On The Stability Of Fixed Points In Topological Spaces

Abedal-Hamza Mahdi Hamza*, and Mohammed Faris Y.**

University of Kufa

Faculty of Mathematics and Computer Sciences

Department of Mathematics

April, 2014

Abstract

In this papers we introduced new concepts of stability: strongly stable, c-stable, and c-strongly stable. We discussed the stability of fixed points with respect to each of a topology and a base for that topology. We obtained that if a fixed point is stable (strongly stable, c-stable, strongly c- stable respectively) with respect to a base for a topology, then it is stable (strongly stable, c-stable, strongly c- stable respectively) with respect to that topology.

Keywords: Fixed points, Stable fixed point, Orbit, Dynamical system, and Convergent sequences.

Introduction

A strong concept of stability for dynamical system were first formulated by Russian scientist N.E. Zhukovskii (1847-1921) [3]. He introduced in 1882 a strong orbital stability concept which is based on a reparametrisation of the time variable [3]. In 1892, the Russian scientist A.M. Lyapunov, defined the concept of stability in his PhD thesis "A general task about the stability of motion" [3]. The work of Lyapunov became famous in Russia and after that in the West. His thesis was translated into French in 1907 , and it is reprinted in Russian in 1950 [3]. An English translation and biography has been published in 1992 [2]. The notation of Lyapunov differs a little from the modern notation. Lyapunov introduces n quantities F_i which are functions of the k trajectories $f_j(t)$ of the positions q_j starting from the unperturbed initial condition q_{j_0}. The quantities Q_i denote these functions for the perturbed trajectories due to perturbations ε_j on the initial position and ε_j' on the initial velocity [3].

1. Preliminaries

1.1 Definition [4]

Let (X,τ) be a topological space. A base B_τ for topology τ is subcollection of τ, such that every nonempty open set in X is a union of some members of B_τ. i.e. :

i) $B_\tau \subseteq \tau$, and

ii) $\forall\, W \in \tau, W \neq \emptyset \implies W = \bigcup_{B_\alpha \epsilon B_\tau} B_\alpha$

1.2 Example

Consider that the topological space (R, τ_u):

τ_u is the usual topology on the real numbers R. $B_{\tau_u} = \{(a,b)\epsilon\tau_u; a,b\epsilon R\}$ is a base for τ_u.

1.3 Definition [5]

Let (X,τ) be a topological space, $\langle a_n\rangle_{n\in Z^+}$ be a sequence in X. Then $\langle a_n\rangle$ is said to be converges to $x_0 \in X$, or x_0 is limit of $\langle a_n\rangle_{n\in Z^+}$ $(a_n \to x_0)$ iff for every open set H of X containing x_0 containing all but finitely many elements of $\langle a_n\rangle$.

1.4 Example

Consider the topological space (R,τ_u), and$\langle a_n\rangle = \langle\frac{1}{n}\rangle, n \in Z^+$.
Let $G = (a,b) \in \tau$ where $0 \in G$. Note that G containing all but finitely many elements of $\langle a_n\rangle$. Hence $a_n \to 0$.

1.5 Definition ([1],[6])

Let $f: X \to X$ be a continuous function The iterates f^n form a group under composition. Then (X,f) or $\{f^n\}_{n\in Z^+ or\ Z}$ is called a discrete dynamical system.

Otherwise if $n \in R$ or any interval of R, then (X,f) or $\{f^n\}_{n\in R}$ is called a continuous dynamical system.

1.6 Example

Consider that the topological space (R,τ_u),

and $f: R \to R$ be a function define by $f(x) = -3x$. The dynamical system define by f is $\{(-3)^n x\}_{n\in Z^+}$.

1.7 Definition [1]

Let (X,τ) be a topological space, and $f: X \to X$ be a continuous function. For all $x \in X$.

The orbit of x under f is the sequence $(x, f(x), f^2(x), \ldots, f^n(x), \ldots)$, and it is denoted by $O(x)$.

