



Chromatic Number and some Properties of Pseudo-Von Neumann Regular graph of Cartesian Product of Rings

Nermen J. Khalel, Nabeel E. Arif

Department of Mathematic, College of Computer Science and Mathematics, Tikrit, Tikrit University, Iraq

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Corresponding Author:

Name: Nermen J. Khalel

E-mail: nrnjamal88@gmail.com

Tel:

ABSTRACT

Let R be a commutative ring, the Pseudo – Von Neumann regular graph of the ring R is define as a graph whose vertex set consists of all elements of R and any two distinct vertices a and b are adjacent if and only if $a = a^2b$ or $b = b^2a$, this graph is denoted by $P-VG(R)$, in this work we got some new results about chromatic number of Pseudo-Von Neumann regular graph of cartesian product of rings.

1- Introduction

Beck [1] studied coloring of commutative rings and studied chromatic number of it is graph such that two different elements x and y are adjacent iff $xy = 0$, Bhavanari et.al. studied prime graph of a ring with some properties of its graph [2], and he studied cartesian product of prime graph with Srinivasulu [3] Kalita [4] computed chromatic number of prime graph of some finite ring, Patra k. et.al [5]. found chromatic number of prime graph of some rings of Z_n , where $n = \prod_{i=1}^r p_i^{\alpha_i}$, Elizabeth [6] studied colorings of zero divisor graphs of commutative rings. The study obtains the chromatic number of Pseudo-Von Neumann regular graph of cartesian product of rings.

2- Primer lay

Definition 2.1: let R be a ring and $a \in R$, a is called regular element if there exist $b \in R$ such that $a = aba$, if any element in R is regular then R is regular ring, if R is commutative then $a = a^2b$ and we say that R is Von Neumann regular ring.

Definition 2.2: A graph G is defined by an ordered pair $(V(G), E(G))$, when $V(G)$ is a non empty set whose elements are called vertices and $E(G)$ is a set (may be empty) of unordered pairs of distinct vertices of $V(G)$. the element of $E(G)$ are called edges of the graph G . we denote by $\overline{uv}$, an edge between two end vertices u and v .

Definition 2.3: A simple graph that has no loops or multiple edges.

Definition 2.4: A graph H is said to be a subgraph of a graph G if all the edges and all the vertices of H are in G , and it is denoted by $H \subset G$.

Definition 2.5: A path is a graph G that contains a list $v_1, v_2, \dots, v_n$ of vertices of G s.t. for $1 \leq i \leq p-1$, there is an edge $\overline{v_i v_{i+1}}$ in G and these are the only edges in G .

Definition 2.6: let v_1 and v_2 be two vertices, $d(v_1, v_2)$ is called the distance from v_1 to v_2 if it is the shortest path from v_1 to v_2 .

Definition 2.7: A close path is called clyce, the degree of each vertex of a cycle graph is two, a cycle with n vertices denoted by C_n .

Definition 2.8: Let $G(V, E)$ be a graph and $C \subset G$, is called clique if the induced subgraph of G induced by C is a complete graph.

The clique is called maximal if there is no clique with more vertices.

Theorem 2.9 :

For circular graph C_n one has

$$\chi(C_n) = \begin{cases} 2 & \text{when } n \text{ is even} \\ 3 & \text{when } n \text{ is odd} \end{cases}$$

Definition 2.10: A h -coloring of the vertex set of a graph G is a function $\gamma: V(G) \rightarrow \{1, 2, \dots, h\}$ such that $\gamma(v_1) \neq \gamma(v_2)$ whenever v_1 is adjacent to v_2 , if a h -coloring of G exists, then G is called h -colorable.

Definition 2.11: The chromatic number of G is defined as $\chi(G) = \min \{h : G \text{ is } h\text{-colorable}\}$ where $\chi(G) = h$, G is called h -chromatic.

Definition 2.12: The Cartesian product $G \times K$ of graphs G and K is a graph such that:

- The vertex set of $G \times K$ is the Cartesian product $V(G) \times V(K)$ and
- The two vertices (u, v) and (s, t) are adjacent in $G \times K$ if and only if either $u = s$ and v is adjacent to t in K or $v = t$ and u is adjacent to s in G .

