Pre-Open Sets In Minimal Bitopological Spaces

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Abstract

Let (X,τ) be a topological space, let M be an M-structure on X then (X,τ,M_X) is called a minimal bitopological space. In this work, I am study pre-open sets in minimal bitopological spaces with some result and definitions separation axioms on minimal bitopological with study some fundamental of their properties.

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1- Introduction

Bitopological space initiated by Kelly [9] is defined as follows. A set X equipped with two topologies τ,σ (denoted by (X,τ,σ)) is called a bitopological space. The notion of pre-open set was introduced by Mashhour et al. [1]. In 2005 Alaa Erees have studied Pre-open set in Bitopological Spaces [6] . In 2000, V. Popa and T. Noiri [18] introduced the concept of minimal structure space. They also introduced the concepts of m_x-open set and m_x-closed set and characterize those sets using m_x-closure and m_x-interior operators, respectively. The notion of biminimal structure space was introduced by C.Boonpok [3] in 2010. Also he studied $m_x^1m_x^2$-closed sets and $m_x^1m_x^2$-open sets in biminimal structure spaces. In this paper, the part two introduce the minimal bitopological space and some basic definitions I need in this paper , part three some result of pre-open sets in minimal bitopological spaces , part four introduce the separation axioms in minimal bitopological spaces and given some theorems about them . If there is no confusion, we will write M instead ofM_X.

2- Minimal Structures

In this section, we introduce the M-structure and define some important subsets associated to the M-structure and the relation between them.

Definition 2.1. [7]

Let X be a nonempty set and let $M \subseteq P(X)$, where $P(X)$ denote the set of power X. We say that M is an M-structure (or a minimal structure) on X, if $\emptyset$ and X belong to M.

The members of the minimal structure M are called M-open sets, and the pair (X, M) is called an M-space. The complement of an M-open set is called M-closed. Given $A \subseteq X$, we define M-interior of A abbreviate $Int_M(A)$ as $\cup\{W | W \in M, W \subseteq A\}$ and the M-closure of A abbreviate $Cl_M(A)$ as $\cap\{F | A \subseteq F, X \setminus F \in M\}$. An immediate consequence of the above Definition is the following theorem.

Definition 2.2. [7]

A minimal structure M on a nonempty set X is said to have property B if the union of any family of subsets belonging to M belongs to M.

Observe that any collection $\emptyset \neq \mathcal{J} \subseteq P(X)$, always is contained in an M-structure that have the property of B, as we know, $M(\mathcal{J}) = \{\emptyset, X\} \cup \{\cup_{N \in J} N : \emptyset \neq J \subseteq \mathcal{J}\}$. In particular, when $\mathcal{J} = M$, we denote by $M' = M(\mathcal{J})$. Clearly $M' = M$, if M have the property of B. Note that if M is an M-structure and $Y \subseteq X$, then $\{N \cap Y : N \in M\}$ is an M-structure on X, and is denoted by $M|Y$, and the pair $(Y, M|Y)$ is called an M-subspace of (X, M).

Lemma 2.3. [16]

Let X be a nonempty set and M an M-structure on X satisfying property B. For a subset A of X, the following properties hold:

(1) $A \in M$ if and only if $Int_M(A) = A$.

(2) A is M-closed if and only if $Cl_M(A) = A$.

(3) $Int_M(A) \in M$ and $Cl_M(A)$ is M-closed

Theorem 2.4. [2]

Let (X, M) be an M-space and A a subset of X. Then $x \in Cl_M(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in M$

containing x . And satisfying the following properties:

1- $Cl_M(Cl_M(A)) = Cl_M(A)$.
2- $Int_M(Int_M(A)) = Int_M(A)$.
3- $Int_M(X \backslash A) = X \backslash (Cl_M(A)$.
4- $Cl_M(X \backslash A) = X \backslash (Int_M(A))$.
5- If $A \subseteq B$ then $Cl_M(A) \subseteq Cl_M(B)$.
6- $Cl_M(A) \cup Cl_M(B) \subseteq Cl_M(A \cup B)$.
7- $A \subseteq Cl_M(A)$ *and* $Int_M(A) \subseteq A$.

Definition 2.5. [4]

Let (X, M) be an M-space. We say that $A \subseteq X$ is M-pre-open set if $A \subseteq Int_M(Cl_M(A))$. Also we say that $A \subseteq X$ is M-pre-closed if $X \setminus A$ is M-preopen.

Definition 2.6.

