Introduction

The problem of Finding the value of the determinant is an old problem appeared for the first time in 1850 by the mathematician scientist Sylvester ;This idea appeared connection with the solution of the system of linear equations which appeared in the physical application a specially in the physical linear system .

There are many ways to find the value of the determinant such as:

(1) Let A be a square matrix of order N*N then the determinant of A is

$$|A| = \sum \mp a_{16(1)} a_{26(2)} \dots a_{n6(n)} \; 6 \in Sn$$

Where S_n is the set of all permutations on {1, 2… n} and the sign is take (+) if 6 is an even permutation and (-) if 6 is an odd permutation. [2]

(2)Let A be a square matrix of order N*N and let Mij be the determinant value for the matrix of order (N-1) * (N-1) by neglecting the row I & the column j from the matrix A, then

$$|A| = a_{1j}c_{1j} + a_{2j}c_{2j} + \dots + a_{nj}c_{nj} \quad \text{For } j=1, 2\dots n$$

$$c_{ij} = (-1)^{i+j} M_{ij.} \quad [2]$$

(3) The following formula which appeared in (1979)

$$|A| = (1/a_{11})^{N-2} \begin{vmatrix} \begin{vmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{vmatrix} \begin{vmatrix} a_{11} a_{13} \\ a_{21} a_{23} \end{vmatrix} \cdots \begin{vmatrix} a_{11} a_{1n} \\ a_{21} a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} a_{12} \\ a_{31} a_{32} \end{vmatrix} \begin{vmatrix} a_{11} a_{13} \\ a_{31} a_{33} \end{vmatrix} \cdots \begin{vmatrix} a_{11} a_{1n} \\ a_{31} a_{3n} \end{vmatrix} \\ \vdots \\ \begin{vmatrix} a_{11} a_{12} \\ a_{n1} a_{n2} \end{vmatrix} \begin{vmatrix} a_{11} a_{13} \\ a_{n1} a_{n3} \end{vmatrix} \cdots \begin{vmatrix} a_{11} a_{1n} \\ a_{n1} a_{nn} \end{vmatrix} \end{vmatrix}$$

In this formula the reduction of size t=2. [4]

(4)A developed formula suggested by a research in (2000) which is

$$\det(A) = |A| = \left(1/|AA|\right)^{N-3}\left(1/|BB|\right)^{N-3}\ldots|Z|. \quad [6]$$

where t=3,|AA|,|BB| is the determinant value of the matrix of order 2x2, $|Z|$ is the determinant value of either 2x2 or 3x3 matrix .

There are many other classical ways to evaluate the determinant value which need a very long time to evaluate the determinant value when the size of the matrix is big while our general new technique can find the determinant value for the matrix of any order with any reduction formula that the researchers need and use it to solve their problems.

1- The Algorithm:

Let A be a matrix of order N*N, t is the value of reduction.
The general formula that we must use is:

$$|A| = \left(1/|L|_{(t-1)*(t-1)}\right)^{N-t} \begin{vmatrix} b_{11}b_{12}\dots b_{1,N-(t-1)} \\ b_{21}b_{22}\dots b_{2,N-(t-1)} \\ \vdots \\ b_{N-(t-1),1}b_{N-(t-1),2}\dots b_{N-(t-1),N-(t-1)} \end{vmatrix} \Big\} = |B|$$

$$| L |= \begin{bmatrix} a1,1 & a1,2 & \dots & a1,t-1 \\ a2,1 & a2,2 & \dots & a2,t-1 \\ . & . & \dots & . \\ at-1,1 & at-1,2 & \dots & at-1,t-1 \end{bmatrix}$$

$b_{11}, b_{12}, \dots, b_{N-(t-1),N-(t-1)}$ Is the determinant value for the matrix of order t*t.

