Almost Stability of Modified Iteration Method with Errors for a Fixed Point of Uniformaly L- Lipschitzian

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ABSTRACT:

In this paper, we prove strong convergence theorem of modified Mann iteration sequence with errors for uniformaly L- Lipschitzian mapping in arbitrary Banach space. Our results improve and gernalize the recent results Osilike , Xu and Xie and many others.

1.INTRODUTION AND PRELIMINARY DEFINITIONS

Let X be arbitrary real Banach space and C is a nonempty subset of X and T is a selfmapping of C. F(T) is set of fixed point of T.

The various mapping appearing in the following Definition (1.1) have been studied widely and deeply be many authors; see e.g., [1-4] for more details.

Definition (1.1): Let X be an arbitrary real Banach space and C be a nonempty subset of X. A mapping T:C→C is called:

(i) Strongly pseudocontractive if there exists $k \in (0,1)$ such that

$$\|x-y\| \le \|x-y+r[(I-T-kI)x-(I-T-kI)y\| \quad .....(1.1)$$

where I is the identity mapping on C and for all $x, y \in C$ and r>0.

(ii) Lipschitz if there exists a constant L>0 such that

$$\|T(x)-T(y)\| \le L\|x-y\| \quad .....(1.2)$$

for all $x, y \in C$.

(iii) Uniformaly L-Lipschitzian if there exists a constant L>0 such that

$$\left\|T^n(x) - T^n(y)\right\| \le L\left\|x - y\right\| \qquad .....(1.3)$$

for all $x, y \in C$ and $n \ge 1$.

We consider the iteration [1]

$x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + c_n u_n \qquad (n \ge 1)$$

Where $\{\alpha_n\}, \{c_n\}$ are sequences in (0,1) and $\{u_n\}$ is sequence in C satisfying $\sum_{n=1}^{\infty}\left\|u_n\right\| \prec \infty$.

This iteration is known Mann iteration sequence with random errors.

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$x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + c_n u_n \qquad (n \ge 1)$$
$$y_{n+1} = (1-\beta_n)x_n + \alpha_n T x_n + e_n v_n$$

Where $\{\alpha_n\}, \{\beta_n\}, \{c_n\}, \{e_n\}$ are sequences in (0,1) and $\{u_n\}, \{v_n\}$ are sequences in C satisfying $\sum_{n=1}^{\infty}\left\|u_n\right\| < \infty$, $\sum_{n=1}^{\infty}\left\|v_n\right\| < \infty$

This iteration is known Ishikawa iteration sequence with random errors.

Let $x_1 \in C$ and $x_{n+1} = f(T, x_n)$ define an iteration procedure which yields a sequence of a points $\{x_n\}$ in C. Suppose that F(T) $\ne \emptyset$ and $\{x_n\}$ converges to a fixed point q of T.

Let $\{y_n\}$ be an arbitrary sequence in C and $\varepsilon_n = \left\|y_{n+1} - f(T, y_n)\right\|$. If $\lim_{n\to\infty}\varepsilon_n = 0$ implies $\lim_{n\to\infty} y_n = q$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T- stable or stable with respect to T. Stability results for several iteration procedures for certain contractive definition have been established in recent papers by several authors (see, [5, 6, 7]).

In 1996, Osilike [8], proved that if X p-uniformly smooth Banach space, C nonempty closed of X and T:C→C is Lipschitz strongly pseudocontractive mapping with fixed point q in C, then both the Mann and Ishikawa iteration schemes are stable. Then he extended the results to arbitrary real Banach space in [6].

In 2001, Zeqing, Lili and Shin [9], show that if X is an arbitrary real Banach space and T:C→C is a Lipschitz strongly pseudocontractive mapping, then under certain conditions the Ishikawa iterative with errors converges strongly to the unique fixed point of T. we also proved that this iteration procedure is stable with respect to T.

In 2004, Xu and Xie [10], proved necessary and sufficient condition for strongly convergence of Mann iteration process with errors to a fixed point of Lipschitz strongly pseudocontractve mapping in real Banach space.

We consider the iteration

$x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n + u_n \qquad (n \ge 1)$$

Where $\{\alpha_n\}$ is sequences in (0,1) and $\{u_n\}$ is sequence in C satisfying $\sum_{n=1}^{\infty}\|u_n\| < \infty.$

This iteration is known Modified Mann iteration sequence with random errors.

We consider the iteration [1]

Next we recall the definition stability. Let $x_1 \in C$ and $x_{n+1} = f(T^n, x_n)$ define an iteration procedure which yields a sequence of a points $\{x_n\}$ in C. Suppose that F(T) $\neq \varnothing$ and $\{x_n\}$ converges to a fixed point q of T.

Let $\{y_n\}$ be an arbitrary sequence in C and $\varepsilon_n = \|y_{n+1} - f(T^n, y_n)\|$. If $\lim_{n\to\infty}\varepsilon_n = 0$ implies $\lim_{n\to\infty} y_n = q,$ then the iteration procedure defined by $x_{n+1} = f(T^n, x_n)$ is said to be T^n-stable or stable with respect of T^n.