1.8 Example

Consider that the topological space (R,τ_u), and $f: R \to R$ be a function define by

$$f(x) = \frac{1}{5}x.$$

$O(0) = (0,0,0,\ldots,0,\ldots)$

$O(1) = (1,\frac{1}{5},\frac{1}{25},\frac{1}{125},\frac{1}{625},\ldots,\left(\frac{1}{5}\right)^n,\ldots)$

$O(-1) = (-1,-\frac{1}{5},-\frac{1}{25},-\frac{1}{125},-\frac{1}{625},\ldots,\left(\frac{1}{5}\right)^n(-1),\ldots)$

$O(2) = (2,\frac{2}{5},\frac{2}{25},\frac{2}{125},\frac{2}{625},\ldots,\left(\frac{1}{5}\right)^n(2),\ldots)$

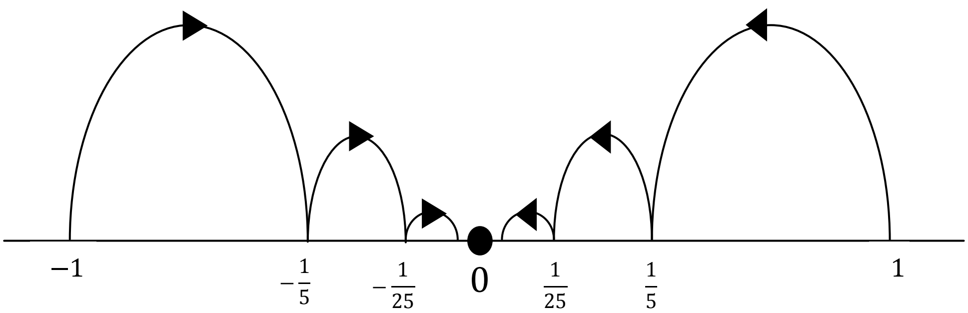

Figure (1): Orbit of 1 and -1 under $f(x) = \frac{1}{5}x$

1.9 Definition [1]

Let (X,τ) be a topological space, $f\colon X \to X$ be a continuous function, and x_0 be a fixed point of f. Then x_0 is called stable if for every open set $U \subseteq X$ containing x_0 there exists an open set $V \subseteq U$ containing x_0 such that, $O(x) \subseteq U \;\; \forall\, x \in V$.Otherwise x_0 is called unstable fixed point.

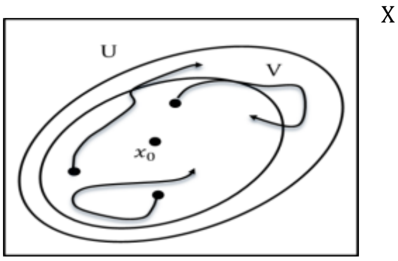

Figure (2): Stable fixed point

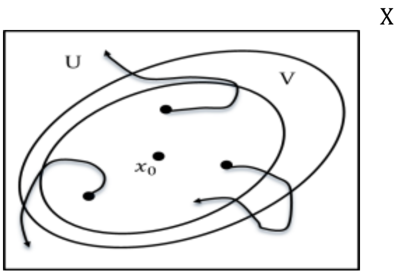

Figure (3): Unstable fixed point

1.10 Example

Consider that the topological space (R,τ_u), and $f\colon R \to R$ be, where

i) $f(x) = \frac{1}{2}x$. The dynamical system define by f is $\left\{(\frac{1}{2})^n x\right\}_{n\in Z^+}$, and 0 is the fixed point of f.

Let $U = (a,b) \in \tau_{\tau_u}$, where $0 \in U$. Choose $V = (-c,c) \in B_{\tau_u}$, where $0 \in V \subseteq U$, $c = \min\{|a|,|b|\}$. Note that

$O(x) \subseteq V \subseteq U \;\; \forall\, x \in V$. Hence 0 is stable.

ii) $f(x) = 3x$. The dynamical system define by f is $\{(3)^n x\}_{n\in Z^+}$ and 0 is the fixed point of f.Let $U = (-1,1) \in \tau_u$, note that $O(x) \not\subset U, x \in V$, $\forall\, V \in \tau_u, 0 \in V \subseteq U$.

Hence 0 is unstable.