Let (X, M) be an M-space and $B \subseteq X$.The M-pre-closure of B denoted by $pCl_M(B)$ is defined to be the intersection of all M-pre-closed sets of (X, M) containing B.

We can observe that the M-pre-closure of a subset B of X satisfies the following properties:

(i) $pCl_M(\emptyset) = \emptyset$.
(ii) $pCl_M(X) = X$.
(iii) If $A \subseteq B$ then $pCl_M(A) \subseteq pCl_M(B)$.
(iv) If $\emptyset \neq B \neq X$. Then $pCl_M(B)$ is not necessarily an M-pre-closed set.
(v) $pCl_M(X \backslash A) = X \backslash pInt_M(A)$.
(vi) $pInt_M(X \backslash A) = X \backslash pCl_M(A)$.

Lemma 2.7.

Let (X, M) be a minimal space . Let A be M-open in X and let $B \subseteq X$ then $A \cap Cl_M(B) \subseteq Cl_M(A \cap B)$.

Proof:-

Let $x \in A \cap Cl_M(B)$. Hence $x \in A$ and $x \in Cl_M(B)$, since $x \in Cl_M(B)$ then $U \cap B \neq \emptyset$ for every $U \in M$ containing x (by theorem 2.4), since A M-open and $x \in U$, $x \in A$ then $A \cap U \neq \emptyset$, so $A \cap U \cap B \neq \emptyset$. Hence $U \cap (A \cap B) \neq \emptyset$ for every $U \in M$ containing x . Therefore ; $x \in Cl_M(A \cap B)$.

Lemma 2.8.

Let (X, M) be a minimal space, $M_Y = \{W \cap Y : W \in M\}$

and let $A \subseteq Y$, then $cl_{M_Y}(A) = cl_M(A) \cap Y$.

Proof:-

Let $x \in cl_{M_Y}(A)$. Hence $x \in Y$, let W be M-open in X containing x. Now $W \cap Y \in M_Y$ also $x \in W \cap Y$, hence $W \cap Y \cap A \neq \emptyset$ which means that $W \cap A \neq \emptyset$ there fore $x \in cl_M(A)$. Hence $x \in cl_M(A) \cap Y$. Therefore; $cl_{M_Y}(A) \subseteq cl_M(A) \cap Y$. Conversely, let $x \in cl_M(A) \cap Y$, let W_1 be M_Y-open in Y containing x. Now $W_1 = W \cap Y$ where W is M-open in X and $x \in W$, but $x \in cl_M(A)$. Hence $W \cap A \neq \emptyset \Rightarrow W \cap Y \cap A \neq \emptyset$, there fore $W_1 \cap A \neq \emptyset$. Hence $x \in cl_{M_Y}(A)$ which means that $cl_M(A) \cap Y \subseteq cl_{M_Y}(A)$. So $cl_{M_Y}(A) = cl_M(A) \cap Y$.

After definition M-structure, recall that some results of M-structure with some properties to interior, closure of M-structure and definition pre-open set on M-structure also with some properties .In [4] different forms of closed sets in m-spaces are introduced and studied. The obtained results are a generalization of many of the results obtained in other article. Now will definition pre-open set in Minimal Bitopological

3- Pre-Open Sets In Minimal Bitopological

In this section, we recall and introduce the basic definitions and facts that we need in this work.

Definition 3.1.

Let X be a non-empty set, τ be a topology on X, let M be a minimal structure on X then the triple (X, τ, M) is called a minimal bitopological space.

Definition 3.2.

Let (X, τ, M) be a minimal bitopological space and A be a sub set of X, A is called pre-open with respect to τ and M if $A \subseteq int_\tau[cl_M(A)]$, where $cl_M(A)$ is the closure of A with respect to M.

Definition 3.3.

The collection of all pre-open sets with respect to τ and M is denoted by $pre.o_M^\tau(X)$.

Definition 3.4.

Let (X, τ, M) be a minimal bitopological space. A subset A of

X is called pre-closed set of X if and only if the complement of A is pre-open set of X , and the collection of all pre-closed sets with respect to τ, M denoted by $pre.C_M^\tau(X)$.

Definition 3.5.

Let (X, τ, M) be a minimal bitopological space, and $x \in X$. A subset N of X is said to be pre-nhd of a point x with respect to τ, M if and only if there exists a pre-open set U with respect to τ, M such that $x \in U \subseteq N$. The set of all pre-nhd of a point x is denoted by $pre_M^\tau - N(x)$.

