

PAIRWISE SEPARATION AXIOMS IN INTUITIONISTIC TOPOLOGICAL SPACES

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Abstract

The concept of intuitionistic topological space was introduced by Çoker .The aim of this paper is to generalize bitopological notions of separation to intuitionistic topological spaces .

الملخص:

في هذا البحث قدمنا تعريف بديهيات الفصل الزوجية T_3, T_4 وتعريف R_3, R_4 للفضاء التوبولوجي الثنائي وناقشنا العلاقة بين فضاء التوبولوجي المضاعف وفضاء التوبولوجي الثنائي من خلال تعميم هذه التعاريف .

1-Introduction

After the introduction of fuzzy sets by Adamek [1] , various mathematicians introduced generalization of the notion of fuzzy set . Among others , Atanassov [2,3] introduced the notion of intuitionistic fuzzy set. Chang [7] used fuzzy sets to introduce the concept of a fuzzy topology . Later this concept was extended to intuitionistic fuzzy topological spaces by Çoker in [8]. The concept of intuitionistic set, which is a classical version of an intuitionistic fuzzy set, was first given by Çoker in [9] . He studied topology on intuitionistic sets in [10]. In this context, Çoker *et. al* .[9,13] studied continuity, connectedness, compactness and separation axioms in intuitionistic fuzzy topological spaces. In this paper we follow the suggestion of J.G. Garcia and S.E. Rodabaugh that (double fuzzy set) is a more appropriate name than (intuitionistic fuzzy set) , and therefore adopt the term (double-set) for the intuitionistic set , and (double-topology) for the intuitionistic topology of Dogan Çoker , (this issue) we denote by **Dbl-Top** the construct (concrete texture over Set) whose objects are pairs (X, τ) where τ is a double-topology on X .In Section 4 we discuss the relations between these bitopologically based on concepts and some previously defined separation axioms for double-topological spaces.

2-Preliminaries

Throughout the paper by X we denote a non-empty set . In this section we shall present various fundamental definitions and propositions. The following definition is obviously inspired by Atanassov [3].

2.1.Definition. [8] A double-set (DS in brief) A is an object having the form $A = \langle X, A_1, A_2 \rangle$, Where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \emptyset$. The set A_1 is called the set of members of A , while A_2 is called the set of non- members of A .
theroughout the remainder of this paper we use the simpler $A = (A_1, A_2)$ for a double-set.

2.2.Remark. Every subset A of X is may obviously be regarded as a double-set having the form $A = (A, A^c)$,
where $A^c = X - A$ is the complement of A in X .
we recall several relations and operations between DS's as follows:

2.3.Definition. [8] Let the DS's A and B on X be the form $A = (A_1, A_2)$,
 $B = (B_1, B_2)$, respectively . Furthermore, let $\{A_j : j \in J\}$ be an arbitrary family of DS's in X, where $A_j = (A_j^{(1)}, A_j^{(2)})$. Then

- (a) $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;
- (b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;
- (c) $\bar{A} = (A_2, A_1)$ denotes the complement of A ;
- (d) $\cap A_j = (\cap A_j^{(1)}, \cup A_j^{(2)})$;
- (e) $\cup A_j = (\cup A_j^{(1)}, \cap A_j^{(2)})$;
- (f) $\square A = (A_1, A_1^c)$;
- (g) $\langle \rangle A = (A_2^c, A_2)$;
- (h) $\phi = (\phi, X)$ and $X = (X, \phi)$.

In this paper we require the following :

- (i) $()A = (A_1, \phi)$ and $(A) = (\phi, A_2)$.

Now we recall the image and preimage of DS' s under a function .

2.4.Definition. [8,13] Let $x \in X$ be a fixed element in X . Then:

- (a) The DS given by $\tilde{x} = (\{x\}, \{x\}^c)$ is called a double-point (DP in brief X) .
- (b) The DS $\tilde{x} = (\phi, \{x\}^c)$ is called a vanishing double-point (VDP in brief X) .

2.5.Definition. [8,13]

- (a) Let $\tilde{x}$ be a DP in X and $A = (A_1, A_2)$ be a DS in X . Then $\tilde{x} \in A$ iff $x \in A_1$.
- (b) Let $\tilde{x}$ be a VDP in X and $A = (A_1, A_2)$ a DS in X . Then $\tilde{x} \in A$ iff $x \notin A_2$.

