Partial Integro-Differential Equations: Classification & Solutions

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Abstract:

The main purpose of this paper is divided into two parts: First is the classification of the partial integro-differential equation(PIDE).Second is the finding the approximate solution for this equation by using Least- square and Collocation methods.

1-Introduction:

The theory and application of integro- differential equation play an important role in the mathematical modeling of many fields: physical, biological phenomena and engineering sciences in which it is necessary to take into account the effect of real world problems.

The advantage of the integro- differential equations representation for a variety of problem is witnessed by its increasing frequency in the literature and in many texts on method of advanced applied mathematics. Also, the suitability of the solution method for machine computation, and the inherent simplicity of the structure of the subject, combine to make the integro-differential equation approach a very valuable one for many applications[4,5].

Definition & classification of integro- differential equation with illustrative examples are given.

An integro-differential equation(IDE) is defined as a functional equation involving unknown function f(x). together with both differential and integral operations on f [4&6].

Remark: If the derivative is always taken with respect to one variable, the integro- differential equation is called ordinary. Other integro-differential equations, on the contrary, which often occur in the mathematical physics, contain derivatives with respect to different variables are called partial integro- differential equations [4].

The classification of the IDE is giving in the following sections.

2- Ordinary integro- differential equations(OIDEs)

2.1 Linear ordinary integro- differential equations:

The general form of the linear ordinary integro- differential equation is:

$$h(x)\frac{\partial f}{\partial x} = g(x) + \lambda \int_{a}^{b(x)} k(x,y)\, f(y)dy \ldots (1)$$

where h and g are known function of x, k(x,y) is a known function of x and y called the kernal of the integro-differential equation given by eq.(1) and a and λ are known parameters and f is the unknown function that must be determined.

Next , we classified two types of the linear (OIDEs), namely Fredholm and Voltera types.

2.1.1 Fredholm linear ordinary integro- differential equations:

If the limit of the integral operator in eq. (1) does not depends on x i.e. if b(x)=b then eq.(1) is called Fredholm linear OIDEs. In this case if h(x)=0 then eq.(1)reduces to the following equation:

$$g(x) = \lambda \int_a^b k(x,y) f(y) dy$$

which is called the Fredholm linear OIDEs of the first kind. If h(x)=1 in eq.(1) then eq.(1) becomes.

$$\frac{\partial f}{\partial x} = g(x) + \lambda \int_a^b k(x,y) f(y) dy$$

which is called the Fredholm linear OIDEs of the second kind.
On the other hand if g(x)=0 then eq.(1) takes the form

$$h(x)\frac{\partial f}{\partial x} = \lambda \int_a^b k(x,y) f(y) dy$$

Which is called the Fredholm linear OIDEs of the third kind.

2.1.2 Volterra linear ordinary integro- differential equations:

If b(x)=x, then eq.(1) is called the Volterra linear OIDEs.
i.e

$$h(x)\frac{\partial f}{\partial x} = g(x) + \lambda \int_a^x k(x,y) f(y) dy$$

and like the Fredholm linear OIDEs, the Volterra linear OIDEs can be classified into first, second and third kinds.

1.2 Nonlinear ordinary integro- differential equations:

The general form of the nonlinear OIDE is:

$$h(x)\frac{\partial f}{\partial x} = g(x) + \lambda \int_a^{b(x)} k(x,y,f(y)) dy \cdots (2)$$

and like the linear OIDEs, the nonlinear OIDEs can be classified into Fredholm, Volterra of the first, second and third kinds.
The OIDEs have been studies by many authors, see for example[4&6].

2- Partial integro- differential equation(PIDE)

The partial integro- differential equation (PIDE) is an integro- differential equation such that the unknown function depends on more than one independent variable like the OIDEs, the partial integro-differential equations (PIDEs) is divided into linear and nonlinear.

2.1 Linear partial integro- differential equations:

The general form of the linear PIDE is:

$$h(x,y)\frac{\partial f(x,y)}{\partial x} = g(x,y) + \lambda \int_a^{b(x)} \int_c^{d(y)} k(x,y,z,m) f(z,m)dzdm \cdots (3)$$

Also if the upper limits of integral sign (b(x)&d(y))in eq.(3) do not depend on x and y respectively, then eq.(3) is called Fredholm linear PIDE. Moreover, if b(x)=x and d(y)=y then eq. (3) is called Volterra linear PIDE.
On the other hand, the Fredholm or Volterra linear PIDEs may be of the first, second or third kinds.