Remark 3.6.

Let (X, τ, M) be a minimal bitopological space, let $x \in X$, $pre_M^\tau - N(x)$ be the collection of all pre-nhds of x , then:

i.) $pre_M^\tau - N(x) \neq \varphi$.

ii.) If $W_1 \in pre_M^\tau - N(x)$ & $W_1 \subseteq W_2$, then $W_2 \in pre_M^\tau - N(x)$.

iii.) If $W_1 \in pre_M^\tau - N(x)$, then there exists $W_2 \in pre_M^\tau - N(x)$ such that $W_2 \subseteq W_1$ and $W_2 \in pre_M^\tau - N(y)$, for all $y \in W_2$.

Definition 3.7.

Let (X, τ, M) be a minimal bitopological space, and $A \subseteq X$. A point $x \in X$ is said to be pre-interior point of A with respect to τ, M if and only if there exists a pre-open set B such that $x \in B \subseteq A$. The set of all pre-interior points of A with respect to τ, M denoted by $pre_M^\tau - int(A)$.(i.e.)

$pre_M^\tau - int\ (A) = \cup\{G \in pre.o_M^\tau - (X) : G \subseteq A\}$

Definition 3.8.

Let (X, τ, M) be a minimal bitopological space. A point $x \in X$ is said to be a pre-boundary point of a subset A of X with respect to τ, M if and only if for all U pre-open in X with respect to τ, M containing x we have $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$. The set of all pre-boundaries points of A will be denoted by $pre_M^\tau - b(A)$.

Definition 3.9.

Let (X, τ, M) be a minimal bitopological space, and $A \subseteq X$, the intersection of all pre-closed sets containing A is called pre-closure of A, and is denoted by $pre_M^{\tau} - cl(A)$. That is $pre_M^{\tau} - cl(A)$ is the smallest pre-closed set in X containing A, also A is pre-closed if and only if $pre_M^{\tau} - cl(A) = A$.

Definition 3.10.

Let (X, τ, M) be a minimal bitopological space, Y be a subset of X , and let $\tau_Y = \{G \cap Y : G \in \tau\}$, $M_Y = \{H \cap Y : H \in M\}$ then (Y, τ_Y, M_Y) is called a subspace of the minimal bitopological space (X, τ, M) .

Some studies on ideal Bitopological Space introduced Pre-open set on ideal Bitopological and definition pre-interior, pre-closure. Hence main result on ideal Bitopological space as obtained by [10]. Now will give some result on Minimal Bitopological Space.

4- Some result of Minimal Bitopological

In this section given some result on Minimal Bitopological Space and subspace of Minimal Bitopological Space.

Proposition 4.1.

Let (X, τ, M) be a minimal bitopological space then the union of any two pre-open sets is pre-open.

Proof:-

Let A and B be two pre-open sets with respect to τ, M . Now $A \subseteq int_{\tau}(cl_M(A))$ and $B \subseteq int_{\tau}(cl_M(B))$, $A \cup B \subseteq int_{\tau}(cl_M(A)) \cup int_{\tau}(cl_M(B)) \subseteq int_{\tau}(cl_M(A) \cup cl_M(B))$, Since $cl_M(A) \cup cl_M(B) \subseteq cl_M(A \cup B)$ by theorem [2.4] . Then $A \cup B \subseteq int_{\tau}(cl_M(A \cup B))$. Hence $A \cup B$ is pre-open with respect to , M .

Theorem 4.2.

Let (X, τ, M) be a minimal bitopological space, and A, B be any subsets of X , then:

1. $pre_M^{\tau} - int(X) = X$, $pre_M^{\tau} - int(\emptyset) = \emptyset$.
2. $pre_M^{\tau} - int(A) \subseteq A$.
3. If $A \subseteq B$, then $pre_M^{\tau} - int(A) \subseteq pre_M^{\tau} - int(B)$.
4. $pre_M^{\tau} - int(A \cap B) \subseteq pre_M^{\tau} - int(A) \cap pre_M^{\tau} - int(B)$.

5. $pre_M^\tau - int\,(A) \cup pre_M^\tau - int\,(B) \subseteq pre_M^\tau - int\,(A \cup B)$.

Proof:-

1, 2, 3 is clear. Proof (4). Since $A \cap B \subseteq A$, and $A \cap B \subseteq B$ .Hence $pre_M^\tau - int(A \cap B) \subseteq pre_M^\tau - int(A)$, $pre_M^\tau int\,(A \cap B) \subseteq pre_M^\tau - int(B)$ .Then $pre_M^\tau - int(A \cap B) \subseteq pre_M^\tau - int(A) \cap pre_M^\tau - int\,(B)$. The proof of (5) is similar to the proof of (4).