It is clear that $\tilde{x} \in A \Leftrightarrow \tilde{x} \subseteq A$ and that $\tilde{x} \in A \Leftrightarrow \tilde{x} \subseteq A$.

2.6.Definition. [10] A double-topology (DT in brief) on a set X is a family τ of DS's in X satisfying the following axioms :

- T1:** $\phi, X \in \tau$,
- T2:** $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,
- T3:** $\bigcup G_j \in \tau$ for any arbitrary family $\{G_j : j \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called a double-topological space (DTS in brief), and any DS in τ is known as a double open set (DOS in brief). The complement $\bar{A}$ of a DOS A in a DTS is called a double closed set (DCS in brief) in X .

2.7.Definition. [10] Let (X, τ) be an DTS and $A = (A_1, A_2)$ be a DS in X .

Then the interior and closure of A are defined by :

$$\text{int}(A) = \bigcup \{G : G \text{ is a DOS in } X \text{ and } G \subseteq A\},$$

$$\text{cl}(A) = \bigcap \{H : H \text{ is a DCS in } X \text{ and } A \subseteq H\},$$

respectively .

It is clear that $\text{cl}(A)$ is a DCS in and $\text{int}(A)$ a DOS in X . Moreover , A is a DCS in X iff $\text{cl}(A) = A$, and A is a DOS in X iff $\text{int}(A) = A$.

2.8. Example. [10] Any topological space (X, τ_0) gives rise to a DT of the form

$\tau = \{A' : A \in \tau_0\}$ by identifying a subset A in X with its counterpart $A' = (A, A^c)$, as in Remark 2.2.

3- The Constructs Dbl-Top and Bitop :

We begin recalling the following result which associates a bitopology with a double topology.

3.1.Proposition. [10] Let (X, τ) be a DTS.

- (a) $\tau_1 = \{A_1 : \exists A_2 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ is a topology on X .

(b) $\tau_2^* = \{A_2 : \exists A_1 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ is the family of closed sets of the topology $\tau_2 = \{A_2^c : \exists A_1 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ on X .

(c) Using (a) and (b) we may conclude that (X, τ_1, τ_2) is a bitopological space.

3.2.Proposition. Let (X, u, v) be a bitopological space. Then the family

$$\{(U, V^c) : U \in u, V \in v, U \subseteq V\}$$

is a double topology on X .

Proof . The condition $U \subseteq V$ ensures that $U \cap V^c = \emptyset$, while the given family contains $\emptyset$ because

$\emptyset \in u, v$, and it contains X because $X \in u, v$. Finally this family is closed under finite intersections and arbitrary unions by Definition 2.3 (d,e) and the corresponding properties of the topologies u and v .

3.3. Definition. Let (X, u, v) be a bitopological space. Then we set

$$\tau_{uv} = \{(U, V^c) : U \in u, V \in v, U \subseteq V\}$$

and call this the double topology on X associated with (X, u, v) .

3.4.Proposition. If (X, u, v) is a bitopological space and τ_{uv} the corresponding DT on X , then

$$(\tau_{uv})_1 = u \text{ and } (\tau_{uv})_2 = v.$$

Proof. $U \in u$ implies $(U, \emptyset) \in \tau_{uv}$ since $U \subseteq X \in v$, so $U \subseteq (\tau_{uv})_1$. Conversely, take $U \in (\tau_{uv})_1$. Then $(U, B) \in \tau_{uv}$ for some $B \subseteq X$, and now $U \in u$. Hence $(\tau_{uv})_1 \subseteq u$, and the first equality is proved.

□

The proof of the second equality may be obtained in a similar way, and we omit the details.

4. Pairwise Separation Axioms in Double Topological Space .

In this section we use the link between bitopological spaces and double topological spaces established above to generalize several bitopological separation properties to the case of double topological spaces. We limit ourselves to the strong forms of the pairwise T_3 , T_4 properties and we will also consider the pairwise R_3 and R_4 axioms.

The (strong) T_0 and T_1 axioms are due to Fletcher, Hoyle and Patty [11], pairwise T_2 due to Kelly [12]:

4.1. Definition. [12] In a bitopological space (X, u, v) , u is said to be T_3 with respect to v if $\forall DCS A \in u, \forall a \in \text{int } A \text{ in } X, \exists B \in u, D \in v \text{ and } a \in B,$

$A \subseteq D \text{ and } B \cap D = \emptyset.$ (X, u, v) is said to be pairwise T_3 , if u is T_3 with respect to v and v is T_3 with respect to u .

