quasi-normal Operator of order n

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Abstract

In this paper, we introduce a new class of operators acting on a complex Hilbert space H which is called quasi-normal operator of order n. An operator T∈B(H) is called quasi-normal operator of order n if $T(T^{*n}T^n)=(T^{*n}T^n)T$, where n is positive integer number greater than 1 and T^* is the adjoint of the operator T, We investigate some basic properties of such operators and study relations among quasi-normal operator of order n and some other operators.

Keywords: quasi-normal operators, normal operators, n-normal operators.

1. Introduction

Through this paper, $\mathcal{B}(\mathcal{H})$ denoted to the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be self-adjoint if $T^* = T$, isomtry if $T^*T = I$, unitary if $T^*T = TT^* = I$, where T^* is the adjoint of T [2]. The operator $T \in \mathcal{B}(\mathcal{H})$ is called normal if $TT^* = T^*T$ [1] , quasi-normal if $T(T^*T) = (T^*T)\,T$ [6] , n-normal if $T^nT^* = T^*T^n$ [1] , quasi n-normal if $T(T^*T^n) = (T^*T^n)T$ and n power quasi normal if $T^n(T^*T) = (T^*T)T^n$ [5], where n is positive integer number. If S and T are bounded liner operator T on a Hilbert space H, then the operator S is said to be unitarily equivalent to T if there is a unitary operator U on H such that $S = UTU^*$ [4] , Let T∈B(H) and M is a closed subspace of H. Then M is called reduce of T if $T(M) \subseteq M$ and $T(M^\perp) \subseteq M^\perp$, that is both M and $M^\perp$ are invariant under T and denoted by T/M [4] .

2. Quasi-normal operator of order n:

In this section, we will study some properties which are applied for the quasi-normal operator of order n .

Definition 2.1:

The operator $T \in \mathcal{B}(\mathcal{H})$ is called quasi-normal operator of order n if $T(T^{*n}T^n)=(T^{*n}T^n)T$, where T^* is the adjoint of the operator T .

Example 2.2:

Let U be a unilateral shift operator on ℓ_2, where ℓ_2 is the space of all sequence $x=(x_1,x_2,\ldots..)$ such that $\sum |x|^2 \leq \infty$ i.e. $U(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3 \ldots)$ .Then U is quasi-normal operator of order n for every n , since $U(U^{*n}U^n)=UI=IU=(U^{*n}U^n)U$

The following example show the adjoint of quasi-normal operator of order n need not to be quasi-normal operator of order n.

Example 2.3:

Let U be the unilateral shift operator on ℓ_2,then the adjoint of U is not quasi-normal operator of order n , since :

$$U^*(U^nU^{*n})(x_1,x_2,\ldots)=(\underbrace{0,0,\ldots..0}_{\text{n-1 times}},x_{n+1},x_{n+2},..)\neq(\underbrace{0,0,\ldots.0}_{\text{n times}},x_{n+2},x_{n+3},\ldots..)=(U^nU^{*n})U^*(x_1,x_2,\ldots)$$

The following remark consequence from definitions of normal operator and quasi-normal operator of order n.

Remark 2.4 :

If T is normal operator then T and T^* are quasi-normal operators of order n

Proposition 2.5 :

If T is invertible quasi-normal operator of order n , then the adjoint of T^{-1} is quasi-normal operator of order n

Proof :

Since T is quasi-normal operator of order n then

$$T(T^{*n}T^n)=(T^{*n}T^n)T$$

Therefor $((T^{-1})^n (T^{-1*})^n)T^{-1}= T^{-1}((T^{-1})^n (T^{-1*})^n)$

by taking adjoint of both side

we have $T^{-1*}((T^{-1})^n (T^{-1*})^n)= ((T^{-1})^n (T^{-1*})^n)\, T^{-1*}$

Thus T^{-1*} is quasi-normal operator of order n

Theorem 2.6 :

If T is quasi-normal operator of order n,then:

αT is quasi-normal operator of order n for every complex number α.

If S is Unitarily equivalence to T then S is quasi-normal operator of order n .

If M is closed subspace of H , then T/M is quasi-normal operator of order n ,where T/M is restriction of T to M that reduce T.