3- Main Results

Definition 3.1[14]: Let R be a commutative ring. A graph $G(V, E)$ is said to be (Pseudo-Von Neumann regular graph) of R if $V(G) = R$ and $E(G) = \{ \overline{ab} / a = a^2b \text{ or } b = b^2a \text{ and } a \neq b \}$ denoted by $P-VG(R)$, shortly P -Von Neumann regular graph.

Example 3.2:

$$Z_2 = \{0, 1\}$$

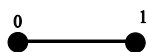


Fig. 1: $P-VG(Z_2)$

$$Z_3 = \{0, 1, 2\}$$

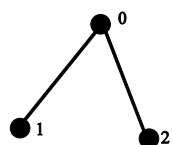


Fig. 2: $P-VG(Z_3)$

$$Z_4 = \{0, 1, 2, 3\}$$

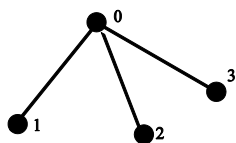


Fig. 3: $P-VG(Z_4)$

$$Z_5 = \{0, 1, 2, 3, 4\}$$

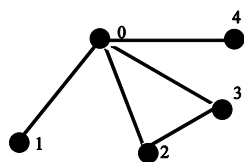


Fig. 4: $P-VG(Z_5)$

Definition 3.2 :

Let $R = R_1 \times R_2$, then the $P-VG(R)$ is define as the vertices set = $\{ (a, b) : a \in R_1 \text{ and } b \in R_2 \}$ then (a, b) and (u, v) are adjacent in $P-VG(R)$ if and only if a adjacent to u in $P-VG(R_1)$ and b adjacent to v in $P-VG(R_2)$, and $(0, 0)$ adjacent to all vertices.

Example 3.3:

1- Let $R = Z_2 \times Z_2$

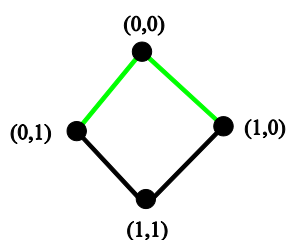


Fig. 5-i: $P-VG(Z_2) \times P-VG(Z_2)$

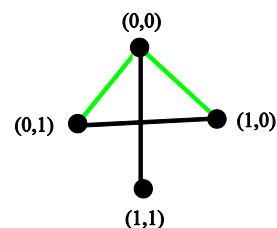


Fig. 5-ii: $P-VG(Z_2 \times Z_2)$

$P-VG(Z_2) \times P-VG(Z_2) \cap P-VG(Z_2 \times Z_2) = 3\text{-star graph.}$

2- Let $R = Z_3 \times Z_3$

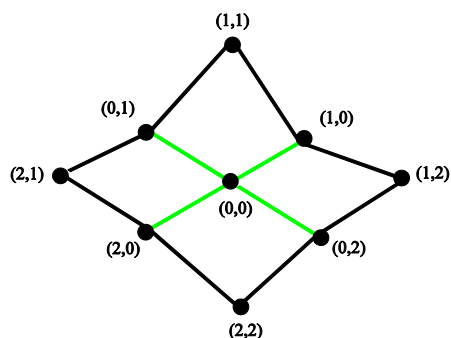


Fig. 6-i: $P-VG(Z_3) \times P-VG(Z_3)$

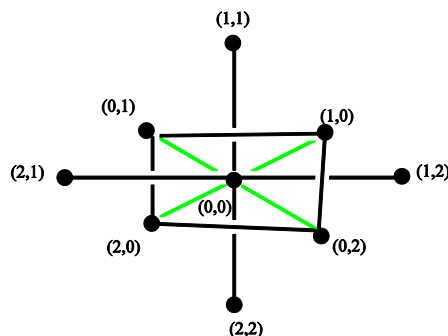


Fig. 6-ii: $P-VG(Z_3 \times Z_3)$

$P-VG(Z_3) \times P-VG(Z_3) \cap P-VG(Z_3 \times Z_3) = 5\text{-star graph.}$

In general $P-VG(Z_n) \times P-VG(Z_n) \cap P-VG(Z_n \times Z_n) = (2n-1)\text{-star graph}$ and the below theorem show that.