Theorem 4.3.

Let (X, τ, M) be a minimal bitopological space, and A , B be subsets of X , then:

i. $pre_M^\tau - cl(\emptyset) = \emptyset$, $pre_M^\tau - cl(X) = X$.

ii. $A \subseteq pre_M^\tau - cl(A)$.

iii. If $A \subseteq B$, then $pre_M^\tau - cl(A) \subseteq pre_M^\tau - cl(B)$.

iv. $pre_M^\tau - cl(A \cup B) = pre_M^\tau - cl(A) \cup pre_M^\tau - cl(B)$.

v. $pre_M^\tau - cl(A \cap B) \subseteq pre_M^\tau - cl(A) \cap pre_M^\tau - cl(B)$.

vi. $pre_M^\tau - cl\left(pre_M^\tau - cl(A)\right) = pre_M^\tau - cl(A)$.

Proof:-

I we will prove (v) only the other parts are clear:

$A \cap B \subseteq A$ The $pre_M^\tau - cl(A \cap B) \subseteq pre_M^\tau - cl(A)$ ….(1) also $A \cap B \subseteq B$ the $pre_M^\tau - cl(A \cap B) \subseteq pre_M^\tau - cl(B)$ …….(2) , then $pre_M^\tau - cl(A \cap B) \subseteq pre_M^\tau - cl(A) \cap pre_M^\tau - cl(B)$. The converse of part (v) is not necessarily true as shown by the following example.

Example 4.4.

Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X.\{d\}\}$, $M = \{\emptyset, X, \{a,c\}, \{b,d\}\}$ and $A = \{a,b,d\}$, $B = \{c,d\}$. $A \cap B = \{d\}$ then $pre_M^\tau - cl(A \cap B) = pre_M^\tau - cl(\{d\}) = \{d\}$ and $pre_M^\tau - cl(A) \cap pre_M^\tau - cl(B) = X \cap \{c,d\} = \{c,d\}$ .So $pre_M^\tau - cl(A) \cap pre_M^\tau - cl(B) \nsubseteq pre_M^\tau - cl(A \cap B)$.

Lemma 4.5. [11]

Let (X, τ) be a topological space, let (Y, τ_Y) be a sub space of (X, τ) and let $A \subseteq Y$. Then $Int_\tau(A) = Int_{\tau_Y}(A) \cap Int_\tau(Y)$.

Theorem 4.6.

Let (X, τ, M) be a minimal bitopological space, $Y \subseteq X$, Y open with respect to τ , M and let (Y, τ_Y, M_Y) be a sub space of (X, τ, M) .If G be pre-open in X with respect to τ , M. Then $G \cap Y$ is pre-open in Y with respect to τ_Y , M_Y .

Proof:-

G is pre-open in X with respect to τ, M , then $G \subseteq int_\tau cl_M(G)$. Now $G \cap Y \subseteq int_\tau cl_M(G) \cap Y$.

$G \cap Y \subseteq int_\tau cl_M(G) \cap int_\tau(Y)$ $(int_\tau(Y) = Y$, Y is τ-open)

$\subseteq int_\tau(cl_M(G) \cap Y)$ (By lemma 2.7)

$\subseteq int_\tau(cl_M(G \cap Y))$

$\subseteq int_{\tau_Y}(cl_M(G \cap Y)) \cap int_\tau(Y)$ (By lemma 4.5)

$\subseteq int_{\tau_Y}(cl_M(G \cap Y)) \cap Y$

$\subseteq int_{\tau_Y}(cl_M(G \cap Y)) \cap int_{\tau_Y}(Y)$

$\subseteq int_{\tau_Y}(cl_M(G \cap Y) \cap Y)$ (By lemma 2.8)

$G \cap Y \subseteq int_{\tau_Y}\left(cl_{M_Y}(G \cap Y)\right)$. So $G \cap Y$ is pre-open in Y with respect to τ_Y , M_Y .

Corollary 4.7.

Let (X, τ, M) be a minimal bitopological space, let (Y, τ_Y, M_Y) be a sub space of (X, τ, M) and let Y open with respect to τ , M . If G be pre-closed in X with respect to τ, M . Then $G \cap Y$ is pre-closed in Y with respect to τ_Y , M_Y .