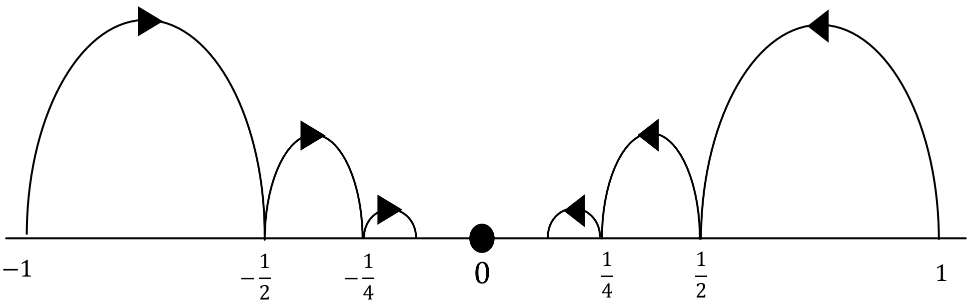

Figure (4): Orbit of 1 and -1 under $f(x) = \frac{1}{2}x$

2. Main Results

2.1 Theorem

Let (X,τ) be a topological space, and B_τ be a base for τ. The sequence $\langle a_n\rangle_{n\in Z^+}$ is convergent with respect to B_τ iff it is convergent with respect to τ.

Proof (⇒)**:** Let $\langle a_n\rangle_{n\in Z^+}$ be a convergent sequence with respect to B_τ, and x_0 be a limit of $\langle a_n\rangle_{n\in Z^+}$. Let H be an open set containing x_0. $H=\bigcup_{\alpha\in\wedge} W_\alpha$ where $W_\alpha\in B_\tau\ \forall\ \alpha\in\wedge$, then $x_0\in\bigcup_{\alpha\in\wedge} W_\alpha \Rightarrow x_0\in W_{\alpha_0}$ for some $W_{\alpha_0}\in B_\tau$. Since W_{α_0} containing all but finitely many elements of $\langle a_n\rangle_{n\in Z^+}$, and $W_{\alpha_0}\subseteq H$. Then H containing all but finitely many elements of $\langle a_n\rangle_{n\in Z^+}$. Hence $\langle a_n\rangle_{n\in Z^+}$ is convergent with respect to τ.

Proof (⇐)**:** let $\langle a_n\rangle_{n\in Z^+}$ be a convergent sequence with respect to τ, and x_0 be a limit of $\langle a_n\rangle_{n\in Z^+}$. Let $H\in B_\tau$ containing x_0. Since H is an open set, so H containing all but finitely many element of $\langle a_n\rangle_{n\in Z^+}$. Hence $\langle a_n\rangle_{n\in Z^+}$ is convergent with respect to B_τ.

2.2 Definition

Let (X,τ) be a topological space, $f:(X,\tau)\to(X,\tau)$ be a continuous function, and x_0 be a stable fixed point. We say that x_0 is strongly stable if there exists an open set G containing x_0 such that $\langle O(x)\rangle\to x_0, \forall\ x\in G$, Otherwise we say that x_0 is not strongly stable.

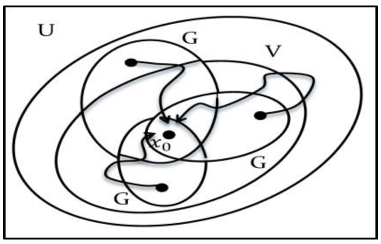

Figure (5): Strongly stable fixed point

2.3 Theorem

Let (X,τ) be a topological space, B_τ is a base for τ, $f:(X,\tau)\to(X,\tau)$ be a continuous function, and x_0 be a fixed point of f. If x_0 is stable with respect to B_τ then x_0 is stable with respect to τ.