Theorem 3.4:

let $R = Z_n$, then the intersection of cartesian product of $P-VG(Z_n)$ and $P-VG(Z_n \times Z_n)$ is equal to $(2n-1)\text{-star graph}$ i.e.

$P-VG(Z_n) \times P-VG(Z_n) \cap P-VG(Z_n \times Z_n) = (2n-1)\text{-star graph.}$

Proof :

Since $\overline{(0,0)(0,a)} \in E(P-VG(Z_n) \times P-VG(Z_n) \cap P-VG(Z_n \times Z_n))$.

Then $P-VG(Z_n) \times P-VG(Z_n) \cap P-VG(Z_n \times Z_n) \neq \emptyset$.

Now, let $H = \{ (a, b), a = 0 \text{ or } b = 0 \}$ and $E(H) = \{ \overline{(0,0)(a,b)}, (a, b) \neq (0,0) \}$:

H is a subgraph and in the same time H is a star graph by the set $E(H)$.

Now, we need to prove this star has $2n-1$ of vertices.

The study focuses on a number have of vertices are equal to $2(n-1)$ because the set of vertices of K graph $\{(0,1), \dots, (0,n-1), (1,0), \dots, (n-1,0)\}$ and $(0,0)$ is a center of star graph then H has $2(n-1) + 1 = 2n-1$ vertices, i.e. H has order $2n-1$

$H = (2n-1) - \text{star graph} \subset P-VG(Z_n) \times P-VG(Z_n)$ and it is in the same time is a sub graph from $P-VG(Z_n \times Z_n)$.

Hence $H \subset P-VG(Z_n) \times P-VG(Z_n) \cap P-VG(Z_n \times Z_n)$.

Let $(a,b)(u,v) \in P-VG(Z_n) \times P-VG(Z_n) \cap P-VG(Z_n \times Z_n)$, and $a,b,u,v \neq 0$.

Implies that $(a,b)(u,v) \in P-VG(Z_n) \times P-VG(Z_n)$ and $(a,b)(u,v) \in P-VG(Z_n \times Z_n)$.

Then (either $a=u$ and $\overline{bv} \in E(P-VG(Z_n))$ or $b=v$ and $\overline{au} \in E(P-VG(Z_n))$ and $(\overline{au}, \overline{bv}) \in E(P-VG(Z_n))$) Hence there are two cases:

Case i: if $(a=u$ and $\overline{bv} \in E(P-VG(Z_n))$ and $\overline{au} \in E(P-VG(Z_n))$), this is a contradiction by the definition of $P-VN$ - Regular graph).

Case ii: if $b=v$ and $\overline{au} \in E(P-VG(Z_n))$ and $\overline{bv} \in E(P-VG(Z_n))$

,also this is a contradiction by the definition of $P-VN$ - Regular graph).

Then $P-VG(Z_n) \times P-VG(Z_n) \cap P-VG(Z_n \times Z_n) = K = (2n-1) - \text{star graph}$.

lemma 3.5:

Let $R = Z_p$, $p > 3$ be a prime number then $\chi(P-VG(R)) = 3$.

Proof: since $P-VG(Z_p)$ has only cycle C_3 then $\chi(P-VG(R)) = 3$.

Theorem 3.6:

Let $R = Z_{p^k}$, $p > 3$ be a prime number, k be a positive integer then $\chi(P-VG(R)) = 3$.

Proof: since $P-VG(Z_{p^k})$ has only cycle C_3 then $\chi(P-VG(R)) = 3$.

Corollary 3.7:

Let $R = Z_n$, then $\chi(P-VG(R)) =$

$$\begin{cases} 2 & n = 2, 3, 4, 8 \\ 4 & \text{if any vertex has invrse.} \\ 3 & \text{other wise} \end{cases}$$

proof:

if $R = Z_n$, $n = 2, 3, 4, 8$ then $P-VG(R)$ is a star graph and $\chi(P-VG(R)) = 2$.

if any vertex $a \in R$ has inverse then $(\overline{pa}, \overline{pa^{-1}}, \overline{aa^{-1}})$ is a cycle C_3 in $P-VG(R)$, and since all vertices in $P-VG(R)$ are adjacent to vertex 0 then $\chi(P-VG(R)) = 4$ now in other wise

Case 1: if n is prime number then $\chi(P-VG(R)) = 3$ (by lemma 3.5)

Case 2: if $n = p^k$ where p is prime number and k a positive integer then $\chi(P-VG(R)) = 3$ since $P-VG(R)$ has only $\frac{(p-1)n}{2p} - 1$ of cycle C_3 .

