

Simulation Analysis Based on Stochastic Delay Differential Equations

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Abstract : This paper interested in studying the problem of the numerical solution for the stochastic delay differential equations (SDDE), we conducted few simulation scenarios based on the ordinary Black-Scholes process, and delay Black- Scholes process. It is well known that the finding of explicit solution of an SDDE is very hard to obtain, therefore, strong Euler-Maryama numerical method provided to find the approximation solution for the SDDE. The simulation examples results displayed by the pathways of the approximation solution for the $s(t)$ process under different values of the process par for parameters, also we sketched the histogram of the $s(t)$ that provides the log normal distribution of the solution values. Furthermore, we obtained some important statistics that described the approximation solution.

Keywords - Wiener process, SDDE, Euler-Maryama, approximation solution.

INTRODUCTION: Stochastic delay differential equation (SDDE) developed to study the behavior of some system. Generally, SDDE assumed as models to understand the behavior and the structure of some phenomena different area of sciences, such as, finance, economics, medicine, ecology, etc. In real life application modeling the phenomena to study the future behavior required the present information and then .We have what is called the stochastic differential equation in the presence of noise term. See Mao (2007) for more details. But, in many real data analysis, modeling the phenomena by modeling its functions and parameters do not appear their effects instantly of their occurrence, Mao (2007) and Aladagli (2017). So based on the past information. The stochastic differential equation becomes realistic in presently the studied phenomenon. Consequently, by adding the time delay term to the SDE. We get what is called the stochastic delay to differential equation. Aladagli (2017) referred financial assets. Scheinkman and Lebaron (1989) stated that the stocks returns depends on the past information of stocks returns. Jassim (2006) proposed analytical solution for the SDE and SDDE with some examples. Stoica (2005) considered that the trader in the stock prices market follows the Black-Scholes diffusion process, but the insider in the market knows that the average rate of change and the dispersion of returns processes are affected by defined events that occur before the trading process period started. Zheng (2015) studied the stochastic delay differential equation (SDDE) in real application of asset pricing through studying the mathematical properties of SDDE ,as well studying the estimation of its parameters from the Bayesian point of view and compared the results with the classical Geometric Brownian Motion model.

Explicit solutions for SDE is hardly to be obtained; which means ,there are no closed form for the analytical solution .Because of that ,various authors put in a lot of effort to find the analytical solution for SDE and SDDE ,see Smith (1999), Muszta(2005). For further information .But some authors works of the numerical solution for SDE, Han (2005), and Mahony (2006). Arriojas et al (2007) developed an explicit formula for the solution of the SDDE with pricing European options considering that the underlying stock price follows nonlinear stochastic differential equation. Reiss (2007) studied the adaptive estimation for the SDDE functions. Non parametric inference developed to solve the estimation problem by using the wavelet estimator. Mohammed (1998) introduced the stochastic differential equation with memory term (delay) which evolution based on the past in formation of the underlying state .The existence of the unique solution have been proved, as well as , the asymptotic stability for the SDDE have developed. Buckwar (2000) developed numerical solution for the stochastic delay differential equation based on Itô formula. Exploit solution has proposed for the SDDE by illustrating some numerical examples using the Euler maruyama method. Ferrante and Rovira (2006) studied the stochastic delay differential equation through the Fractional Brownian motion with Hurst parameter $> \frac{1}{2}$. The developed the existence and uniqueness solution of SDDE considering that the coefficient of the SDDE are sufficiently regular.