Proof: Let x_0 be a fixed point of f, and it is stable with respect to B_τ. Let U an open set containing x_0. Since $U=\bigcup_{\propto\in\wedge} W_\propto$, where $W_\propto\in B_\tau\ \forall\propto\in\wedge$, then $x_0\in\bigcup_{\propto\in\wedge} W_\propto \Longrightarrow x_0\in W_{\propto_0}$ for some $W_{\propto_0}\in B_\tau$. Since is stable with respect to B_τ then $\exists\ V\in B_\tau$ such that $x_0\in V\subseteq W_{\propto_0}$,and $O(x)\subseteq W_{\propto_0}\forall x\in V$. Note that $V\in\tau$ and $W_{\propto_0}\subseteq U$, so $O(x)\subseteq W_{\propto_0}\subseteq U\ \forall x\in V$. Hence x_0 is stable with respect to τ.

2.4 Theorem

Let (X,τ) be a topological space, B_τ is a base for τ, $f:(X,\tau)\to(X,\tau)$ be a continuous function, and x_0 be a fixed point of f. If x_0 is strongly stable with respect to B_τ then x_0 is strongly stable with respect to τ.

Proof: Let x_0 be a fixed point of f, and it is strongly stable with respect to B_τ. From Theorem(2.3) x_0 is stable with respect to τ, suppose that $\exists G\in\tau$, such that $x_0\in G$. Since $G=\bigcup_{\propto\in\wedge} W_\propto$ where $W_\propto\in B_\tau\ \forall\propto\in\wedge$, then $x_0\in\bigcup_{\propto\in\wedge} W_\propto \Longrightarrow x_0\in W_{\propto_0}$ for some $W_{\propto_0}\in B_\tau$. Since $\langle O(x)\rangle\to x_0$ with respect to $B_\tau\ \forall x\in W_\propto\subseteq U$. Hence x_0 is strongly stable with respect to τ.

2.5 Example

Consider the topological space (R,τ_u), and $f:R\to R$ be a function define by

i) $f(x)=\frac{1}{3}x$

The dynamical system define by f is $\left\{\left(\frac{1}{3}\right)^n x\right\}_{n\in Z^+}$ and 0 is the fixed point of f.

Let $U = (a, b) \in B_{\tau_u}$, where $0 \in U$. Choose $V = (-c, c) \in B_{\tau_u}$, where $0 \in V \subseteq U$, $c = \min\{|a|, |b|\}$. Note that $O(x) \subseteq V \subseteq U\ \forall x \in V$. Then; 0 is stable. Let $G = (-\frac{1}{3}, \frac{1}{3}) \in B_{\tau_u}$ where $0 \in G$. Then $\langle O(x)\rangle \to 0\ \forall\ x \in G$. Hence, 0 is strongly stable.