It is our purpose in this paper to show that if X is an arbitrary real Banach space and T:X→X is uniformly L- Lipschitzian mapping , then under certain condition the Modified Mann iterative method with errors converges strongly to the unique fixed point of T. we also prove that this iteration procedure is stable with respect of T^n . Our results generalize most of the results that have appared recently.

For our result we need the following lemma:

Lemma 1.2. [1] Let $\{\alpha_n\}$ be a nonnegative sequence that satisfies the inequality

$$\alpha_{n+1} \le (1-t_n)\alpha_n + b_n + c_n \qquad n > 1$$

$$where\ \ t_n \in [0,1]\ \ for\, each\, n \in N, \sum_{n\in N} t_n = \infty\ \ and\, b_n = 0(t_n),\ \sum_{n\in N} c_n < \infty.\ Then\ \alpha_n \to 0\ \ as\ \ n \to \infty$$

2.MAIN RESULT

Theorem 2.1. Let X be an arbitrary real Banach space and $T: X \to X$ is uniformly L-Lipschitzian mapping and T satisfies the condition

$$\|x-y\| \le \|x-y+r[(I-T^n-kI)x-(I-T^n-kI)y\| \quad .....(2.1)$$

where I is the identity mapping on X and for all $x, y \in C,\ n \ge 1$, r>0 and $k \in (0,1)$.

If q is a fixed point of T and for arbitrary $x_1 \in X$, the Modified Mann iterative sequence with errors defined by (1.4) satisfying

$$0 < \alpha < \alpha_n \le k[2(L^2+3L+3)] \quad ...(2.2)$$

where L>1 is Lipschitz constant of T . Then

(1) $\{x_n\}$ converges strongly to unique fixed point q of T,

(2) $\{y_n\}$ any sequence in X. Then $\{y_n\}$ converges strongly to fixed point q of T if and

only if ε_n converges to 0.

Proof(1): using (1.4), we have

$$\begin{aligned}
x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T^n x_n - u_n \\
&= x_{n+1} + \alpha_n x_n - \alpha_n T^n x_n - x_n + 2\alpha_n x_{n+1} - 2\alpha_n x_{n+1} + k\alpha_n x_{n+1} - k\alpha_n x_{n+1} + \alpha_n T^n x_{n+1} - \alpha_n T^n x_{n+1} - u_n \\
&= (1+\alpha_n)x_{n+1} + \alpha_n x_n + \alpha_n (I - T^n - kI)x_{n+1} - (2-k)\alpha_n x_{n+1} + \alpha_n (T^n x_{n+1} - T^n x_n) - u_n \\
&= (1+\alpha_n)x_{n+1} + \alpha_n (I - T^n - kI)x_{n+1} - (2-k)\alpha_n[(1-\alpha_n)x_n + \alpha_n T^n x_n + u_n] \\
&\quad + \alpha_n x_n + \alpha_n (T^n x_{n+1} - T^n x_n) - u_n \\
&= (1+\alpha_n)x_{n+1} + \alpha_n (I - T^n - kI)x_{n+1} - (2-k)(\alpha_n - \alpha^2{}_n)x_n - (2-k)\alpha^2{}_n T^n x_n \\
&\quad + (2-k)\alpha_n u_n + \alpha_n x_n + \alpha_n (T^n x_{n+1} - T^n x_n) - u_n \\
&= (1+\alpha_n)x_{n+1} + \alpha_n (I - T^n - kI)x_{n+1} - 2\alpha_n x_n - 2\alpha^2{}_n x_n + k\alpha_n x_n - k\alpha^2{}_n x_n \\
&\quad - 2\alpha^2{}_n T^n x_n + k\alpha^2{}_n T^n x_n + (2-k)\alpha_n u_n + \alpha_n x_n + \alpha_n T^n x_{n+1} - \alpha_n T^n x_n - u_n \\
&= (1+\alpha_n)x_{n+1} + \alpha_n (I - T^n - kI)x_{n+1} - \alpha_n x_n + (2-k)\alpha^2{}_n (x_n - T^n x_n) \\
&\quad + k\alpha_n x_n + \alpha_n (T^n x_{n+1} - T^n x_n) + (2-k)\alpha_n u_n - u_n
\end{aligned}$$

therefore

$$x_n = (1+\alpha_n)x_{n+1} + \alpha_n(I - T^n - kI)x_{n+1} - (1-k)\alpha_n x_n + (2-k)\alpha^2{}_n(x_n - T^n x_n)$$
$$+ \alpha_n(T^n x_{n+1} - T^n x_n) + (2-k)\alpha_n u_n - u_n.$$

Therefore

$$x_n - q = (1+\alpha_n)(x_{n+1} - q) + \alpha_n[(I - T^n - kI)x_{n+1} - (I - T^n - kI)q]$$
$$+ \alpha_n(T^n x_{n+1} - T^n x_n) - (2-k)\alpha^2{}_n(T^n x_n - q) - (1-k)\alpha_n(x_n - q)$$
$$+ (2-k)\alpha^2{}_n(x_n - q) - [1 + (2-k)\alpha_n]u_n.$$