Proof :

$(\alpha T)\,((\alpha T)^{*n}(\alpha T)^n)= \alpha(\overline{\alpha})^n\alpha^n\, T(T^{*n}T^n)$

$= \alpha(\overline{\alpha})^n\alpha^n\,(T^{*n}T^n)T = ((\alpha T)^{*n}(\alpha T)^n)\,(\alpha T)$

Since S is unitarily equivalence to T . then there exist a unitary operator U such that $S=UTU^*$, so that $S^n=UT^nU^*$ and $S^{*n}=UT^{*n}U^*$

$(S^{*n}S^n)S = UT^{*n}U^*UT^nU^*UTU^*$

$= U(T^{*n}T^n)TU^*$

$= UT(T^{*n}T^n)U^* \ldots\ldots\ldots$ ❶

$S(S^{*n}S^n) = UTU^*UT^{*n}U^*UT^nU^*$

$= UT(T^{*n}T^n)U^* \ldots\ldots\ldots$ ❷

from ❶ and ❷ we get : $(S^{*n}S^n)S = S(S^{*n}S^n)$

(3) $(T/M)\,((T/M)^{*n}(T/M)^n)$

$= (T/M)\,((T^{*n}/M)(T^n/M))$

$= (T(T^{*n}T^n))/M$

$= ((T^{*n}T^n)T)/M$
$= ((T^{*n}/M)\,(T^n/M))\,(T/M)$
$= ((T/M)^{*n}\,(T/M)^n)\,(T/M)$

Therefor T/M is quasi-normal operator of order n

The following example shows that the product of two quasi-normal operators of order n is not necessary quasi-normal operator of order n .

Example 2.7 :

The operators $T=\begin{pmatrix} i & 3 \\ 0 & -i \end{pmatrix}$ and $S=\begin{pmatrix} i & -i \\ 0 & -i \end{pmatrix}$ on two dimensional Hilbert space $\mathbb{C}^2$ are quasi-normal operators of order 2 but:

$[(ST)\quad ((ST)^{*2}(ST)^2)]=\begin{pmatrix} -21 & 43-119i \\ 2+6i & -41 \end{pmatrix} \neq \begin{pmatrix} -1 & 3-9i \\ 2+6i & -61 \end{pmatrix} =[((ST)^{*2}(ST)^2)\,(ST)]$

Therefor ST is not quasi-normal operator of order 2 .

Proposition 2.8 :

let S be normal operator and T is quasi-normal operator of order n , If ST=TS then ST is quasi-normal operator of order n.

Proof :-

since ST=TS and S is normal operator then by Fugled-Putnam-Rosenblum Theorem [3] $S^*T=TS^*$ ……… ❶

By take adjoint for both side of equation ❶ we get : $T^*S=ST^*$ …….. ❷

Also by taking n-power for both side of equation ❶ we get :

$T^nS^{*n}=S^{*n}T^n$ ….. ❸

Now : $(ST)\,((ST)^{*n}(ST)^n) = (ST)\,(T^{*n}S^{*n}(S^nT^n))$
$= (ST)\,(T^{*n}\,(S^{*n}T^n)S^n)$
$= (ST)\,(T^{*n}\,(T^nS^{*n})S^n)$
$= (S\,(T(T^{*n}T^n)S^{*n}S^n)$
$= (S\,((T^{*n}T^n)T)S^{*n}S^n)$ since T is quasi-normal operator of order n
$= ((T^{*n}S)\,T^nTS^{*n}S^n)$
$= (T^{*n}S\,T(T^nS^{*n})S^n)$
$= (T^{*n}S\,T(S^{*n}T^n)S^n)$
$= (T^{*n}S\,(TS^{*n})T^nS^n)$
$= (T^{*n}S\,(S^{*n}T)T^nS^n)$
$= (T^{*n}(SS^{*n})TT^nS^n)$
$= (T^{*n}(S^{*n}S)TT^nS^n)$ since S is normal operator
$= (T^{*n}S^{*n}(ST^n)(TS^n))$
$= (T^{*n}S^{*n}(T^nS)(\,S^nT))$ since ST=TS
$= ((ST)^{*n}(ST)^n)\,(ST)$

Therefore (ST) is quasi-normal operator of order n

The following example shows that the sum of two quasi-normal operators of order n is not necessary quasi-normal operator of order n .