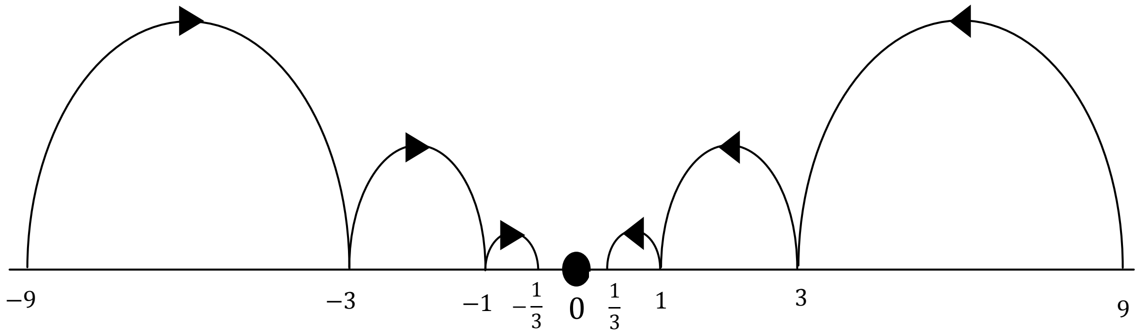

Figure (6): Orbit of 9 and -9 under $f(x) = \frac{1}{3}x$

ii) $f(x) = x^2$

The dynamical system define by f is $\{x^{2^n}\}_{n\in Z^+}$ and $-1, 0,$ and 1 are the fixed points of f.0 is strongly stable and 1 is unstable.To show that 0 is strongly stable and 1 is unstable.Let $U = (a, b) \in B_{\tau_u}$, where $0 \in U$. Let $c = \min\{|a|, |b|\}$. If $c > 1$, $V = (-1,1)$ else if $c \leq 1$, $V = (-c, c)$, where $0 \in V \subseteq U$, Note that $O(x) \subseteq V \subseteq U\ \forall x \in V$. Hence 0 is stable.Let $G = (-1,1) \in B_{\tau_u}$ where $0 \in G$. Then; $\langle O(x)\rangle \to 0\ \forall\ x \in G$. Hence, 0 is strongly stable.Now, let $U = (\frac{1}{2}, 2) \in B_{\tau_u}$. Note that $O(x) \not\subset U\ \forall\ V \in B_{\tau_u}$, $x \in V$, $1 \in V \subseteq U$. Hence 1 is unstable.

iii) $f(x) = -x$

The dynamical system define by f is $\{(-1)^n x\}_{n\in Z^+}$ and 0 is the fixed point of f.

Let $U = (a, b) \in B_{\tau_u}$, where $0 \in U$, choose $V = (-c, c) \in B_{\tau_u}$, where $0 \in V \subseteq U$, $c = \min\{|a|, |b|\}$, such that $O(x) \subseteq V \subseteq U$, $\forall x \in V$. Then 0 is stable.Let $G = (a, b) \in B_{\tau_u}$, where $0 \in G$. Let $x \in G$, and $H = (-x, x) \in B_{\tau_u}$ where $0 \in H$.

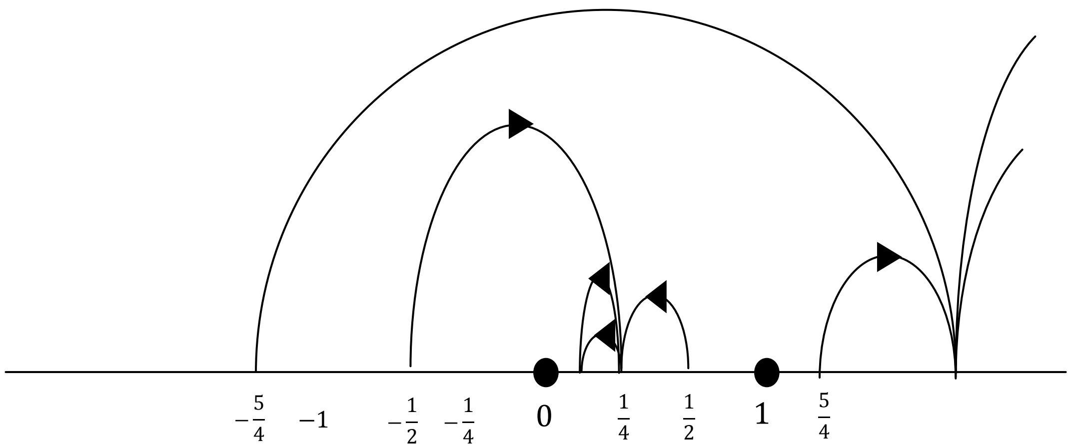

Figure (7): Orbit of $\frac{1}{2}, -\frac{1}{2}, \frac{5}{4}$ and $-\frac{5}{4}$ under $f(x) = x^2$

Note that H does not contain any element of $O(x)$, so $\langle O(x)\rangle \nrightarrow 0\ \forall\, x \in G$. Hence 0 is not strongly stable.