2.2 Nonlinear partial integro- differential equations:

The general form of the nonlinear PIDE is :

$$h(x,y)\frac{\partial f(x,y)}{\partial x} = g(x,y) + \lambda \int_a^{b(x)} \int_c^{d(y)} k(x,y,z,m) f(z,m)dzdm \cdots (4)$$

And like the linear PIDEs, the nonlinear PIDEs can be classified into Fredholm, Volterra of the first, second and third kinds.
Now, we shall light on the solution of the Fredholm first order PIDEs

$$\frac{\partial f(x,y)}{\partial x} = g(x,y) + \lambda \int_0^1 \int_0^1 k(x,y,z,m) f(z,m)dzdm \cdots (5)$$

Here, we use the Least-square and Collocation methods to solve eq.(5).

3.Least-square method:

The Least-square method is one of the most common methods used to approximate the solution of the integral equations[2&3].
This method is based on approximation the solution f(x,y) of eq.(5) as a linear combination of the function $B_{ij}(x,y)$ that is write

$$f(x,y) \approx \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}B_{ij}(x,y) \cdots (6)$$

By substituting eq.(6) into eq.(5) one can get

$$\sum\sum a_{ij}\frac{\partial B_{ij}(x,y)}{\partial x} = g(x,y) + \lambda\int_0^1\int_0^1 k(x,y,z,m)\sum\sum a_{ij}B_{ij}(z,m)dzdm + R(x,y) \cdots (7)$$

where R(x,y) is the error in the approximation of eq.(6)

Let $I(a_{ij}) = \int_0^1\int_0^1 [R(x,y)]^2 w(x,y)dxdy$

Where w(x, y) is a positive function defined over the domain of the integration which is usually called the weight function. So, if $I(a_{ij})$ is positive for all function $B_{ij}(x,y)$ except the solution of the integral equation (5) in which case it is zero. The quality of the approximation of the solution f(x,y) can be measured by the smalleness of $I(a_{ij})$, and a decreasing of $I(a_{ij})$ is equivalent to improving the approximation.

Let w(x,y)=1 , $I(a_{ij}) = \int_0^1\int_0^1 [R(x,y)]^2 dxdy$

So,

$$I(a_{IJ}) = \int_0^1\int_0^1 \left[\sum\sum a_{IJ}\frac{\partial B_{IJ}(x,y)}{\partial x} - g(x,y) - \lambda\int_0^1\int_0^1 k(x,y,z,m)\sum\sum a_{ij}B_{IJ}(z,m)dzdm\right]^2 dxdy \cdots (8)$$

So, finding the values of a_{ij} (i=1,…,n, j=1,…,m) which minimize I gives the best approximation to the solution of PIDE of the form of eq.(5). The values of a_{ij} (i=1,…,n, j=1,…,m) which minimize I can given by the solution of the following system.

$$\frac{\partial I}{\partial a_{ij}} = 0 \quad \forall i=1,\ldots,n,\ j=1,\ldots,m$$

$$\left[\int_0^1\int_0^1 \sum\sum a_{ij}\frac{\partial B_{ij}(x,y)}{\partial x} - g(x,y) - \lambda\int_0^1\int_0^1 k(x,y,z,m)\sum\sum a_{ij}B_{ij}\,j(z,m)dzdm\right] *$$

$$\left\{\frac{\partial^2 B_{ij}(x,y)}{\partial a_{rs}\partial x} - \lambda\int_0^1\int_0^1 k(x,yz,m)B_{rs}(z,m)dzdm\right\}dxdy = 0 \qquad (9)$$

$$\Rightarrow \int_0^1\int_0^1 R(x,y)\left\{\frac{\partial^2 B_{rs}(x,y)}{\partial a_{rs}\partial x} - \lambda\int_0^1\int_0^1 k(x,yz,m)B_{rs}(z,m)dzdm\right\}dxdy = 0 \qquad (10)$$

The above system of the non linear equations can be solved by using MathCad software package to find the values of $\{aij\}_{i,j=1}^{nm}$

To illustrate Least-square method consider the following example:

Example(1): Consider the first order PIDE [7]

$$\frac{\partial f}{\partial x}(x,y) = -y - x - \frac{1}{6} + \int_0^1\int_0^1 (x+y+z+m)\, f(z,m)dzdm \qquad (11)$$

Approximate the solution of this example by

$f(x, y)= a_1+ a_2x+a_3y+a_4xy$

substituting this solution into eq.(10) one can get

$$\int_0^1\int_0^1 \left[(a_2 + a_4 y) - (y+x) + \frac{1}{6} - \int_0^1\int_0^1 (x+y+z+m)(a_1 + a_2 z + a_3 m + a_4 zm)\right]*$$
$$\left[\frac{\partial}{\partial a_i}(a_2 + a_4 y) - \int_0^1\int_0^1 (x+y+z+m)\frac{\partial}{\partial a_i}(a_1 + a_2 z + a_3 m + a_4 zm)dzdm\right]dxdy = 0 \cdots\cdots\cdots\cdots\cdots (12)$$