I. Stochastic Delay Differential Equation :

Suppose that $(\Omega, \mathcal{F}_t, P)$ be a complete probability space, which is defined to describe any random experiment, with a filtration $(\mathcal{F}_t)_{t \geq 0}$. In this paper we will study the convergence of the Euler numerical method assuming that the functions of stochastic delay differential equation such that,

$$X(t) \in L^2(\Omega, \mathcal{F}_t, P), \quad \text{if} \quad E[\sup \|X(t)\|^2] < \infty.$$

See Arnold (1974), Karatzas and Shreve (1991), and Williams (1991) for the background of probability fundamental the Brownian motion and stochastic differential equation properties. Now, let $T \in [t_0 = 0, \infty]$ and $w(t)$ is the Brownian motion process .One dimensional, thus under the filtered probability space $(\Omega, \mathcal{F}_t, P)$, let us consider we have the following SDDE,

$$\begin{cases} dx(t) = g(x(t), x(t-\lambda), t)dt + u(x(t), x(t-\lambda), t)dw(t) & ; 0 \leq t \leq T \\ x(t) = \psi(t) & ; -\lambda \leq t \leq 0 \end{cases} \dots (1)$$

Where $\lambda \geq 0$ is the time –delay parameter, g and u are functions defined as follows:

$$\begin{aligned} g: R \times R \times R_+ &\rightarrow R \\ u: R \times R \times R_+ &\rightarrow R \\ R_+ &= [0, \infty) \end{aligned}$$

Also, $\psi(t)$ is the initial solution defined as measurable valued random variable defined on $[-\lambda, 0] \rightarrow R^n$ that satisfy the condition $E\|\psi\|^2 < \infty$. In equation (1), the function $g(\cdot)$ can be considered as the drift coefficient, $u(\cdot)$ as volatility coefficient, and $w(t)$ is the Wiener process. Moreover, the equation (1) can be rewritten in terms of integration from as follows:

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^t g(x(s), x(s-\lambda))ds + \int_{t_0}^t u(x(s), x(s-\lambda))dw(s) & ; 0 \leq t \leq T \\ x(t) = \psi(t) \text{ for } -\lambda \leq t \leq 0 \end{cases} \quad (2)$$

The following definitions are necessary for understanding the behavior of SDDE, see Mohammed (1984), Mao (1997), Stoica (2005), and Zlenny (2015).

Definition (1): For the SDDE in (1), the strong solution is R -valued stochastic process,

$$x(t): [-\lambda, T] \times \Omega \rightarrow R$$

Which is measurable and continuous process? Such that $x|_{[0, T]}$ is $F_{0 \leq t \leq T}$ is adaptive process and $x(\cdot)$ Satisfies (1) and (2) almost surely and satisfies the condition of the initial solution $x(t) = \psi(t)$; $t \in [-\lambda, 0]$. Hence, the solution of the process $x(t)$ is called path-wise unique if

$$P[x(t) = \hat{x}(t)] = 1 \quad ; \forall t \in [-\lambda, T]$$

Definition (2): The functions g and u in (2) satisfies the local Lipchitz condition if for every integer $i \geq 1$, there is positive constant K_i such that:

$$|g(x_1, y_1, t) - g(x_2, y_2, t)| \vee |u(x_1, y_1, t) - u(x_2, y_2, t)| \leq K_i[|x_1 - x_2| + |y_1 - y_2|]$$

and

$$|x_1| \vee |y_1| \vee |x_2| \vee |y_2| \leq i; \quad x_1, y_1, x_2, y_2 \in R,$$

Where $|x| \vee |y| = \max(|x|, |y|)$.

Definition (3): The functions g and u in (2) satisfies the linear growth condition, such that

$$|g(x, y, t)| \vee |u(x, y, t)| \leq K(1 + |x| + |y|)$$

Where K is appositve constant, and $(x, y, t) \in R \times R \times R_+$

Definitions (1) and (2) provides path-wise unique strong solution to equation (1).

Numerical solution for SDDE

Euler–Maruyama (E-M) is one of most popular numerical method that provides the strong solution for SDDE. E-M employed the stochastic calculus to find the numerical solution for the SDDE in (1)

Theorem (2): Let an SDDE is defined as follows,

$$\begin{aligned} dx(t) &= g(t, x(t), x(t-\lambda))dt + u(t, x(t), x(t-\lambda))dw(t) \quad ; t \in [a, b] \\ x(t_0) &= \psi_0(t) \quad ; t_0 - \lambda \leq t \leq t_0 \end{aligned}$$

and suppose that the interval $[a, b]$ partitioned as follows

$$\begin{aligned} 0 &= t_0 < t_1 < \dots < t_N = t \\ \Delta t_{n+1} &= t_{n+1} - t_n = h \end{aligned}$$

Where Δt_{n+1} represent the time increments and h is uniform step size defined as $h = \frac{b-a}{N}$.