4.2. Proposition. If (X, u, v) is pairwise T_3 then (X, τ_{uv}) satisfies the following condition:

Given $DCS A \subseteq X, A \in u, a \in \text{int } A$ there exists $G, H \in \tau_{uv}$ with $a \in G, a \notin H,$

$$A \subseteq H \text{ and } G \subseteq \overline{(\)H}.$$

Proof . Let (X, u, v) be pairwise T_3 and $DCS A \subseteq X$, such that $A \in u, a \in \text{int } A$, then we have $B \in u$ and $D \in v$ and $a \in B, A \subseteq D$ and $B \cap D = \emptyset$ that is $B \subseteq D^c$.

Now defining $G = (B, \emptyset), H = (\emptyset, D^c)$ gives $G, H \in \tau_{uv}, a \in G$ and $a \notin H$

$$\text{then } a \notin H, A \not\subseteq D^c \text{ so } A \subseteq H \text{ and } G = (B, \emptyset) \subseteq \overline{(\)H} = (D^c, \emptyset).$$

□

4.3. Definition . The DTS (X, τ) is called pairwise T_3 if $\forall DCS A \in \tau, a \in \text{int } A \text{ in } X$ there exists $G, H \in \tau$ satisfying $a \in G, a \notin H$

$$, A \subseteq H \text{ and } G \subseteq \overline{(\)H}.$$

4.4. Proposition. If (X, τ) is pairwise T_3 then (X, τ_1, τ_2) is pairwise T_3 .

Proof . Let A be a DCS in $X, a \in \text{int } A, \exists G = (B, C), H = (D, F) \in \tau$ with

$$a \in G, a \notin H, A \subseteq H \text{ and } G \subseteq \overline{()H}. \text{ clearly } a \in B \in \tau_1, \text{ from } G \subseteq \overline{()H} = (F, \phi)$$

, $B \subseteq F$ and since $a \notin H$ then $a \in F \Rightarrow a \notin F^c \in \tau_2$ and $A \subseteq H$ where $A = (A_1, A_2)$

then $A_1 \subseteq D, F \subseteq A_2$, let $F^c = D$ then $A_1 \subseteq F^c$ and $B \cap F^c = \phi$, we prove that

τ_1 is T_3 with respect to τ_2 , in the same way we prove τ_2 is T_3 with respect to τ_1 , then

(X, τ_1, τ_2) is pairwise T_3 . $\square$

4.5. Corollary. (X, u, v) is pairwise T_3 iff (X, τ_{uv}) is pairwise T_3 .

Proof . Necessity follows from Proposition 4.2 and sufficiency from proposition 4.4 and proposition 3.4. $\square$

4.6. Definition. In a bitopological space (X, u, v) , u is said to be R_3 with respect to v , if for every DCS $A \in u, a \in \text{int } A$ in X and $a \notin cl_u(A), A \not\subseteq cl_v(\{a\})$ then

$\exists B \in u, D \in v$ such that $a \in B, A \subseteq D, B \cap D = \phi$. (X, u, v) is pairwise R_3 , if u is R_3 with respect to v and v is R_3 with respect to u .

4.7. Proposition. If $(X, u, v) R_3$ then (X, τ_{uv}) satisfies the condition : $\forall$ DCS $A, a \in \text{int } A$ in X , if there exists $M \in \tau_{uv}$ with $a \in M, A \not\subseteq M (A \subseteq M, a \in M)$ then there exists

$$G, H \in \tau_{uv} \text{ such that } a \in G, A \subseteq H \text{ and } G \subseteq \overline{()H}.$$

Proof . If $a \in M, A \not\subseteq M$ then $M = (P, Q), a \in P \in u, A = (A_1, A_2) \not\subseteq M,$

$A_1 \not\subseteq P$ so $a \notin cl_u(A)$ likewise, $A \subseteq M, a \in M \Rightarrow A_1 \subseteq P$ and $a \notin Q \in v$ so

$A \not\subseteq cl_v(\{a\})$ so in either case we have $B \in u, D \in v$ satisfying

$a \in B, A \subseteq D$ and $B \cap D = \phi$ that is $B \subseteq D^c$. Now defining

$$G = (B, \phi), H = (\phi, D^c) \text{ gives } G, H \in \tau_{uv}, a \in G, A \not\subseteq D^c \Rightarrow A \subseteq H, \text{ and } G \subseteq \overline{()H}.$$