Case 2: if $n = pk$ where p is prime number and k a positive integer, if any vertex in R has no inverse then $\chi(P-VG(R)) = 3$.

Theorem 3.8:

Let $R = Z_p \times Z_p \times \dots \times Z_p$, (n times of Z_p) $p > 3$ is prime number then $\chi(P-VG(R)) = \chi(P-VG(Z_p)) + 1$.

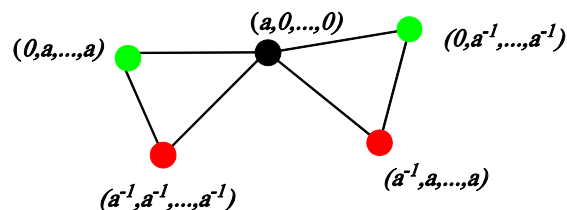
Proof:

Let $(a, 0, \dots, 0) \in R$ and $a \neq 0$, then $(a, 0, \dots, 0)$ is adjacent each to vertices

$(a^{-1}, a^{-1}, \dots, a^{-1})$, $(0, a^{-1}, a^{-1}, \dots, a^{-1})$, $(0, a, a, \dots, a)$ and $(a^{-1}, a, a, \dots, a)$ only. since $a \in Z_p$ and Z_p is a field.

But $(a^{-1}, a, a, \dots, a)$ and $(0, a^{-1}, a^{-1}, \dots, a^{-1})$ are adjacent and also

$(0, a, a, \dots, a)$ and $(a^{-1}, a^{-1}, \dots, a^{-1})$ are adjacent



Then, the graph has cycle of length 3, therefore, we colored it by 3 colors.

But by definition of $P-VG$ -regular graph of cartesian product of a rings all vertices are adjacent to $(0, 0, \dots, 0)$ then $\chi(P-VG(R)) = 4$

$\chi(P-VG(Z_p)) + 1 = 3 + 1 = 4 = \chi(P-VG(R))$.

Theorem 3.9:

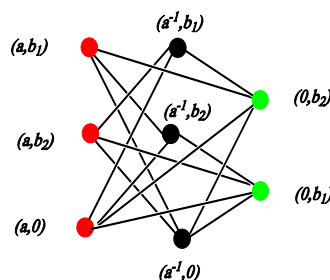
Let $R = Z_m \times Z_n$ and $n, m \geq 3$, then

1- $\chi(P-VG(R)) = \chi(P-VG(Z_m)) + \chi(P-VG(Z_n)) - 2$ if

i- m and n are a prime.

ii- m is prime and n not prime and $\chi(P-VG(Z_n)) = 3$

iii- m and n are not a prime and $\chi(P-VG(Z_m)) = \chi(P-VG(Z_n)) = 3$



2- $\chi(P-VG(R)) = \chi(P-VG(Z_m)) + \chi(P-VG(Z_n)) - 3$ if

i. m and n are not prime.

ii. m is prime and n not prime and $\chi(P-VG(Z_n)) = 4$

Proof:

1-i:

Let $R = Z_m \times Z_n$, m and n are a prime then $\chi(P-VG(Z_m))$ and $\chi(P-VG(Z_n))$ equal to 3, (by lemma 3.5)

Now, let $(a, b) \in R$ and $a \in Z_m$, $b \in Z_n$, then (a, b) is adjacent only to (a^{-1}, b^{-1}) , $(a^{-1}, 0)$ and $(0, b^{-1})$.