And $\Delta w_n = w(t_{n+1}) - w(t_n)$

Represents the standard Brownian process with $n = 0, 1, \dots, N-1$, and $t_n = a + nh$

Then, the sequence of numerical solutions given as follows

$$x_{n+1} = x_n + g(t_n, x_n, \psi_{n-\lambda})h + u(t_n, x_n, \psi_{n-\lambda})\Delta w_n \quad ; 0 \leq n \leq N-1$$

See Jassim (2006) for further details

Definition (4): Local error is define as the difference between the approximation exact solution, and represent the sequence of the following random variables

$$S_n = x(t_n) - \bar{x}(t_n) \quad ; n = 1, \dots, N$$

Where $\bar{x}(t_n)$ is the approximation variable.

II. Simulations Analysis:

In this section we will use the numerical Euler-Maruyama method to study the behavior of sample path by conducting four simulation scenarios (note that there is no closed form for the SDDE solution), as well as, to see the impact of the initial values on the following SDDE.

1- Simulation example one

Let us consider we have the following ordinary Black- Scholes process:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

Where $\mu=0.03$, $\sigma=0.05$, $y_0 = 10$. The following figure (1) illustrates the pathway of the ordinary Black-Scholes process also figure (2) displays that the generated process follows the log normal distribution.

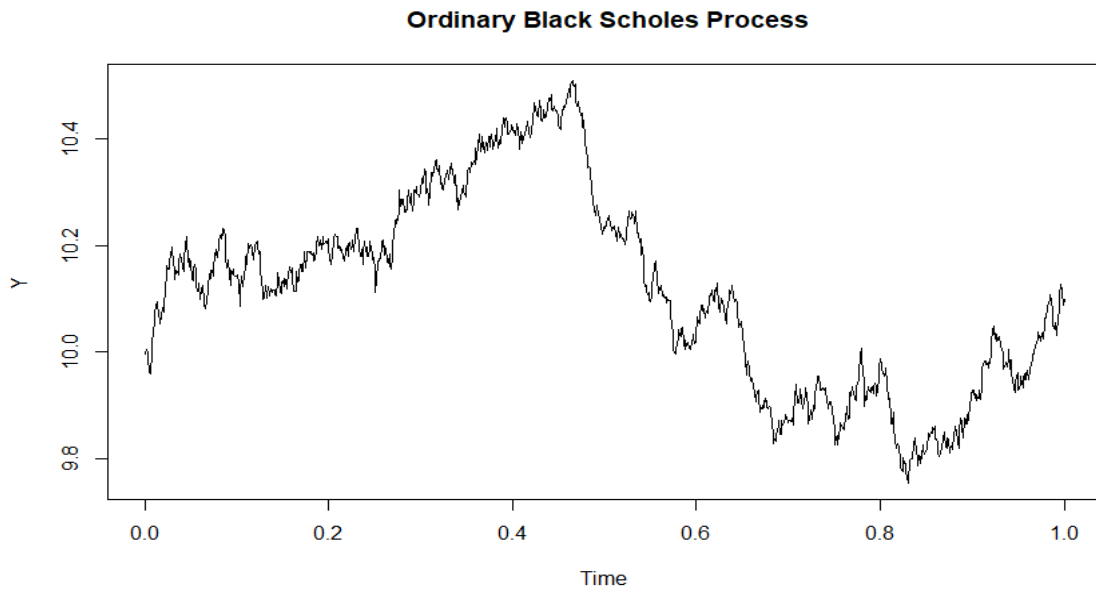


Figure 1: Sample path for the ordinary Black-Scholes values.