Proof:-

To prove by theorem 4.6 . Let G be Pre-closed in X with respect to , M . So G^c is pre-open in X with respect to τ , M by the theorem above $G^c \cap Y$ is pre-open in Y with respect to τ_Y , M_Y , to proof $G \cap Y$ is pre-closed in Y. $Y \subseteq X$, so $= Y \cap X = Y \cap (G \cup G^c) = (Y \cap G) \cup (Y \cap G^c)$. So $G \cap Y$ is pre-closed in Y with respect to τ_Y , M_Y .

In [5] Carlos C., E. R. and M. S., notion of M-operator for an M-structure M_X on a set X is introduced Also several separation forms for points of a set X are described and characterized, in a not necessarily topological context، also studies different relationships between these separation properties, and establish conditions in regards to the operator which determine the equivalence between these separation forms .Now we introduce Separation Axioms in Minimal Bitopological Space.

5- Separation Axioms in Minimal Bitopological space.

In this section, we introduce and study different Separation Axioms on a Minimal Bitopological Space. We also look for the existent relation between this Axioms

Definition 5.1.

Let (X,τ,M) be a minimal bitopological space, then (X,τ,M) is called $pre-T_0$ space if and only if for each pair of points $x,y \in X$, such that $x \neq y$, there exists pre $-$open set G containing x but not y or pre $-$open set H containing y but not x.

Theorem 5.2.

A minimal bitopological space (X,τ,M) is $pr-T_o$-space if and only if Pr_M^τ-closure of distinct points are distinct..

Proof:-

Let $x,y \in X$, $x \neq y$ implies $pr_M^\tau - cl(\{x\}) \neq pre_M^\tau - cl(\{y\})$. Then there exists at least one point z , such that , $z \in pre_M^\tau - cl(\{x\})$ but $Z \notin pre_M^\tau - cl(\{y\})$ Let $x \in pr_M^\tau - cl(\{y\})$, then $pr_M^\tau - cl(\{x\}) \subset pr_M^\tau - cl(\{y\})$ which is a contradiction that $z \notin pr_M^\tau - cl(\{y\})$. Hence $x \in X - pr_M^\tau - cl(\{y\})$ but $pr_M^\tau - cl(\{y\})$ is $pre-$closed, so $X - pr_M^\tau - cl(\{y\})$ is $pre-$open which contains x but not y. It follows that (X,τ,M) is $pr-T_o$-space.

Conversely, since (X,τ,M) is $pr-T_o$-space, then for each two distinct points x, $y \in X$, $x \neq y$ there exists $pre-$open set G such that $x \in G$, $y \notin G$. $X-G$ is $pr-$closed set which does not contain x but contains y. By Definition (3.9) ,

$pre_M^\tau - cl(\{y\})$ is the intersection of all $pr-$closed sets which contain $\{y\}$.Thus , $pre_M^\tau - cl(\{y\}) \subset X - G$, then $x \notin X - G$. This implies that $x \notin pre_M^\tau - cl(\{y\})$, but $x \in pre_M^\tau - cl(\{x\})$, $X \notin pre_M^\tau - cl(\{y\})$. Therefore $pre_M^\tau - cl(\{x\}) \neq pre_M^\tau - cl(\{y\})$.

Theorem 5.3.

Every subspace of $pr - T_o$-space is $pr - T_o$-space .

Proof:

Let (Y, τ_Y, M_Y) be a subspace of $pr - T_o$-space (X, τ, M).To prove that the subspace is $pr - T_o$-space, let $y_1 \neq y_2 \in Y$. Since $Y \subset X$, then $y_1 \neq y_2 \in X$ and (X, τ, M) is $pr - T_o$-space . There exists $pre-$open set G in X, such that $y_1 \in G$, $y_2 \notin G$. So $G \cap Y$ is $pre-$open set in Y and $y_1 \in G \cap Y$, $y_2 \notin G \cap Y$. Hence (Y, τ_Y, M_Y) is $pr - T_o$-space

Definition 5.4.

A minimal bitopological space (X, τ, M) is called $pr - T_1$-space if and only if for each pair of distinct points x, y in X, there exist two $pre-$open sets G, H such that G contains x but not y and H contains y but not x.

Remark 5.5.

Every $pr - T_1$-space is $pr - T_o$-space. But The converse is not true , as the following example .

Example 5.6.

Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{c\}\}$ and $M = \{X, \phi, \{a\}\}$, (X, τ) is a topological space and (X, M) is a minimal space , then (X, τ, M) is a minimal bitopological space , hence: $pre.O_M^\tau(X) = \{X, \phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}\}$. If we take a, and b, $a \neq b$, then we cannot find two $pre-$open sets , such that one of them contains a but not b and the other contains b but not a. Therefore, (X, τ, M) is not $pr - T_1$-space, but it is clear that (X, τ, M) is a $pr - T_o$-

space . The $pr-T_1$-space is hereditary property .

Theorem 5.7.

Every subspace of $pr-T_1$-space is $pr-T_1$-space.

Proof:

Let (Y,τ_Y,M_Y) be a subspace of $pr-T_1$-space (X,τ,M).To prove that the subspace Y is $pr-T_1$-space . Let $y_1 \neq y_2$, since $Y \subset X$, then $y_1 \neq y_2 \in X$ and since (X,τ,M) is $pr-T_1$-space. Then there exist two $pre-$open sets G, H in X, such that $y_1 \in G$, but $y_2 \notin G$; and $y_2 \in H$, but $y_1 \notin H$. Then we obtain two sets $G_1 = G \cap Y$, $H_1 = H \cap Y$ are $pre-$open sets in Y , we have $y_1 \in G_1$, but $y_2 \notin G_1$; $y_2 \in H_1$, but $y_1 \notin H_1$. Hence (Y,τ_Y,M_Y) is $pr-T_1$-space.

Theorem 5.8.

A minimal bitopological space (X,τ,M) is an $pr-T_1$-space If and only if every singleton subset $\{x\}$ of (X,τ,M) is $pre-$closed set.

Proof:

Let $x, y \in X$ and $x \neq y$. If $\{x\}$ and $\{y\}$ are the $pre-$closed sets of x and y respectively such that $\{x\} \neq \{y\}$, then $\{x\}^C$ and $\{y\}^C$ are $pre-$open sets such that $y \in \{x\}^C$ and $x \notin \{x\}^C$ also $x \in \{y\}^C$ and $y \notin \{y\}^C$. Then X is an $pr-T_1$-space . Conversely, suppose X is an $pr-T_1$-space. Then for $x, y \in X$ and $x \neq y$, there are two $pre-$open sets G and H such that $x \in H$, $y \notin H$ and $y \in G$ and $x \notin G$. So $G \subset \{x\}^C$ also $\cup\{G/y \neq x\} \subseteq \{x\}^C$ and $\{x\}^C \subseteq \cup\{G/y \neq x\}$. Hence $\{x\}^C \subseteq \cup\{G/y \neq x\}$, which is a $pre-$open set. Then {x} is a $pre-$closed .

Definition 5.9.

A minimal bitopological space (X,τ,M) is called a $pr-T_2$-space ($pre-$ Hausdorff) if and only if for each pair of distinct points x, y in X, there exist two $pre-$open sets G, H such that $x \in G$, $y \in H$, $G \cap H = \phi$.

Remark 5.10.

Every $pr-T_2$-space is a $pr-T_1$-space. The converse is not true , the following example shows that.

Example 5.11.

Let $X=\{a,b,c\}$,$\tau=\{X,\phi,\{a\},\{b,c\}\}$ and $M=\{X,\phi,\{a,c\},\{a,b\},\{a\}\}$. (X,τ) is two topological spaces and (X,M) is a minimal space , then (X,τ,M) is a minimal bitopological space , then: $pre.O_M^{\tau}(X)=$ $\{X,\phi,\{a\},\{a,b\},\{a,c\}\{b,c\}\}$. Clearly (X,τ,M) is a $pr-T_1$-space, but not $pr-T_2$-space, because there exists $b\neq c$, there is only $pre-$open sets $\{a,b\}$, $\{a,c\}$, such that $b\in\{a,b\}$, $c\in\{a,c\}$, but $\{a,b\}\cap\{a,c\}\neq\phi$. The $pr-T_2$-space is hereditary property .

Theorem 5.12.

Every subspace of $pr-T_2$-space is a $pr-T_2$-space.

Proof:

Let (X,τ,M) be a $pr-T_2$-space, and let $Y\neq\phi$ be a subset of X, and $x\neq y\in Y$, then x and $y\in X$, since (X,τ,M) is $pr-T_2$-space, there exist two $pre-$open sets G,H such that $x\in G$ and $y\in H$, $G\cap H=\phi$. So $G\cap Y$, $H\cap Y$ are $pre-$open sets in Y and $x\in G\cap Y$, $y\in H\cap Y$; and $(G\cap Y)\cap(H\cap Y)=$ $(G\cap H)\cap Y=\phi$. Hence (X,τ_Y,M_Y) is a $pr-T_2$-space

Definition 5.13.