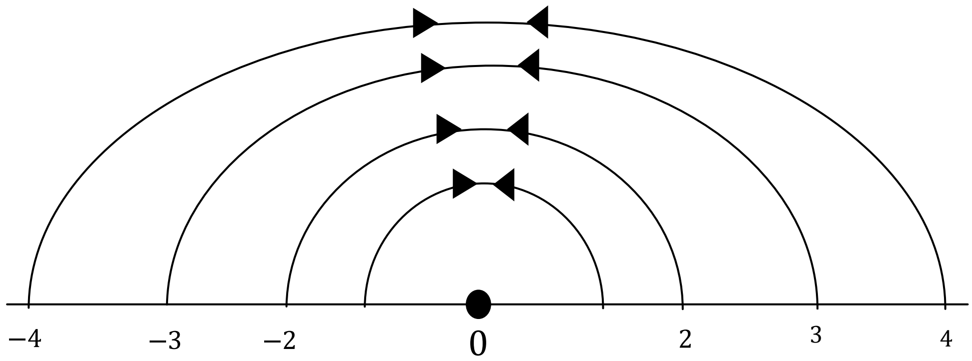

Figure (8): Orbit of 2, 3 and 4 under $f(x) = -x$

2.6 Definition

Let (X, τ) be a topological space, $f: X \to X$ be a continuous function, and x_0 be a fixed point of f. Then we say that x_0 is c-stable if for every open set $U \subseteq X$ containing x_0 there exists an open set $V \subseteq U$ containing x_0 such that, $O(x) \subseteq \overline{U}\ \ \forall\, x \in V$.

Otherwise we say that x_0 is not c-stable fixed point.

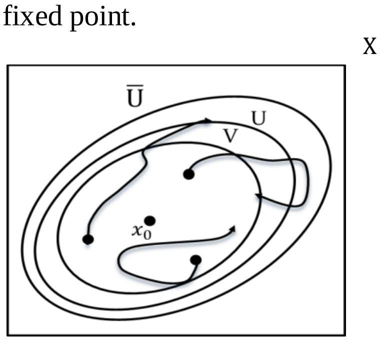

Figure (9): c-stable fixed point

2.7 Definition

Let (X, τ) be a topological space, $f: (X, \tau) \to (X, \tau)$ be a continuous function, and x_0 be a c-stable fixed point. We say that x_0 is strongly c-stable if there exists an open set G containing x_0 such that $\langle O(x)\rangle \to x_0\ \ \forall\, x \in G$.Otherwise we say that x_0 is not strongly c-stable.

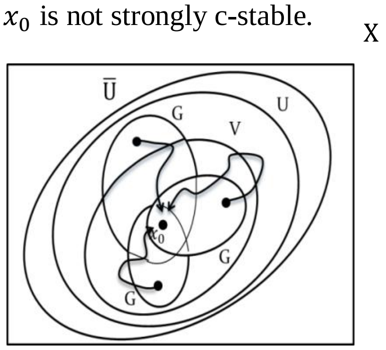

Figure (10): strongly c- stable fixed

2.8 Theorem

Let (X, τ) be a topological space, B_τ is a base for τ, $f: (X, \tau) \to (X, \tau)$ be a continuous function and x_0 is a fixed point of f. If x_0 is stable then it is c-stable.

Proof: Since $U \subseteq \overline{U}$, then $O(x) \subseteq \overline{U}\ \ \forall x \in V$ by definition (1.10).

2.9 Theorem

Let (X, τ) be a topological space, B_τ is a base for τ, $f: (X, \tau) \to (X, \tau)$ be a continuous function and x_0 is a fixed point of f. If x_0 is c-stable with respect to B_τ then x_0 is c-stable with respect to τ.

Proof: Let x_0 be a fixed point of f, and it is c-stable with respect to B_τ. Let U an open set containing x_0. Since $U = \bigcup_{\propto \in \wedge} W_\propto$, where $W_\propto \in B_\tau$ $\forall \propto \in \wedge$, then

$x_0 \in \bigcup_{\propto \in \wedge} W_\propto \implies x_0 \in W_{\propto_0}$ for some $W_{\propto_0} \in B_\tau$. Since is stable with respect to B_τ then $\exists\, V \in B_\tau$ such that $x_0 \in V \subseteq W_{\propto_0}$,and $O(x) \subseteq W_{\propto_0} \forall x \in V$. Note that $V \in \tau$ and $W_{\propto_0} \subseteq U \in \bar{U}$, so $O(x) \subseteq W_{\propto_0} \subseteq \bar{U}\ \forall x \in V$. Hence x_0 is c-stable with respect to τ.

2.10 Theorem

If x_0 is strongly c- stable with respect to B_τ then x_0 is strongly c- stable with respect to τ.