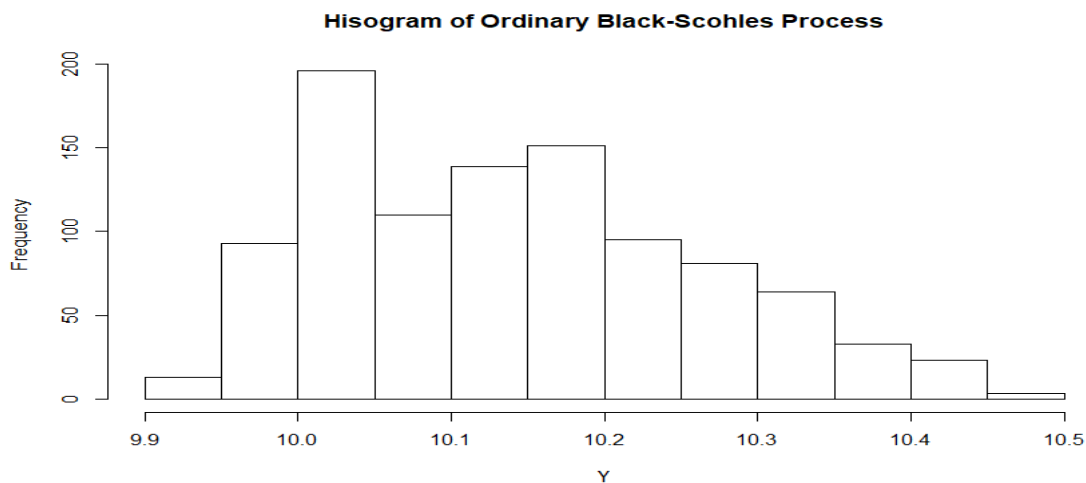


Figure (1) the Histogram of the stock price model in ordinary Black-Scholes
The following table explains some important values for the ordinary Black-Scholes process.

Table (1) some important statistics

Min.	1 st . Qu.	Median	Mean	3 rd . Qu.	Max.
9.841	10.018	10.195	10.195	10.372	10.550

2- Simulation example two

Let us consider we have the following delay Black- Scholes process,

$$ds_t = (\mu s_t + 2e^{-1}(s_t - \lambda)) + 3.2e^{-1}dt + \sigma dw_t,$$

Where μ is a real number to see the impact of the delay time in drift term only, and σ is a constant (diffusion). In Figure (3), we sketched one sample path with $\lambda = -0.6$, $\sigma = 0.5$, $T=1$.

