}, P)$, then $X \perp_B Y$ if and only if there exist $g \in L^p \setminus \{0\}$ such that $|g(X)| = \|g\|_p \|X\|_p$.

Lemma(2.3)

Let $E : L^p(\Omega, \mathcal{F}, P) \rightarrow \bar{R}$ be a conditional expectation, then E is bounded linear operator.

Proof:

Since $E(aX + bY | \mathcal{F}) = aE(X | \mathcal{F}) + bE(Y | \mathcal{F})$ then E is linear operator and

$\|E(X | \mathcal{F})\|_p \leq \|X\|_p$ that is E is bounded by $K = 1$.

Theorem(2.4)

Let $X_n \in L^p(\Omega, \mathcal{F}, P)$ and $(X_n, \mathcal{F}_n)$ be a zero – mean martingale then $X_n \perp_B X_{n+1}$.

Proof:

If $g(X_n) = E(X_n | \mathcal{F}_n)$, we define

$$\begin{aligned} |g(X_n)| &= \|g(X_n)\|_p \\ |g(X_n)| &= \|E(X_n | \mathcal{F}_n)\|_p \\ &= \|X_n E(1 | \mathcal{F}_n)\|_p \quad (\text{since } X_n \text{ is } \mathcal{F}_n \text{-measurable}) \\ &= |E(1 | \mathcal{F}_n)| \|X_n\|_p \quad (\text{by property of norm}) \\ &= \|E(1 | \mathcal{F}_n)\|_p \|X_n\|_p \\ &= \|E\|_p \|X_n\|_p \\ &= \|g\|_p \|X_n\|_p. \end{aligned}$$

Since $|g(X_n)| = \|g\|_p \|X_n\|_p$ and $E(X_{n+1} | \mathcal{F}_n) = 0$,

then (by (2.2)) $\sum_{i=1}^m \alpha_i \|\beta_i X + \gamma_i Y\|_p^2 = 0$

$$X_n \perp_B X_{n+1}$$

Definition(2.5)[8]

In a normed space $L^p(\Omega, \mathcal{F}, P)$, Z is said to be singer orthogonal to W and denoted by $Z \perp_S W$ if either $\|Z\|_p \|W\|_p = 0$ or $\|Z + W\|_p = \|Z - W\|_p$.

Theorem(2.6)

If $E(|X_n|^p) < \infty$ for all n , X_n is a martingale and independent if $E(X_{n+1}^p | \mathcal{F}_n) = 0 \forall P \in [\mathcal{A}]$, then

$$X_n \perp_S X_{n+1}.$$

Proof:

$$\begin{aligned} \|X_n\|_p^p \|X_{n+1}\|_p^p &= E(|X_n|^p) E(|X_{n+1}|^p) \\ &= E[X_n^p X_{n+1}^p] \quad (\text{by independence}) \\ &= E[E(X_n^p X_{n+1}^p | \mathcal{F}_n)] \\ &= E[X_n^p E(X_{n+1}^p | \mathcal{F}_n)] \\ &= 0 \quad (\text{Since } X_{n+1}^p \text{ is } \mathcal{F}_n \text{-measurable}). \end{aligned}$$

$$= 0$$

Therefore $\|X_n\|_p \|X_{n+1}\|_p = 0$

Hence $X_n \perp_S X_{n+1}$.

Definition(2.7)[4]

In a normed space $L^p(\Omega, \mathcal{F}, P)$, Z is said to be Roberts orthogonal to W and denoted by $Z \perp_R W$ if the equality

$$\|Z + \alpha W\|_p = \|Z - \alpha W\|_p$$

holds for any real number α .

Theorem(2.8)

If X_n belong to a Hilbert space $L^2(\Omega, \mathcal{F}, P)$ and $(X_n, \mathcal{F}_n)$ is a martingale then martingale differences are orthogonale in the sense of Roberts.

Proof:

$$\begin{aligned} &\|(X_m - X_{m-1}) + \alpha(X_n - X_{n-1})\|_2^2 \\ &= E[(X_m - X_{m-1}) + \alpha(X_n - X_{n-1})]^2 \\ &= E[(X_m - X_{m-1})^2 + 2\alpha(X_m - X_{m-1})(X_n - X_{n-1}) + \alpha^2(X_n - X_{n-1})^2] \\ &= E(X_m - X_{m-1})^2 + 2\alpha E[(X_m - X_{m-1})(X_n - X_{n-1})] + \alpha^2 E(X_n - X_{n-1})^2 \\ &= E(X_m - X_{m-1})^2 + 2\alpha E[E[(X_m - X_{m-1})(X_n - X_{n-1}) | \mathcal{F}_n]] + \alpha^2 E(X_n - X_{n-1})^2 \\ &= E(X_m - X_{m-1})^2 + 2\alpha E[(X_m - X_{m-1})E[(X_n - X_{n-1}) | \mathcal{F}_n]] + \alpha^2 E(X_n - X_{n-1})^2 \end{aligned}$$

Since $X_m - X_{m-1}$ is $\mathcal{F}_n$ -measurable.