Example 2.9 :

The operators $T=\begin{pmatrix} i & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -i \end{pmatrix}$ and $S=\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}$ on three dimensional Hilbert space $\mathbb{C}^3$ are quasi-normal operator of order 2,but

$(T+S)\,((T+S)^{*2}\,(T+S)^2)=\begin{pmatrix} 24+40i & 0 & 24+8i \\ 0 & 1 & 0 \\ 40+24i & 0 & 24-8i \end{pmatrix} \neq \begin{pmatrix} 40+24i & 0 & 24+8i \\ 0 & 1 & 0 \\ 8+24i & 0 & 8+8i \end{pmatrix} = ((T+S)^{*2}\,(T+S)^2)\,(T+S)$

Therefor (T+S) is not quasi-normal operator of order 2.

The following example shows that quasi-normal operator of order n need not to be quasi-normal operator of order n+1

Example 2.10 :

The operator $T=\begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}$ on two dimensional Hilbert space $\mathbb{C}^2$ is quasi-normal operator of order 2,but

$T(T^{*3}T^3)=\begin{pmatrix} i & 2 \\ -1 & -i \end{pmatrix} \neq \begin{pmatrix} 2i & 3 \\ 1 & -2i \end{pmatrix}=(T^{*3}T^3)T$

Therefore T is not quasi-normal operator of order 3.

Theorem 2.11:

Let $T_1,T_2,\ldots\ldots,T_k$ be quasi-normal operators of order n in B(H) then the direct sum $(T_1\oplus T_2\oplus\ldots\ldots\oplus T_k)$ and tensor product $(T_1\otimes T_2\otimes\ldots\ldots\otimes T_k)$ are quasi-normal operators of order n

Proof:

$(T_1\oplus T_2\oplus\ldots\ldots\oplus T_k)$
$[(T_1\oplus T_2\oplus\ldots\ldots\oplus T_k)^{*n}(T_1\oplus T_2\oplus\ldots\ldots\oplus T_k)^n]$
$= (T_1\oplus T_2\oplus\ldots\ldots\oplus T_k)\;[(T_1^{*n}\;\oplus T_2^{*n}\;\oplus\ldots\ldots\oplus T_k^{*n})\,(T_1^n\oplus T_2^n\oplus\ldots\ldots\oplus T_k^n)$
$= (T_1\oplus T_2\oplus\ldots\ldots\oplus T_k)\;(T_1^{*n}\;T_1^n\;\oplus T_2^{*n}\;T_2^n\oplus\ldots\ldots\oplus T_k^n\,T_k^{*n})$
$= (T_1(T_1^{*n}\;T_1^n)\oplus T_2(T_2^{*n}\;T_2^n)\oplus\ldots\ldots\oplus T_k(T_k^n\,T_k^{*n}))$

Since $T_1,T_2,\ldots\ldots,T_k$ are quasi-normal operators of order n, then

$= ((T_1^{*n}\;T_1^n)T_1\oplus(T_2^{*n}\;T_2^n)T_2\oplus\ldots\ldots\oplus(T_k^n\,T_k^{*n})T_k)$
$= [(T_1\oplus T_2\oplus\ldots\ldots\oplus T_k)^{*n}(T_1\oplus T_2\oplus\ldots\ldots\oplus T_k)^n]\,(T_1\oplus T_2\oplus\ldots\ldots\oplus T_k)$

Also

$(T_1\otimes T_2\otimes\ldots\ldots\otimes T_k)$
$[(T_1\otimes T_2\otimes\ldots\ldots\otimes T_k)^{*n}(T_1\otimes T_2\otimes\ldots\ldots\otimes T_k)^n]$
$= (T_1\otimes T_2\otimes\ldots\ldots\otimes T_k)\;[(T_1^{*n}\;\otimes T_2^{*n}\;\otimes\ldots\ldots\otimes T_k^{*n})\,(T_1^n\otimes T_2^n\otimes\ldots\ldots\otimes T_k^n)$
$= (T_1\otimes T_2\otimes\ldots\ldots\otimes T_k)\;(T_1^{*n}\;T_1^n\;\otimes T_2^{*n}\;T_2^n\otimes\ldots\ldots\otimes T_k^n\,T_k^{*n})$
$= (T_1(T_1^{*n}\;T_1^n)\otimes T_2(T_2^{*n}\;T_2^n)\otimes\ldots\ldots\otimes T_k(T_k^n\,T_k^{*n}))$

Since $T_1,T_2,\ldots\ldots,T_k$ are quasi-normal operators of order n, then

$= ((T_1^{*n}\;T_1^n)T_1\otimes\;(T_2^{*n}\;T_2^n)T_2\otimes\ldots\ldots\otimes\;(T_k^n\,T_k^{*n})T_k)$
$(T_1\otimes T_2\otimes\ldots\ldots\otimes T_k)$
$[(T_1\otimes T_2\otimes\ldots\ldots\otimes T_k)^{*n}(T_1\otimes T_2\otimes\ldots\ldots\otimes T_k)^n]$

The following proposition shows that the relation between n-normal operators and quasi-normal operators of order n.