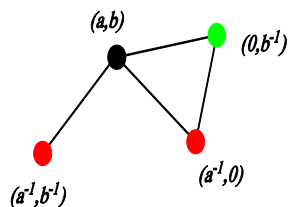
But $(a^{-1}, 0)$ and $(0, b^{-1})$ are adjacent therefore we colored it by 3 colors, and since all vertices are adjacent to the vertex $(0, 0)$, then $\chi(P-VG(R)) = 3 + 1 = 4$.

$$\mathcal{X}(P-VG(Z_m)) + \mathcal{X}(P-VG(Z_n)) - 2 = 3 + 3 - 2 = 4 = \mathcal{X}(P-VG(R))$$

1-ii :

Let m is prime then $\mathcal{X}(P-VG(Z_m))$ equal to 3, and n is not prime and $\mathcal{X}(P-VG(Z_n))$ equal to 3

Then we have three sets



$\{(a, b_1), (a, b_2), (a, 0)\}, \{(a^{-1}, b_1), (a^{-1}, b_2), (a^{-1}, 0)\}$ and $\{(0, b_1), (0, b_2)\}$ where $a \in Z_m$ and $b_1, b_2 \in Z_n$ such that $\overline{b_1 b_2} \in E(P - VG(Z_n))$

such that the vertices in the same set not adjacent to each other, then we have 3- partite graph and to colored it we need 3 colors, but all vertices are adjacent to vertex $(0,0)$ then $\mathcal{X}(P-VG(R)) = 3+1=4$
 $\mathcal{X}(P-VG(Z_m)) + \mathcal{X}(P-VG(Z_n)) - 2 = 3 + 3 - 2 = 4 = \mathcal{X}(P-VG(R))$.

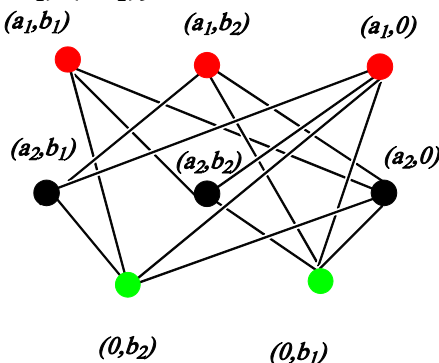
1-iii:

Let m and n are not prime and $\mathcal{X}(P-VG(Z_m))$ and $\mathcal{X}(P-VG(Z_n))$ are equal to 3

Let $a_1, a_2 \in Z_m$ such that $\overline{a_1 a_2} \in E(P - VG(Z_m))$, $b_1, b_2 \in Z_n$ such that $\overline{b_1 b_2} \in E(P - VG(Z_n))$

Now, we have three sets

$\{(a_1, b_1), (a_1, b_2), (a_1, 0)\}, \{(a_2, b_1), (a_2, b_2), (a_2, 0)\}$ and $\{(0, b_1), (0, b_2)\}$



such that the vertices in the same set not adjacent to each other, then we have 3- partite graph and to colored it we need 3 colors, but all vertices are adjacent to vertex $(0,0)$ then $\mathcal{X}(P-VG(R)) = 3+1=4$
 $\mathcal{X}(P-VG(Z_m)) + \mathcal{X}(P-VG(Z_n)) - 2 = 3 + 3 - 2 = 4 = \mathcal{X}(P-VG(R))$.

2-i :

Let m and n are not prime and $n, m \neq 4, 8$ then $\mathcal{X}(P-VG(Z_m))$ and $\mathcal{X}(P-VG(Z_n))$ equal to 3 or 4

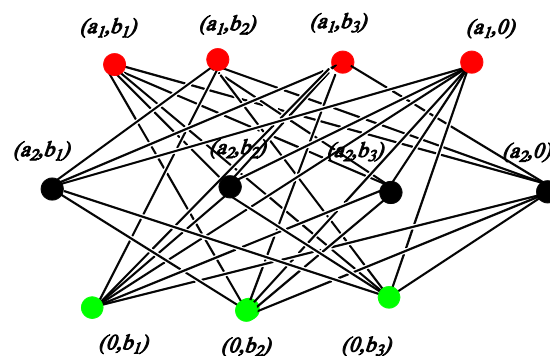
Then either $\mathcal{X}(P-VG(Z_m)) = 3$ and $\mathcal{X}(P-VG(Z_n)) = 4$ or $\mathcal{X}(P-VG(Z_m))$ and $\mathcal{X}(P-VG(Z_n))$ equal to 4

case 1:

