The Open Limit Point Compactness

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Abstract

In this paper, we gave a new topological concept and we called it the open limit point compactness.We have proved that each of the compactness, and the limit point compactness is stronger than of the open limit point compactness., that is, compactness implies open limit point compactness, also limit point compactness implies open limit point compactness, but the converse is not true. Also we have shown that the continuous image of an open limit point compact is an open limit point compact and so this property is a topological property .This property is not a hereditary property. Connected spaces are open limit point spaces. The one-point compactification of a space X is an open limit point compact. Finally, we have shown that if $X \times Y$ is an open limit point compact, then each of X, and $Y$is an open limit point compact.

keywords: product space, limit point, compact, limit point compact.

Introduction

” Every bounded infinite set of real numbers has a limit point”. It is the “Bolzano-Weierstrass Theorem”. Firstly it was proved by Bolzano in 1817 as a Lemma in the proof of the intermediate value theorem,then fifty years later the result was identified as significant in its own right, and proved again by Weierstrass.In 1906, Frechet used this property when he generalized Weierstrass's theorem to topological spaces. We now know this property as the Bolzano-Weierstrass property, or the limit point compactness [4]

This paper gave a new topological concept which we called it the open limit point compactness.The section one included the fundamental topological concepts such as, topological property, hereditary property, and limit point compactness. In section two, we gave the main results of this paper which are a new topological concepts, an open limit point compactness.

1. Fundamental Concepts

1.1 Definition[2].

A property of a topological space $(X, \mathcal{T})$ is said to be a topological

property if any a topological space $(Y, \mathcal{S})$ which is homeomorphic to $(X, \mathcal{T})$ has that property.

1.2 Definition[3].

A property of a topological space $(X, \mathcal{T})$ is said to be a hereditary property if any subspace of $(X, \mathcal{T})$ has that property.

1.3 Theorem[2].

A subset of a topological space is closed if, and only if, it contains all its limit points.

1.4 Definition[1].

A space X is said to be a limit point compact if every infinite subset of X has a limit point.

1.5 Examples.

i)The real numbers $\mathbb{R}$ with the cofinite topology is a limit point compact.

ii)The real numbers $\mathbb{R}$ with the usual topology $(\mathbb{R}, \mathcal{T}_u)$ is not a limit point compact.

1.6 Definition[2].

A space X is said to be compact if every open cover for X has a finite subcover.

1.7 Definition[2].

A space X is said to be Hausdorff space if for any two elements $x, y \in X$, there exist disjoint open sets containing them.

1.8 Definition[5].

A compact Hausdorff space Y is called a compactification of a space X, if there is an embedding $f: X \longrightarrow Y$ such that $f(X)$ is dense in Y. If $Y \backslash X$ is a singleton set then Y is called a one-point compactification of X.

1.9 Theorem[1].

A one-point compactification of a space X is compact.

2. The Main Results

2.1 Definition.

A topological space $(X, \mathcal{T})$ is said to be an open limit point compact space if, and only if, any an infinite proper open subset of X has a limit point. A subset A of X is said to be an open limit point compact if, and only if, any an infinite proper open subset of A has a limit point in A.

2.2 Examples:

i) The cofinite space $(X, \mathcal{T}_c)$; X is an infinite set; is an open limit point compact space:

Any an open set of X is of the form $X \backslash F$,where F is a finite set.Since $X \backslash F$ is an infinite set, then every element of X is a limit point of $X \backslash F$.

ii) $(\mathbb{R}, \mathcal{T}_u)$ is an open limit point compact space.

iii) Consider the space of the integer numbers Z with the topology that generated by

$$B = \{(-n, n) \mid n \in Z^+\}$$

Where $(-n, n) = \{m \in Z, -n < m < n\}$.

This space is an open limit point compact space for every non-empty subset of Z has a limit point.[1]

iv) A discrete topological space $(X, \mathfrak{D})$ of more than one point is not an open limit point compact space.

2.3 Theorem.

The continuous image of an open limit point compact is an open limit point compact.

Proof.

Let $(X,\mathcal{T})$ be an open limit point compact space, $(Y,\mathcal{S})$ be a space, and $f: X \longrightarrow Y$ be a continuous function.Let G be an infinite proper open subset of $f(X)$.Then $f^{-1}(G)$ is an infinite proper open subset of X.