Proposition 2.12:

Every n-normal operator is quasi-normal operator of order n for every positive integer number n .

Proof:

Let T be n-normal operator operator ,then

$T^nT^*=T^*T^n$ by taking adjoint for both side we get $TT^{*n}=T^{*n}T$

Now

$T(T^{*n}T^n)=T^{*n}TT^n$

$=(T^{*n}T^n)T$

Therefor T is quasi-normal operator of order n

The following example shows that the convers of above proposition is not true .

Example 2.13:

The Unilateral shift U is quasi-normal operator of order n but:

$$U^nU^*(x_1,x_2,\ldots)=(\underbrace{0,0,\ldots,0}_{n\text{-times}},x_2,x_3,\ldots)\neq(\underbrace{0,0,\ldots,0}_{n-1\text{ times}},x_1,x_2,\ldots)=U^*U^n(x_1,x_2,\ldots)$$

Therefore U is not n-normal operator .

The following proposition shows that the relation between quasinormal operators and quasi-normal operators of order n.

Proposition 2.14:

Every quasinormal operator is quasi-normal operator of order n for every positive integer number greater than 1 .

Proof:

$(T^{*n}T^n)T = T^{*n-1}((T^*T)T^{n-1})T$

$= T^{*n-1}(T^{n-1}(T^*T))T$ since T is quasi-normal operator

$= T^{*n-2}((T^*T)\,T^{n-1})\,(T^*T)$

$= T^{*n-3}(T^*T)T^{n-2}\,(T^*T)\,(T^*T)\,(T^*T)$

$= T^{*n-4}\,(T^*T)T^{n-3}\,(T^*T)\,(T^*T)\,(T^*T)\,(T^*T)$

.

.

$=\underbrace{T(T^*T)\,(T^*T)\,(T^*T)\,(T^*T)\,(T^*T)\,\ldots\ldots\ldots(T^*T)}_{n\text{-times}}\ \ldots$ ❶

$T(T^{*n}T^n) = T(T^{*n-1}((T^*T)T^{n-1})$

$= T(T^{*n-1}(T^{n-1}(T^*T))$

$= T(T^{*n-2}(T^*T)T^{n-2}(T^*T)$

.

.

$=\underbrace{T(T^*T)\,(T^*T)\,(T^*T)\,(T^*T)\,(T^*T)\,\ldots\ldots\ldots(T^*T)}_{n\text{-times}}\ldots$ ❷

.

therefor from ❶ and ❷ we get T is quasi-normal operator of order n.

the following example shows that the convers of (proposition 2.14) is not true.

Example 2.15:

The operator $T=\begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}$ on two dimensional Hilbert space $\mathbb{C}^2$ is quasi-normal operator of order 2 ,but

$T(T^*T)=\begin{pmatrix} 2i & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} i & 0 \\ -1 & i \end{pmatrix}=(T^*T)T$

Therefor T is not quasinormal operator.

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الموثر شبه القياسي من الرتبة n

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الملخص

في هذا البحث , قدمنا نوع جديد من المؤثرات المعرفة على فضاء هيلبرت H والتي تسمى المؤثر شبه القياسي من الرتبة n . يقال للمؤثر T بانه مؤثر شبه قياسي من الرتبة n اذا كان $T(T^{*n}\,T^n)=(T^{*n}\,T^n)T$ حيث n عدد صحيح موجب اكبر من واحد و T^* هو المؤثر المجاور للمؤثر T. قدمنا بعض الخواص الاساسية لهذا المؤثر و درسنا العلاقات بين بين المؤثر شبه قياسي من الرتبة n ومع بعض المؤثرات الاخرى.

الكلمات المفاتيحية: المؤثرات شبه القياسية ,المؤثرات القياسية , المؤثرات n القياسية .