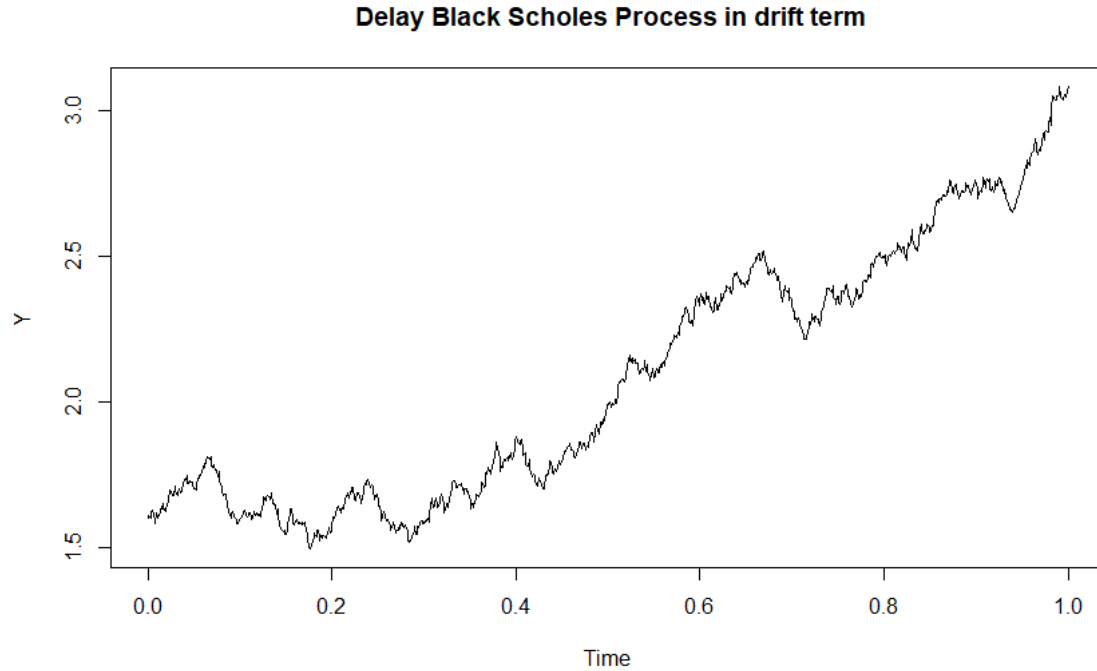


Figure 3: Sample path with of delay-drift Black-Scholes process.

The Figure (3) displays one sample path for the different initial values with $\lambda = -0.6$, $\mu = 0.003$, $\sigma = 0.5$, $T=1$. With time $t = 0$, the sample path decreases while path with time $t = 1$ increases. At time $t = 0$, it is seen that the graph take the value, 1.7 and after time $t = 0$, we observed that graph shows increases values for the sample paths between $t = 0$ and $t = 1$. Also, figure (4) displays that the generated process follows the log normal distribution.

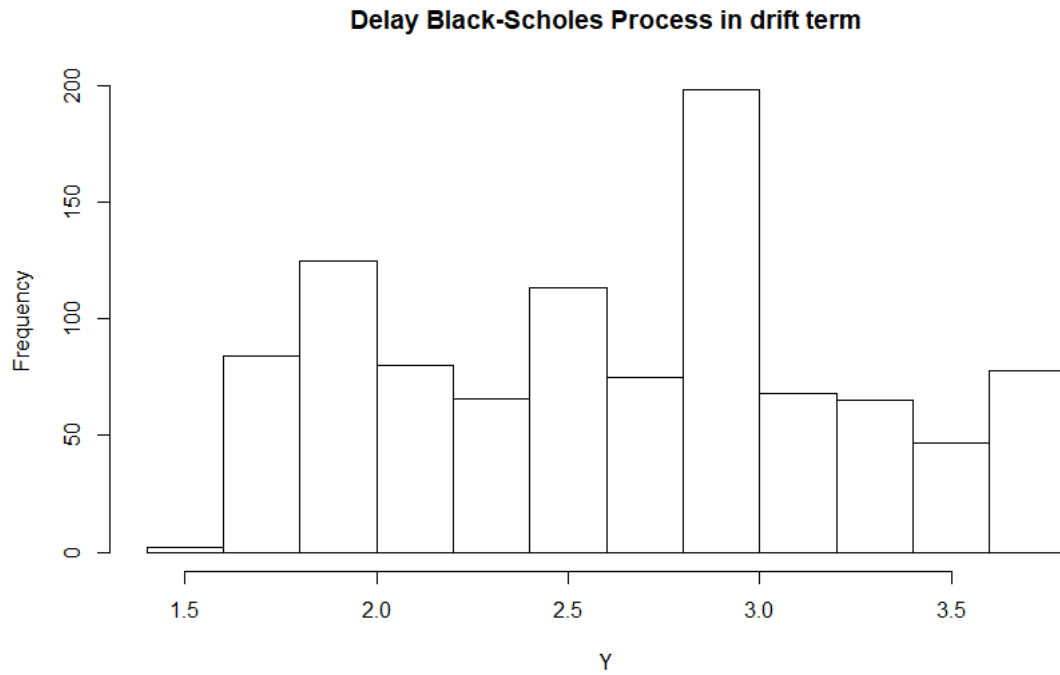


Figure (4) the Histogram of the stock price model in delay-drift Black-Scholes
The following explains some important values for the delay-drift Black-Scholes process.