Since $(X,\mathcal{T})$ is an open limit point compact space, then $f^{-1}(G)$ has a limit point, say x_0. We claim that $f(x_0)$ is a limit point of G:

Let V be an open set of $f(X)$ containing $f(x_0)$ .Then

$(V \setminus \{f(x_0)\}) \cap G$

$\supseteq f\{f^{-1}((V \setminus \{f(x_0)\}) \cap G)\}$
$= f\{(f^{-1}(V \setminus \{f(x_0)\})) \cap f^{-1}(G)\}$

$= f\{(f^{-1}(V) \setminus f^{-1}(\{f(x_0)\})) \cap f^{-1}(G)\}$
$= f\{(f^{-1}(V) \setminus \{x_0\}) \cap f^{-1}(G)\}$

$\neq \phi$, for x_0 is limit point of $f^{-1}(G)$.

So $f(x_0)$ is a limit point of G. Hence $f(X)$ is open limit point compact.

2.4 Corollary.

Open limit point compactness property is a topological property.

2.5 Remark.

An open limit point compact is not a hereditary property:

$(\mathbb{R},\mathcal{T}_u)$ is an open limit point compact but Z as a subspace of $(\mathbb{R},\mathcal{T}_u)$ is not an open limit point compact.

2.6 Theorem.

A limit point compact space is an open limit point compact space.

Proof. It's clear.

2.7 Theorem.

A compact space is an open limit point compact space.

Proof.

Let $(X,\mathcal{T})$ be a compact pace.

Suppose that $(X,\mathcal{T})$ is not an open limit point compact.

This means that there exists an infinite proper open subset of X, say G, which has no a limit point.Then each $x \in G$ is not a limit point of G.

So for each $x \in G$, there exists an open set U_x containing x such that $U_x \cap G = \{x\}$ .Note that G is a closed set .Then G^c is an open set.

Now,

$\{U_x\}_{x \in G} \bigcup \{G^c\}$ is an open cover of X.But this cover has no a finite subcover for each U_x contains one element of G and $G \bigcap G^c = \phi$.This is a contradiction.

Hence $(X,\mathcal{T})$ must be an open limit point compact.

2.8 Remark.

The converse of above theorems is not true. For example $(\mathbb{R},\mathcal{T}_u)$ is an open limit point compact but it is neither compact nor limit point compact.

2.9 Theorem.

Every a connected space is an open limit point compact space.

Proof. Let $(X,\mathcal{T})$ be a connected space.

Suppose that $(X,\mathcal{T})$ is not an open limit point compact.

This means that there exists an infinite proper open subset of X, say G, which has no a limit point. So G is a closed set and this implies that $(X,\mathcal{T})$ is disconnected, which is a contradiction.

Hence $(X,\mathcal{T})$ must be an open limit point compact.

2.10. Theorem.

If $X \times Y$ is an open limit point compact, then each of X, and Y is an open limit point compact.

Proof:

Since the projection functions

$$p_X : X \times Y \to X \text{ ,and } p_y : X \times Y \to Y$$

are continuous onto functions, then each of X, and Y is an open limit point compact (Theorem 2.3).

2.11. Theorem.

A one-point compactification of a space X is an open limit point compact.

Proof:

Since a one-point compactification of a space X is compact [Theorem 1.9] , then it is an open limit point compact [Theorem 2.7].

References

[1] Adams C., and Franzosa R., "***Introduction to Topology Pure and Applied***", Person Printice Hall, (2008).

[2] Munkres J. R, "***Topology***", Person Prentice Hall, Second Edition, (2009).

[3] Morris S. A. , "***Topology***", Without Tears, Springer, New York, (2007).

[4] Raman M., "***A Pedagogical History of Compactness***", arXiv: 1006.4131v1 , [math.HO] 21 Jun 22, (2010).

[5] Swamy U. M., Santhi Sundar Raj Ch., Venkateswarlu B., and Ramesh

S., "***Compactifications of an infinite discrete space***", Int. J. Contemp. Math. Sciences, Vol. 4, no. 22, 1079 – 1084, (2009).