For all $n \geq 1$. *Furthermore,*

$$\|x_n - q\| \geq (1+\alpha_n)\left\|(x_{n+1} - q) + \frac{\alpha_n}{1+\alpha_n}[(I - T^n - kI)x_{n+1} - (I - T^n - kI)q]\right\|$$
$$+ \alpha_n\|T^n x_{n+1} - T^n x_n\| - (2-k)\alpha^2{}_n\|T^n x_n - q\| - (1-k)\alpha_n\|x_n - q\|$$
$$+ (2-k)\alpha^2{}_n\|x_n - q\| - [1 + (2-k)\alpha_n]\|u_n\| \qquad ...(2.3)$$

By virtue of (1.1) *and* T *is uniformly* L – *Lipschitzian, we have*

$$\|x_n - q\| \geq (1+\alpha_n)\|(x_{n+1} - q)\| + L\alpha_n\|x_{n+1} - x_n\| - (1-k)\alpha_n\|x_n - q\|$$
$$+ (2-k)\alpha^2{}_n\|x_n - q\| - 3M$$
$$\geq (1+\alpha_n)\|(x_{n+1} - q)\| - (1-k)\alpha_n\|x_n - q\| - (L+1)(L+2)\alpha^2{}_n\|x_n - q\|$$
$$-(3+L)M \qquad ...(2.4)$$

where $M = \sup\{\|u_n\| : n = 1, 2, ...\}$

It follows for (2.4) *and the condition* (2.2)

$$\|x_{n+1} - q\| \leq (1 - \alpha_n + \alpha^2{}_n)\|x_n - q\| + (1-k)\alpha_n\|x_n - q\| + (L+1)(L+2)\alpha^2{}_n\|x_n - q\|$$
$$+ (3+L)M$$
$$\leq (1 - k\alpha_n)\|x_n - q\| + \alpha_n\left[\alpha_n(L^2 + 3L + 3) - \frac{k}{2}\right]\|x_n - q\|$$
$$+ (3+L)M$$
$$\leq (1 - \frac{k\alpha_n}{2})\|x_n - q\| + (3+L)M \qquad ...(2.5)$$

$$Set \quad t_n = \frac{k\alpha_n}{2},\ \alpha_n = \|x_n - q\| \ and\ c_n = (3+L)M$$

Then we have

$$\alpha_{n+1} \le (1-t_n)\alpha_n + b_n + c_n$$

According to the above argument, it is easy seen that

$$\sum_{n=1}^{\infty} t_n = \infty,\ b_n = 0(t_n),\ \sum_{n=1}^{\infty} c_n < \infty$$

and so, by lemma (1.2), *we have* $\lim_{n\to\infty}\|x_n - q\| = 0$, *i.e.,* $\{x_n\}$ *converges strongly to fixed point* q *of* T.

If p *also is a fixed point* T, *putting* $r = 1$ *in* (2.1) *we obtain*

$$\|q-p\| \le (1-k)\|q-p\|.$$

It implies that $q = p$.

Proof(2): Suppose that $\lim_{n\to\infty}\varepsilon_n = 0$, *then*

$$\begin{aligned}
\|y_{n+1}-q\| &= \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T^n y_n - u_n + (1-\alpha_n)y_n + \alpha_n T^n y_n + u_n - q\| \\
&\le \varepsilon_n + \|(1-\alpha_n)(y_n - q) + \alpha_n(T^n y_n - q) + u_n\| \\
&\le \varepsilon_n + \|(1-\alpha_n)(y_n - q) + \alpha_n(T^n y_n - q) + u_n\| \\
&\le (1-\frac{k\alpha_n}{2})\|y_n - q\| + (3+L)M + \varepsilon_n
\end{aligned}$$

$$Set \quad t_n = \frac{k\alpha_n}{2},\ \alpha_n = \|x_n - q\| \ and\ c_n = (3+L)M + \varepsilon_n$$

Then we have

$$\alpha_{n+1} \le (1-t_n)\alpha_n + b_n + c_n$$

According to the above argument, it is easy seen that

$$\sum_{n=1}^{\infty} t_n = \infty,\ b_n = 0(t_n),\ \sum_{n=1}^{\infty} c_n < \infty$$

and so, by lemma (1.2), *we have* $\lim_{n\to\infty}\|y_n - q\| = 0$,

i.e., $\{y_n\}$ *converges strongly to fixed point* q *of* T.

Then the iterative process defined by $x_{n+1} = f(T^n, x_n)$ *is* T^n-*stable.*

On the coontrary, let $\{y_n\}$ *converges strongly to fixed point* q *of* T. *Then*

$$\varepsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T^n y_n - u_n\|$$
$$\leq \|y_{n+1} - q\| + (1-\alpha_n)\|y_n - q\| + L\alpha_n \|y_n - q\| + M \quad \to 0.$$

This implies that $\lim_{n\to\infty} \varepsilon_n = 0$. $\oplus$

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