Proof: Let x_0 be a fixed point of f, and it is strongly c- stable with respect to B_τ. From Theorem(2.9) x_0 is c-stable with respect to τ, suppose that $\exists G \in \tau$, such that $x_0 \in G$. Since $G = \bigcup_{\propto \in \wedge} W_\propto$ where $W_\propto \in B_\tau$ $\forall \propto \in \wedge$, then $x_0 \in \bigcup_{\propto \in \wedge} W_\propto \implies x_0 \in W_{\propto_0}$ for some $W_{\propto_0} \in B_\tau$. Since $\langle O(x)\rangle \to x_0$ with respect to B_τ $\forall x \in W_\propto \subseteq U$. Hence x_0 is strongly c-stable with respect to τ.

2.11 Example

Consider the topological space (R, τ_u), and $f: R \to R$ be a function define by

i) $f(x) = \frac{1}{4}x$

The dynamical system define by f is $\left\{(\frac{1}{4})^n x\right\}_{n \in Z^+}$, and 0 is the fixed point of f.

Let $U = (a, b) \in B_{\tau_u}$, where $0 \in U$. Choose $V = (-c, c) \in B_{\tau_u}$, where $0 \in V \subseteq U$, $c = \min\{|a|, |b|\}$. Note that $O(x) \subseteq V \subseteq \bar{U}\ \forall x \in V$. Then 0 is c-stable.

Let $G = (-\frac{1}{4}, \frac{1}{4}) \in B_{\tau_u}$ where $0 \in G$.Then $\langle O(x)\rangle \to 0\ \forall\, x \in G$.

Hence 0 is strongly c- stable.

ii) $f(x) = -2x$

The dynamical system define by f is $\{(-2)^n x\}_{n \in Z^+}$, and 0 is the fixed point of f.

Let $U = (-1,1) \in B_{\tau_u}$. Note that $O(x) \not\subset U$ $\forall\, V \in B_{\tau_u}, x \in V, 0 \in V \subseteq U$.

Hence 0 is not c-stable fixed point.

iii) $f(x) = 1 - x$

The dynamical system define by f is $\{\begin{cases} 1 - x & if\ \ n\ odd \\ x & if\ \ n\ even \end{cases}\}_{n \in Z^+}$, and $\frac{1}{2}$ is the fixed point of f.

Let $U = (a, b) \in B_{\tau_u}$ where $\frac{1}{2} \in U$. Let $c = \left|\frac{1}{2} - a\right|$, and $d = \left|b - \frac{1}{2}\right|$. If $c \leq d$, choose $V = (\frac{c}{2}, \frac{1}{2} + r) \in B_{\tau_u}$ where $r = \left|\frac{1-c}{2}\right|$, if $c > d$, choose $V = (\frac{1}{2} - l, \frac{d}{2}) \in B_{\tau_u}$ where $l = \left|\frac{1-d}{2}\right|$, where $\frac{1}{2} \in V \subseteq U$. Note that $O(x) \subseteq V \subseteq \bar{U}\ \ \forall\, x \in V$. Then $\frac{1}{2}$ is c-stable.

Let $G = (a, b) \in B_{\tau_u}$, where $\frac{1}{2} \in U$. Let $x \in G$, and $H = (-x, x) \in B_{\tau_u}$ where $\frac{1}{2} \in H$.

Note that H does not contain any element of $O(x)$, so $\langle O(x)\rangle \nrightarrow \frac{1}{2}\ \forall\, x \in G$.

Hence $\frac{1}{2}$ is not strongly c- stable.