$$E[(X_n - X_{n-1}) | \mathcal{F}_n] = X_n - X_{n-1} = 0$$

(by property of martingale)

$$\begin{aligned} &= E(X_m - X_{m-1})^2 + \alpha^2 E(X_n - X_{n-1})^2 \\ &= \|X_m - X_{m-1}\|_2^2 + \alpha^2 \|X_n - X_{n-1}\|_2^2 \quad \dots(1) \end{aligned}$$

Similarity

$$\begin{aligned} &\|(X_m - X_{m-1}) - \alpha(X_n - X_{n-1})\|_2^2 \\ &= \|X_m - X_{m-1}\|_2^2 + \alpha^2 \|X_n - X_{n-1}\|_2^2 \quad \dots(2) \end{aligned}$$

From (1) and (2) we have,

$$\begin{aligned} &\|(X_m - X_{m-1}) + \alpha(X_n - X_{n-1})\|_2 \\ &= \|(X_m - X_{m-1}) - \alpha(X_n - X_{n-1})\|_2 \\ &(X_m - X_{m-1}) \perp_R (X_n - X_{n-1}). \end{aligned}$$

Definition(2.9)[9]

Let X, Y belong to a normed space $L^p(\Omega, \mathcal{F}, P)$, and m be a positive integer. Then X is said to be orthogonal in the sense of Carlsson to Y and denoted by

$$X \perp_C Y \text{ if and only if } \sum_{i=1}^m \alpha_i \|\beta_i X + \gamma_i Y\|_p^2 = 0.$$

Where $\alpha_i, \beta_i, \gamma_i$ are real number such that

$$\sum_{i=1}^m \alpha_i \beta_i^2 = \sum_{i=1}^m \alpha_i \gamma_i^2 = 0, \text{ and } \sum_{i=1}^m \alpha_i \beta_i \gamma_i = 1.$$

Theorem(2.10)

Let $E(|X_n|^p) < \infty$ and X_n be a martingale such that $E(X_{n+1}^{p-K} | \mathcal{F}_n) = 0, \forall P \in [1, \infty)$ and $K = 0, 1, \dots, P$ then $X_{n+1} \perp_C X_n$.

Proof:

$$\begin{aligned} \sum_{i=1}^m \alpha_i \|\beta_i X_{n+1} + \gamma_i X_n\|_p^2 &= \sum_{i=1}^m \alpha_i \left\{ E(\beta_i X_{n+1} + \gamma_i X_n)^p \right\}^{\frac{1}{p}} \Bigg\}^2 \\ &= \sum_{i=1}^m \alpha_i \left\{ E \left[\sum_{K=0}^P \binom{P}{K} (\beta_i X_{n+1})^{p-K} (\gamma_i X_n)^K \right] \right\}^{\frac{2}{p}} \end{aligned}$$

(by binomial theorem)

$$= \sum_{i=1}^m \alpha_i \left\{ \sum_{K=0}^P \binom{P}{K} \beta_i^{P-K} \gamma_i^K E [X_{n+1}^{P-K} X_n^K] \right\}^{\frac{2}{P}}$$

$$= \sum_{i=1}^m \alpha_i \left\{ \sum_{K=0}^P \binom{P}{K} \beta_i^{P-K} \gamma_i^K E (E [X_{n+1}^{P-K} X_n^K | \mathcal{F}_n]) \right\}^{\frac{2}{P}}$$

$$= \sum_{i=1}^m \alpha_i \left\{ \sum_{K=0}^P \binom{P}{K} \beta_i^{P-K} \gamma_i^K E (X_n^K E [X_{n+1}^{P-K} | \mathcal{F}_n]) \right\}^{\frac{2}{P}}$$

$$= 0$$

Since X_n^K is $\mathcal{F}_n$ - measurable.Hence $X_{n+1} \perp_C X_n$.

References

- [1] Ash, R. B. (1972). Real Analysis and Probability. Academic press, INC, New York.
- [2] Williams, D. (1961). Probability with Martingales, Cambridge University Press.
- [3] Ojha, B. P. (2016). Some New Types of Orthogonalities in Normed Spaces and Application in Best Approximation. *Journal of Advanced College of Engineering and Management*, 2: 1-4.
- [4] Roberts, B. D. (1934). On the geometry of abstract vector spaces. *Tohoku Mathematical Journal, First Series*, 39, 42-59.
- [5] Kanu, R. U. and Rauf, K. (2014). On Some Results on Linear Orthogonality Spaces. *Asian Journal of Mathematics and Applications*, 14: 155-166.
- [6] Taylor, A.E. (1958). Introduction To Functional Analysis, New York.
- [7] Birkhoff, G. (1935). Orthogonality in linear metric spaces. *Duke Mathematical Journal*, 1(2): 169-172.
- [8] Singer, I. (1957). Angles abstraits et fonctions trigonométriques dans les espaces de Banach. *Acad. RP Romine. Bui. § ti. Sec}. § ti. Mat. Fiz*, 9, 29-42.
- [9] Carlsson, S. O. (1962). Orthogonality in normed linear spaces. *Arkiv för Matematik*, 4(4): 297-318.

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الملخص

التعامد من المفاهيم المهمة في الرياضيات، لذلك نوقش في هذا البحث تعامد المارتينكل حسب مفهوم بيرخوف، روبرتس، سنكر، وكارلسون للتعامد وماهي الشروط الواجب توفرها للحصول على التعامد.