$\square$

4.8. Definition. The DTS (X, τ) is called pairwise R_3 , if it satisfies the condition :

$\forall$ DCS $A, a \in \text{int } A$ in X , if there exists $M \in \tau$ with $a \in M,$

$A \not\subseteq M (A \subseteq M, a \in M)$ then there exists $G, H \in \tau$ such that $a \in G, A \subseteq H$ and $G \subseteq \overline{()H}$. **4.9.**

Proposition. If (X, τ) is pairwise R_3 then (X, τ_1, τ_2) pairwise R_3 .

Proof . First to prove τ_1 is R_3 with respect to τ_2 take a DCS $A, a \in \text{int } A$ in X with $a \notin cl_{\tau_1}(A)$.

Now we have $a \in P \in \tau_1$ with $A_1 \not\subseteq P, A = (A_1, A_2)$ so for some $G \subseteq X$ we have

$(P, Q) \in \tau$. sitting $M = (P, Q)$ gives $a \in M, A \not\subseteq M$ so we have $G, H \in \tau$ satisfying

$$a \in G, A \subseteq H \text{ and } G \subseteq \overline{()H}. \text{ By a similar argument } G, H \text{ satisfying the same conditions also be}$$

obtain if $A \not\subseteq cl_{\tau_2}(\{a\})$ the existence of τ_1, τ_2 open sets separating a and A as follow ,

let $G = (B, C), H = (D, F)$ with $a \in G, A \subseteq H, A = (A_1, A_2)$ clearly $a \in B \in \tau_1$

$A \subseteq H$ then $A_1 \subseteq D, F \subseteq A_2$ so $A_1 \not\subseteq F$

$$\Rightarrow A_1 \subseteq F^c \in \tau_2 \text{ while } (B, C) \subseteq \overline{()H} = (F, \phi)$$

$\Rightarrow B \subseteq F$ so that $C \cap F^c = \phi$. in a same way we prove that τ_2 is R_3 with respect to τ_1 then

(X, τ_1, τ_2) is pair wise R_3 . $\square$

4.11. Corollary . (X, u, v) is pairwise R_3 iff (X, τ_{uv}) is pairwise R_3 .

Proof . Necessity follows from proposition 4.7 and sufficiency from proposition 4.9 and proposition 3.4 .

4.12. Definition. The bitopological space (X, u, v) is called pairwise T_4 if for every two DCS A and B in X

and $A \cap B = \phi$ then there exist $G \in u, H \in v$ and $A \subseteq G, B \subseteq H, G \cap H = \phi$.

4.13. Proposition. If (X, u, v) is pairwise T_4 then (X, τ_{uv}) has the property: Given A, B as DCS's in X , there exists $N, M \in \tau_{uv}$ with $A \subseteq N, B \subseteq M$ and $N \subseteq \overline{M}$.

Proof . Take A, B DCS in X , s.t $A \cap B = \phi$ and $G \in u, H \in v$ satisfying

$A \subseteq G, B \subseteq H$ and $G \cap H = \phi$ that is $G \subseteq H^c$ then $A \subseteq H^c$.

Now defining $N = (G, \phi), M = (\phi, H^c)$ gives $N, M \in \tau_{uv}$, $A \subseteq N, B \subseteq H$ then

$B \subseteq H$ so that $B \not\subseteq H^c \Rightarrow B \subseteq M$ and $N \subseteq \overline{M}$. □

4.14. Definition.[12] The DTS (X, τ) is called pairwise T_4 if

$\forall$ DCS A, B in X there exists $N, M \in \tau$ satisfying $A \subseteq N, B \subseteq M$ and $N \subseteq \overline{M}$.

4.15. Proposition. If (X, τ) is pairwise T_4 then (X, τ_1, τ_2) is pairwise T_4 .

Proof . $\forall$ DCS A, B in X , take $N = (C, D), M = (G, H) \in \tau$ with

$A \subseteq N \in \tau_1, B \subseteq M$ and $N \subseteq \overline{M}$. clearly

$A_1 \subseteq C \in \tau_1$ for $A = (A_1, A_2)$ and $B \subseteq M$ then $B \not\subseteq \overline{M}$ and $B_1 \not\subseteq H$

for $B = (B_1, B_2)$ then $B_1 \subseteq H^c \in \tau_2$ and

$N \subseteq \overline{M}$ then $(C, D) \subseteq (H, \phi) \Rightarrow C \subseteq H$ so $C \cap H^c = \phi$. □

4.16. Corollary. (X, u, v) is pairwise T_4 iff (X, τ_{uv}) is pairwise T_4 .

Proof . Necessity follows from proposition 4.13 and sufficiency from proposition 4.15 and proposition 3.4.