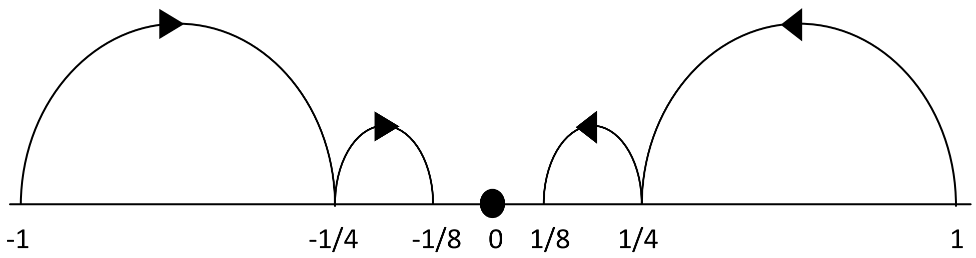

Figure (11): Orbit of 1 and -1 under $f(x) = \frac{1}{4}x$

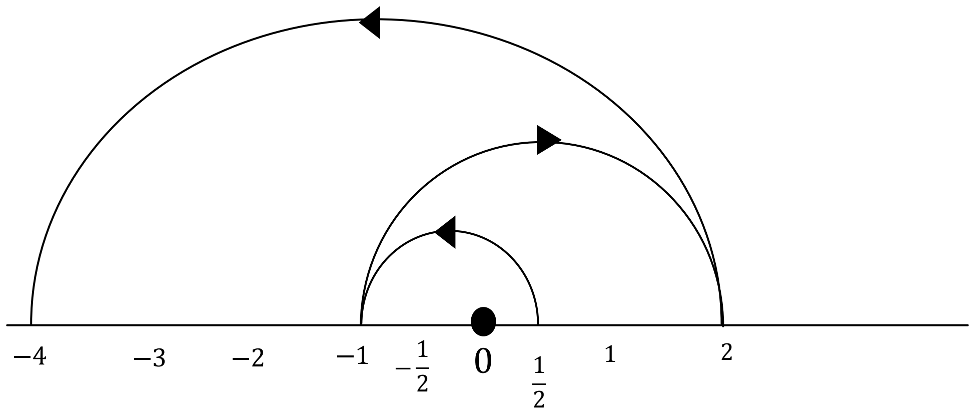

Figure (12): Orbit of $\frac{1}{2}$ under $f(x) = -2x$

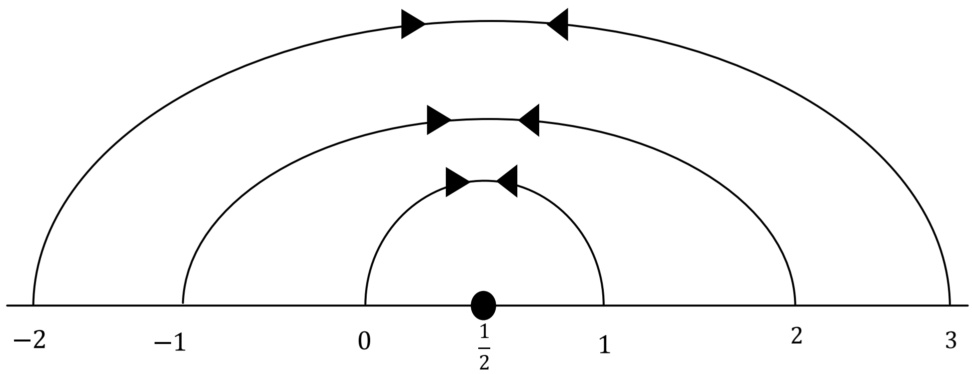

Figure (13): Orbit of $1, 2$, and 3 under $f(x) = 1 - x$

References

[1] Adam C. and Franzosa S., “*Introduction to Topology Pure and Applied*”, Pearson Prentice Hell, (2008).

[2] Fuller A.T. "*Lyapunov centenary issue*", *Int. J. Control 55(3), 521–527* (1992).

[3] Leine R. I., "*The historical development of classical stability concepts: Lagrange, Poisson and Lyapunov stability*", *Springer Science + Business Media B.V.* (2009).

[4] Munkers J. R., "*topology*", 2ed Ed., PHI, (2009).

[5] Sharma J. N., "*topology*", Published by Krishna Prakashan, Mandir Meerut printed at Manoj printers, Meerut, (1977).

[6] Sharma P. , and Nagar, A. , *Topological dynamics on hyperspaces*, Applied General Topology, Volume 11, No. 1, (2010).

* Email: abdulhamza.hamza@uokufa.edu. iq

** Email: mfymath@gmail.com