Table (2) some important statistics

Min.	1 st . Qu.	Median	Mean	3 rd . Qu.	Max.
1.598	2.037	2.742	2.630	3.091	3.730

3- Simulation example three

Let us consider we have the following delay Black- Scholes process,

$$ds_t = (\mu s_t)dt + (\sigma s_t + b(s_t - \lambda))dwt,$$

Where σ is a real number to see the impact of the delay time in diffusion term only, and μ is a constant (drift). In Figure (5), we sketched one sample path with $\lambda = -0.5$, $\sigma = 0.5$, $T=1$.

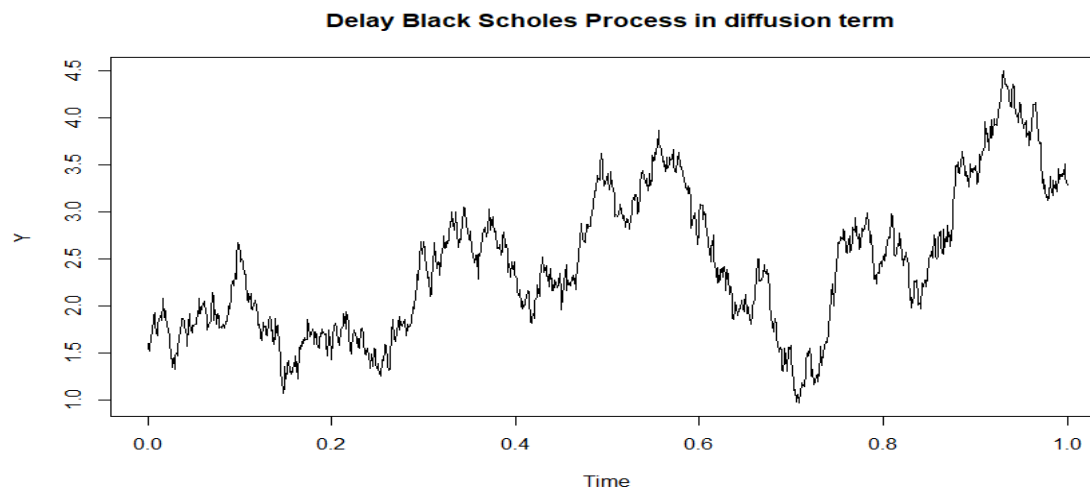


Figure 5: Sample path with different drift and diffusion values.

The Figure (5) displays one sample path for the different initial values with $\lambda = -0.6$, $\mu = 0.003$, $\sigma = 0.5$, $T=1$. With time $t = 0$, the sample path decreases while path with time $t = 1$ increases. At time $t = 0$, it is seen that the graph take the value, 1.7 and after time $t = 0$, we observed that graph shows increases values for the sample paths between $t = 0$ and $t = 1$.