A minimal bitopological space (X,τ,M) is called $pre-$regular space if and only if for each $pre-$closed set F in X, and each $x\notin F$; there exist $pre-$open sets U, V such that $x\in U$, $F\subset V$ and $U\cap V=\phi$.

Theorem 5.14.

Let (X,τ,M) be a Minimal bitopological space, then (X,τ,M) is a $pre-$regular iff for each $pre-$open set U and $x\in U$, there exists $pre-$open set V

such that $x \in V$, $pre_M^{\tau} - cl(V) \subset U$.

Proof:

Let (X,τ,M) be a $pre-$regular, $x \in U$ where U is a $pre-$open set . Let $H = X - U$, then H is $pre-$closed, $x \notin H$. Then there exist $pre-$open sets W and V such that: $x \in V$, $H \subset W$, $V \bigcap W = \phi$. Then $V \subset X - W$, $pre_M^{\tau} - cl(V) \subset pre_M^{\tau} - cl(X - W) = X - W$ …..(2-1) $H \subset W$, then $X - W \subset X - H = U$, then $X - W \subset U$ …….(2-2) From (2-1),(2-2) we have, $X \in V$, $pre_M^{\tau} - cl(V) \subset U$ .Conversely , let H be $pre-$closed set and $x \notin H$. Let $U = X - H$, then U is a $pre-$open and $x \in U$. By hypothesis, there exists $pre-$open set V such that $x \in V$, $pre_M^{\tau} - cl(V) \subset U$, $H \subset (X - (pre_M^{\tau} - cl(V)))$ .Since $x \in V$, $V \bigcap (X - (pre_M^{\tau} - cl(V))) = \phi$. Hence (X,τ,M) is $pre-$regular space

Theorem 5.15.

Every subspace of a $pre-$regular space is a $pre-$regular space.

Proof:

Let (X,τ,M) be a $pre-$regular space , (Y,τ_Y,M_Y) be a subspace of Y. To prove (Y,τ_Y,M_Y) is $pre-$regular space . Let $q \in Y$ and A be $pre-$closed set in Y, such that $q \notin A$, then $pre-cl_Y(A) = pre-cl_X(A) \bigcap Y$, and since A is $pre-$closed set in Y, so $pre-cl_Y(A) = A$ Then $A = pre-cl_X(A) \bigcap Y$. Since $q \notin A$, then $q \notin pre-cl_X(A) \bigcap Y$, $q \notin pre-cl_X(A)$, thus $pre-cl_X(A)$ is $pre-$closed in X, and since (X,τ,M) is $pre-$regular , then there exist two disjoint $pre-$open sets G,H in X, such that $q \in G$, $pre-cl_X(A) \subset H$ and $G \bigcap H = \phi$. $q \in G \bigcap Y$ and $pre-cl_X(A) \bigcap Y \subset H \bigcap Y$, $A \subset H$, since G,H are $pre-$open in X, then $G \bigcap Y$, $H \bigcap Y$ are $pre-$open set in Y. Since $G \bigcap Y = \phi$, then $(G \bigcap Y) \bigcap (H \bigcap Y) = (G \bigcap H)$ $\bigcap Y = \phi \bigcap Y = \phi$. So (Y,τ_Y,M_Y)

is $pre-$regular subspace of (X,τ,M)

Definition 5.16.

A minimal bitopological space (X,τ,M) is called $pre-$normal space if and only if for each pair of $pre-$closed sets G,H in X, such that $G\bigcap H=\phi$, there exist $pre-$open sets U,V such that $G\subset U$, $H\subset V$ and $U\bigcap V=\phi$.

Theorem 5.17.

Let (X,τ,M) be a minimal bitopological space, then (X,τ,M) is a $pre-$normal if and only if for each $pre-$closed set H in X and $pre-$open set U in X containing H, there exists $pre-$open set V such that $H\subset V\subset pre_M^{\tau}-cl(V)\subset U$.