4.17. definition. The bitopological space (X, u, v) is called pairwise R_4 if

$\forall$ DCS A, B in $X, A \cap B = \phi, A \not\subseteq cl_u(B), B \not\subseteq cl_v(A)$ implies, there exists $G \in u, H \in v$ with $G \cap H = \phi$ and $A \subseteq G, B \subseteq H$.

4.18. Proposition . If (X, u, v) is pairwise R_4 then (X, τ_{uv}) satisfies the condition :

$\forall$ DCS A, B in X

$A \cap B = \phi$, if there exists $M \in \tau_{uv}$ with $A \subseteq M, B \not\subseteq M (A \not\subseteq M, B \subseteq M)$

Then there exists $G, H \in \tau_{uv}$ s.t $A \subseteq G, B \subseteq H$ and $G \subseteq \overline{H}$.

Proof . If we have $M \in \tau_{uv}$ with

$A \subseteq M$ & $B \not\subseteq M$ then $M = (P, Q), A = (A_1, A_2) \subseteq M, A_1 \subseteq P \in u, Q \subseteq A_2,$

$B = (B_1, B_2) \not\subseteq M, B_1 \not\subseteq P, Q \not\subseteq B_2$ so $A \not\subseteq cl_u(B)$. Likewise

$B \subseteq M, A \not\subseteq M, B_1 \subseteq P$ & $A_1 \not\subseteq P$ so $B \not\subseteq cl_v(A)$, so in either case we have

$N_1 \in u, N_2 \in v$ satisfying $A \subseteq N_1, B \subseteq N_2$ & $N_1 \cap N_2 = \phi$ that is $N_1 \subseteq N_2^c$

Now defining $G = (N_1, \phi), H = (\phi, N_2^c)$ gives $G, H \in \tau_{uv}$, $A \subseteq G, B \subseteq H$

and $G \subseteq \overline{H}$. □

4.19. Definition. The DTS (X, τ) is called pairwise R_4 if it satisfies the condition: $\forall$ DCS A, B in X

$A \cap B = \phi$, if there exists $M \in \tau_{uv}$ with $A \subseteq M, B \not\subseteq M (A \not\subseteq M, B \subseteq M)$

Then there exists $G, H \in \tau_{uv}$ s.t $A \subseteq G, B \subseteq H$ and $G \subseteq (\overline{H})$.

4.20.Propositon. If (X, τ) is pairwise R_4 then (X, τ_1, τ_2) is pairwise R_4 .

Proof . First take two DCS A and B in X with $A \not\subseteq cl_{\tau_1}(B)$. Now we have

$A_1 \subseteq P \in \tau_1$ for $A = (A_1, A_2)$ so for some $Q \subseteq X$ we have $(P, Q) \in \tau$

Sitting $M = (P, Q)$ gives $A \subseteq M, B \not\subseteq M$ so we have $G, H \in \tau$ satisfying

$A \subseteq G, B \subseteq H$ and $G \subseteq (\overline{H})$. By a similar argument G, H satisfying the same conditions may also be obtain if $B \not\subseteq cl_{\tau_2}(A)$ existence of τ_1, τ_2 open

Sets . Separating A and B as follow

Let $G = (D, C), H = (F, L)$ with $A \subseteq G, B \subseteq H$ clearly $A_1 \subseteq D \in \tau_1, B \subseteq H$ so that

$B_1 \subseteq F, L \subseteq B_2$ then $B_1 \not\subseteq L$ so $B_1 \subseteq L^c \in \tau_2$ while $(D, C) \subseteq (\overline{H}) = (L, \phi)$

Then $D \subseteq L$ and $D \cap L^c = \phi$.

□

4.21.Corollary. (X, u, v) is pairwise R_4 iff (X, τ_{uv}) is pairwise R_4 .

Proof . Necessity follows from proposition 4.18 and sufficiency from proposition 4.20 and proposition 3.4.

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