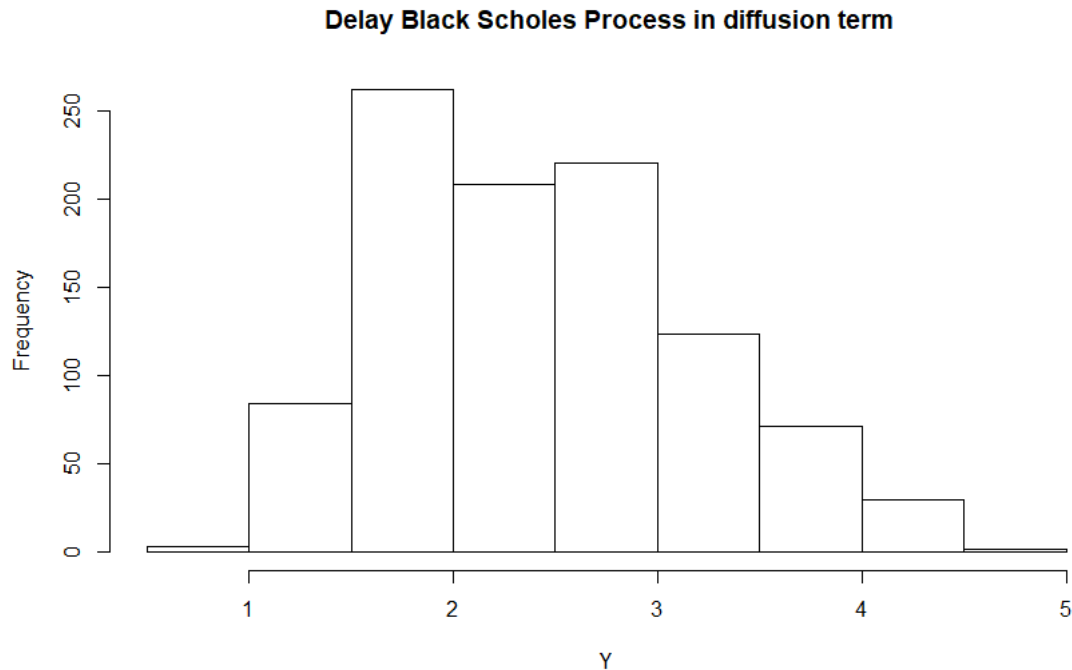


Figure (6) the Histogram of the stock price model in delay - diffusion Black-Scholes
The following explains some important values for the ordinary Black-Scholes process.

Table (3) some important statistics

Min.	1 st . Qu.	Median	Mean	3 rd . Qu.	Max.
0.974	1.809	2.354	2.439	2.930	4.502

4- Simulation example four

Let us consider we have the following delay Black- Scholes process,

$$ds_t = (\mu s_t + 2e^{-1}(s_t - \tau) + 3.2e^{-1})dt + \sigma^2(s_t + 3(s_t - \tau))dwt,$$

Where μ and σ are real numbers to see the impact of the delay time in drift and diffusion terms. In Figure (7), we sketched one sample path with $\lambda = -0.6$, $\sigma =$, $T=1$ and the initial values are ...

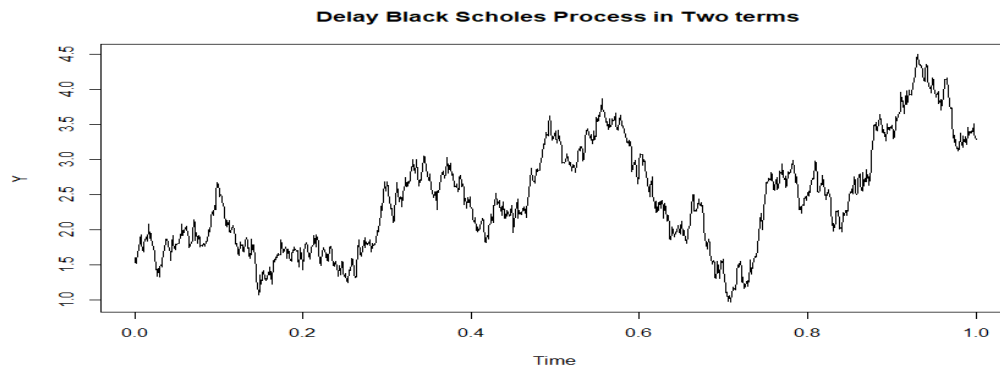


Figure 7: Sample path with different drift and diffusion values.

The Figure (7) displays one sample path for the different initial values with $\lambda = -0.6$, $\mu =$, $\sigma =$, $T=1$. With time $t = 0$, the sample path decreases while path with time $t = 1$ increases. At time $t = 0$, it is seen that the graph take the value, 1.7 and after time $t = 0$, we observed that graph shows increases values for the sample paths between $t = 0$ and $t = 1$.

The following explains some important values for the ordinary Black-Scholes process.

Table (4) some important statistics

Min.	1 st . Qu.	Median	Mean	3 rd . Qu.	Max.
0.975	1.809	2.354	2.440	2.931	4.504

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