$i = 1,2,3,4$

After simple computation

$$\frac{25}{6}a_1 + (a_2 + 9a) + \frac{1}{8}a_4 = \frac{15}{6}$$

$$3a_1 + \frac{1}{12}(7a_2 + 19a_3) - \frac{1}{6}a_4 = \frac{13}{6}$$

$$27a_1 + \frac{1}{12}(19a_2 + 175a_3) + \frac{5}{6}a_4 = \frac{97}{6}$$

$$\frac{1}{2}a_1 - (14a_2 + 2a_3) + \frac{5}{24}a_4 = \frac{-4}{3}$$

solving the above system by MathCad software package with initial value $(a_1, a_2, a_3, a_4) = (1,1,0,1)$, we get:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -.923 \\ 0 \\ 2.923 \\ -1.846 \end{pmatrix}$$

4.Collocation method :

The collocation method is also one of then most important methods that can be used to approximate the solution of the integral equation[1&3].

Here, we use this method to solve the PIDE(5).This method is starts by considering eq.(7) and requiring that the error defined in eq.(7) vanishes. To do so, this method requires nm conditions by assuming that the error giving in eq.(7) vanishes at them. In other word, choose nm points say $(x_k, y_k)_{k=1}^{nm}$ in the domain D where D={(x,y):0≤ x≤1, 0≤ y≤ 1} such that R(x, y)=0 and substituting these points into eq.(7) to give the system of the nonlinear equations and use MathCad software package to find the values of $\{aij\}_{i,j=1}^{nm}$

To illustrate collocation method consider the following example:

Example(2):

Recall example(1)

Now, to find a_1, a_2, a_3, a_4 we choose four points in domain D where D={(x,y):0≤ x ≤1, 0≤ y≤ 1} say the points (x_1, y_1) =(0,0), (x_2, y_2)=(1,0), (x_3, y_3)=(1/2,0)and (x_4, y_4) =(1,1) at which the error defined in eq.(7) will be vanished to get the system of the equations.

$$a_1 - \frac{1}{12}(5a_2 - 7a) - \frac{2}{3}a_4 = \frac{1}{6}$$

$$2a_1 + \frac{1}{12}(a_2 + 13a_3) + \frac{7}{12}a_4 = \frac{7}{6}$$

$$3a_1 - \frac{1}{6}(2a_2 - 10a_3) + \frac{11}{12}a_4 = \frac{4}{3}$$

$$3a_1 + \frac{1}{12}(7a_2 + 19a_3) - \frac{1}{12}a_4 = \frac{13}{6}$$

By using MathCad software package to solve above system with initial value (a_1, a_2, a_3, a_4)=(1, 1,0,-1), we get:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0.444 \\ 0.926 \\ 0.189 \\ 0 \end{pmatrix}$$

Conclusion:

The numerical solutions of PIDEs were introduced using several of the numerical techniques. Least-square & Collocation methods were used to treat the Fredholm first order PIDEs and Collocation method is easier in computations than the Least –square method.

References:

1-Abdul J. Jerri ,"Introduction to Integral Equations with Applications", Clarkson University, Potsdam New york 1985.

2-Chambers.L ,"Integral Equation, A short Course", International text Book company limited 1976.

3-Dlves L.M & Mohammad J. ,"Computational Methods for Integral Equations ", Cambridge University Prees 1985.

4-Lakshmikantham,V. and Rao, M.R.M.," Theory of Integro-Differential Equations ",Gordon and Breach publishers 1995.

5-Linz,P.,"Analytical and Numerical Methods for Volterra Equation ",Siam Philadelphia 1985.

6-Shazad A.," Numerical Solutions of Linear Volterra Integro-Differential Equations", Msc thesis,Technology University 2002.

7-Zeena H.," About the Linear and Nonlinear Generalized Eigenvalue Problem ", Msc thesis, Al-Nahrain University 2003.

المعادلات التفاضلية التكاملية الجزئية: تصنيفها و حلولها

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الخلاصة:

الغرض من هذا البحث هو اولاً تصنيف المعادلات التفاضلية التكاملية الجزئية ومن ثم ايجاد الحلول التقريبية لهذه المعادلات باستخدام طريقة المربعات الصغرى(Least-Square Method) وكذلك باستخدام (Collocation Method) .