$\mathcal{X}(P-VG(Z_m)) = 3$ and $\mathcal{X}(P-VG(Z_n)) = 4$ then we have three sets

$\{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_1, 0)\},$
 $\{(a_2, b_1), (a_2, b_2), (a_2, b_3), (a_2, 0)\}$ and

$\{(0, b_1), (0, b_2), (0, b_3)\}$ such that where $a_1, a_2 \in Z_m$ such that $\overline{a_1 a_2} \in E(P - VG(Z_m))$, $b_1, b_2, b_3 \in Z_n$ such that $\overline{b_1 b_2}, \overline{b_1 b_3}, \overline{b_2 b_3} \in E(P - VG(Z_n))$.



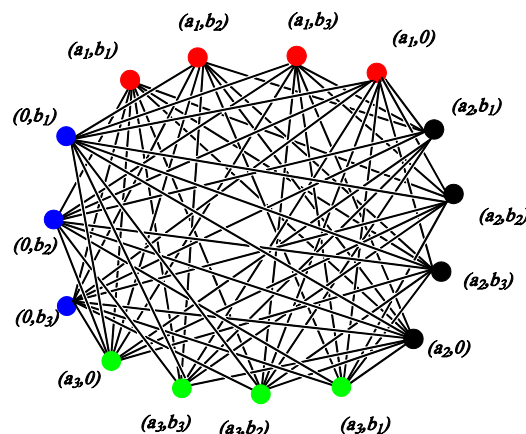
The vertices in the same set not adjacent to each other, then we have 3- partite graph and to colored it we need 3 colors, but all vertices are adjacent to vertex $(0,0)$ then $\mathcal{X}(P-VG(R)) = 3+1=4$

$$\mathcal{X}(P-VG(Z_m)) + \mathcal{X}(P-VG(Z_n)) - 3 = 3 + 4 - 3 = 4 = \mathcal{X}(P-VG(R))$$

Case2:

$\mathcal{X}(P-VG(Z_m))$ and $\mathcal{X}(P-VG(Z_n))$ are equal to 4 then we have four sets

$\{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_1, 0)\},$
 $\{(a_2, b_1), (a_2, b_2), (a_2, b_3), (a_2, 0)\},$
 $\{(a_3, b_1), (a_3, b_2), (a_3, b_3), (a_3, 0)\},$
 $\{(0, b_1), (0, b_2), (0, b_3)\}.$



where $a_1, a_2, a_3 \in Z_m$ such that $\overline{a_1 a_2}, \overline{a_1 a_3}, \overline{a_2 a_3} \in E(P - VG(Z_m))$, $b_1, b_2, b_3 \in Z_n$ such that $\overline{b_1 b_2}, \overline{b_1 b_3}, \overline{b_2 b_3} \in E(P - VG(Z_n))$.

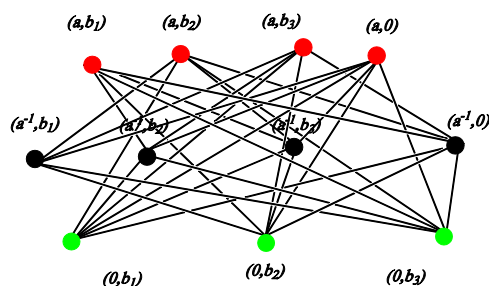
The vertices in the same set not adjacent to each other, then we have 4- partite graph and to colored it we need 4 colors, but all vertices are adjacent to vertex $(0,0)$ then $\mathcal{X}(P-VG(R)) = 4+1=5$.

$$\mathcal{X}(P-VG(Z_m)) + \mathcal{X}(P-VG(Z_n)) - 3 = 4 + 4 - 3 = 5 = \mathcal{X}(P-VG(R))$$

2-ii:

Let m is prime and n not prime and $\mathcal{X}(P-VG(Z_n)) = 4$ then we have three sets