Proof:

Let (X,τ,M) be $pre-$normal. Let H be $pre-$closed in X and U is $pre-$open in X, such that $H\subset U$. then $X-U$ is $pre-$closed in X and $H\bigcap(X-U)=\phi$. So there exist $pre-$open sets K and V such that, $X-U\subset K$, $H\subset V$, $V\bigcap K=\phi$, $X-K\subset U$, $V\subset X-K$. This implies that $pre_M^{\tau}-cl(V)\subset pre_M^{\tau}-cl(X-K)=X-K$. The $H\subset V\subset pre_M^{\tau}-cl(V)\subset U$ $V\subset pre_M^{\tau}-cl(V)\subset U$.

Conversely, H is a $pre-$closed set in X and U is a $pre-$open in X, then $X-U$ is a $pre-$closed set in X, then $X-U$ is a $pre-$closed in X, and $H\bigcap(X-U)=\phi$. By hypothesis, there exists $pre-$open set V such that $H\subset V\subset pre_M^{\tau}-cl(V)\subset U$, then $X-U\subset(X-(pre_M^{\tau}-cl(V)))$. So we have $H\subset V$, $X-U\subset(X-(pre_M^{\tau}-cl(V)))$, and $V\bigcap(X-(pre_M^{\tau}-cl(V)))=\phi$. Hence (X,τ,M) is $pre-$normal space

Corollary 5.18.

A minimal bitopological space (X,τ,M) is a $pre-$normal if and only if for each $pre-$closed set H in X and each $pre-$open set U in X containing H, there exists a subset A of X, such that $H\subset pre_M^{\tau}-\operatorname{int}(A)\subset pre_M^{\tau}-cl(A)\subset U$.

Proof:

By using the Theorem (5.17) and taking $A=V$, the proof is satisfied

Theorem 5.19.

Every $pre-$closed subspace of a $pre-$normal space is a $pre-$ normal.

Proof:

Let (X,τ,M) be a $pre-$normal space and (Y,τ_Y,M_Y) be any $pre-$closed subspace of X. I have to show that (Y,τ_Y,M_Y) is also $pre-$normal space. Let L^*, N^* be disjoint $pre-$closed subsets of Y. Then there exist $pre-$closed subsets L,N of X, such that $L^*=L\bigcap Y$ and $N^*=N\bigcap Y$. Since Y is $pre-$closed , it follows that L^*,N^* are disjoint $pre-$closed subsets in X. Then by $pre-$normality of X, there exist $pr-$open sets G,H in X, such that $L^*\subset G$, $N^*\subset H$ and $G\bigcap H=\phi$. Since $L^*\subset Y$ and $N^*\subset Y$, these relations imply $:L^*\subset G\bigcap Y, M^*\subset N\bigcap Y$ and $(G\bigcap Y)\bigcap H\bigcap Y)=\phi$. Setting $G\bigcap Y=G^*$ and $H\bigcap Y=H^*$, we see that G^*,H^* are $pre-$open subset of Y such that : $L^*\subset G^*$, $N^*\subset H^*$ and $G^*\bigcap H^*=\phi$. Accordingly (Y,τ_Y,M_Y) is a $pre-$normal space

Definition 5.20.

A minimal bitopological space (X,τ,M) is called a $pr-T_3$-space if and only if X is a $pr-T_1$ and a $pre-$regular space.

Remark 5.21.

Every $pr-T_3$- space is a $pre-$regular and the converse is not true as we will show in the following example.

Example 5.22.

Let $X=\{a,b,c\}$, $\tau=\{X,\phi,\{b,c\}\}$ and $M=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\}\}$, let (X,τ) is topological spaces and (X,M) is minimal , then (X,τ,M) is a minimal bitopological space , so: $pre.O_M^\tau(X)=\{X,\phi,\{b,c\}\}$, and $pre.C_M^\tau(X)=\{X,\phi,\{a\}\}$.
Clearly , (X,τ,M) is a $pre-$regular, but it is not a $pr-T_1$-space . Hence

(X,τ,M) is not a $pr-T_3$-space.

Theorem 5.23.

Every subspace of $pr-T_3$-space is a $pr-T_3$-space.

Proof:

Let X be a $pr-T_3$-space and Y be a subspace of X. N X is a $pre-$regular as well as a $pr-T_1$-space and we have shown that both these properties are hereditary. It follows that Y is a $pre-$regular as well as a $pr-T_1$-space. Hence Y is a $pr-T_3$-space

Definition 5.24.

A minimal bitopological space (X,τ,M) is called a $pr-T_4$ space if and only if X is a $pre-$normal and a $pr-T_1$-space.

Remark 5.25.

Every $pr-T_4$-space is a $pre-$normal and the converse is not true as we will show in Example (5.22).

6- References

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