$\{(a, b_1), (a, b_2), (a, b_3), (a, 0)\},$
 $\{(a^{-1}, b_1), (a^{-1}, b_2), (a^{-1}, b_3), (a^{-1}, 0)\}$ and
 $\{(0, b_1), (0, b_2), (0, b_3)\}$



where $a \in Z_m$ and $b_1, b_2, b_3 \in Z_n$ such that $\overline{b_1 b_2}, \overline{b_1 b_3}, \overline{b_2 b_3} \in E(P - VG(Z_n))$, the vertices in the same set are not adjacent to each other, then we have 3- partite graph and to colored it we need 3 colors, but all vertices are adjacent to vertex $(0,0)$ then $\chi(P - VG(R)) = 3+1=4$.
 $\chi(P - VG(Z_m)) + \chi(P - VG(Z_n)) - 3 = 3 + 4 - 3 = 4 = \chi(P - VG(R))$.

Example 3.10:

$$R = Z_3 \times Z_5$$

$$\chi(P - VG(R)) = \chi(P - VG(Z_3)) + \chi(P - VG(Z_5)) - 2 = 2 + 3 - 2 = 3$$

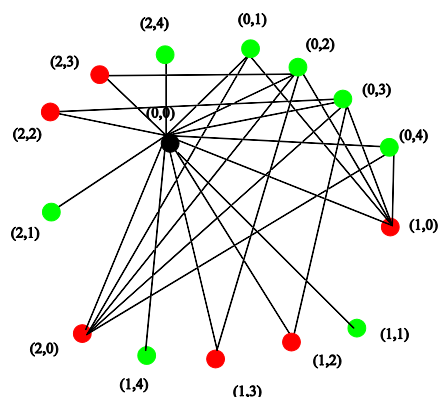


Fig. 7 : $\chi(P - VG(Z_3 \times Z_5))$

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Example 3.11:

$$R = Z_6 \times Z_9$$

$$\chi(P - VG(R)) = \chi(P - VG(Z_6)) + \chi(P - VG(Z_9)) - 2 = 3 + 3 - 2 = 4$$

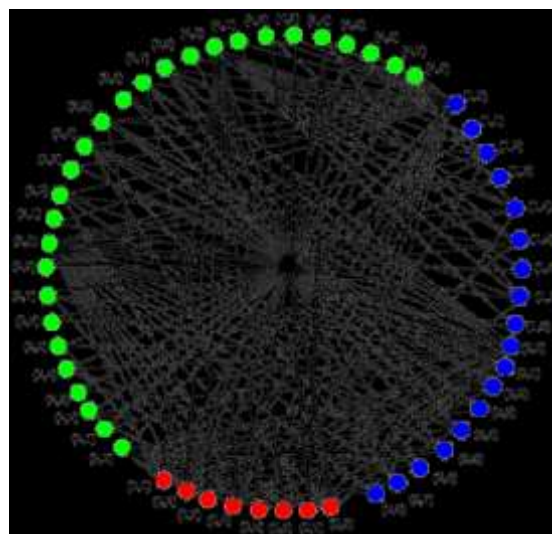


Fig. 8: $\chi(P - VG(Z_6 \times Z_9))$

4- Conclusion

In this paper we gave a definition of Pseudo-Von Neumann regular graph of Cartesian product of commutative rings and found the relation between it and Cartesian product of Pseudo-Von Neumann regular graph of commutative ring. Also found the chromatic number $\chi(P - VG(R))$ where R is commutative ring, and R is Cartesian product of commutative ring.

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العدد اللوني و بعض خصائص البيان المنتظم الزائف للضرب الديكارتي للحلقات الابدالية

نرمين جمال خليل ، نبيل عز الدين عارف

قسم الرياضيات ، كلية علوم الحاسوب والرياضيات ، جامعة تكريت ، تكريت ، العراق

الملخص

لنكن R حلقة ابدالية، فأن البيان المنتظم الزائف للحلقة R يعرف على انه البيان الذي مجموعة رؤوسه هي جميع عناصر الحلقة R واي رأسين مختلفين a و b في R يكونان متجاوران اذا وفقط اذا $b = b^2a$ او $a = a^2b$ ، نرمز لهذا البيان بالرمز $P-VG(R)$ ، في بحثنا حصلنا على بعض النتائج حول العدد اللوني للبيان المنتظم الزائف بالنسبة للضرب الديكارتي لبعض الحلقات الابدالية .