Where

$$b_{11} = \begin{vmatrix} a_{11}a_{12}\dots a_{1,t} \\ a_{21}a_{22}\dots a_{2,t} \\ \vdots \\ a_{t1}a_{t2}\dots a_{t,t} \end{vmatrix} \qquad b_{12} = \begin{vmatrix} a_{11}a_{13}\dots a_{1,t} \\ a_{21}a_{23}\dots a_{2,t} \\ \vdots \\ a_{t1}a_{t3}\dots a_{t,t} \end{vmatrix}$$

$$b_{1,N-(t-1)} = \begin{vmatrix} a_{11}\dots a_{1,t-1}a_{1,t} \\ a_{21}\dots a_{2,t-1}a_{2,t} \\ \vdots \\ a_{t1}\dots a_{t,t-1}a_{t,t} \end{vmatrix} \qquad b_{N-(t-1),N-(t-1)} = \begin{vmatrix} a_{11}\dots a_{1,t-1}a_{1,t} \\ \vdots \\ a_{t-1,1}\dots a_{t-1,t-1}a_{t-1,t} \\ a_{t,1}\dots a_{t,t-1}a_{t,t} \end{vmatrix}$$

(2)IF the value of |L| is zero we must change the columns to avoid this case but in this case the sing of the determinant must be changed.

(3)IF |B| is of order 2x2 or 3x3 we evaluate it by using one of the classical methods, if not

$$|B| = \left(1/|L1|\right)^{N1-t} \begin{vmatrix} c_{11}c_{12}\cdots c_{1,N1-(t-1)} \\ c_{21}c_{22}\cdots c_{2,N1-(t-1)} \\ \vdots \\ c_{N1-(t-1),1}c_{N1-(t-1),2}\cdots c_{N1-(t-1),N1-(t-1)} \end{vmatrix} \} = |C|$$

Where N_1 is the size of the new matrix B & t is the reduction value which can take the same value of the previous step or different value.

(4)IF $|C|$ are 2x2 or 3x3 we stop, if not we continue as in the step (3) until we reach in the following case:

$$|A| = \left(1/|L|\right)^{N-t}\left(1/|L1|\right)^{N1-t}\left(1/|L2|\right)^{N2-t}\ldots|Z|$$

Where $|Z|$ is either determinant value of order 2x2 or 3x3.

(5)IF we take the value of reduction t=3, then we use the approach explain in [6], if t>3 we use the previous technique as a subroutine to evaluate each determinant value.

2-Solving the system of n linear equations by using general new technique:

Consider we have the following system of linear equations

$a_{11}x_1 + a_{12}\, x_2 + \ldots + a_{1n}\, x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}\, x_n = b_2$
....
$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn.}\, x_n = b_n$
This can be written in matrix notation as:
AX=B, Where

$$A = \begin{bmatrix} a_{11}a_{12}\ldots a_{1n} \\ a_{21}a_{22}\ldots a_{2n} \\ \vdots \\ a_{n1}a_{n2}\ldots a_{nn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If we solve this system by using crammers rule, we have

$$X1 = \begin{vmatrix} b_1 a_{12}\ldots a_{1n} \\ b_2 a_{22}\ldots a_{2n} \\ \vdots \\ b_n a_{n2}\ldots a_{nn} \end{vmatrix} \Bigg/ \begin{vmatrix} a_{11}a_{12}\ldots a_{1n} \\ a_{21}a_{22}\ldots a_{2n} \\ \vdots \\ a_{n1}a_{n2}\ldots a_{nn} \end{vmatrix}$$

$$X2 = \begin{vmatrix} a_{11}b_1\ldots a_{1n} \\ a_{21}b_2\ldots a_{2n} \\ \vdots \\ a_{n1}b_n\ldots a_{nn} \end{vmatrix} \Bigg/ \begin{vmatrix} a_{11}a_{12}\ldots a_{1n} \\ a_{21}a_{22}\ldots a_{2n} \\ \vdots \\ a_{n1}a_{n2}\ldots a_{nn} \end{vmatrix}$$

•
•
•

$$X_n = \begin{vmatrix} a_{11}a_{12}\ldots b_1 \\ a_{21}a_{22}\ldots b_2 \\ \vdots \\ a_{n1}a_{n2}\ldots b_n \end{vmatrix} \Big/ \begin{vmatrix} a_{11}a_{12}\ldots a_{1n} \\ a_{21}a_{22}\ldots a_{2n} \\ \vdots \\ a_{n1}a_{n2}\ldots a_{nn} \end{vmatrix}$$

Each determinant in the above formula evaluate by using our general new formula.

3-Some examples:

The following examples took from ref. [1], [3]

1-Find the determinant value of the following matrix

$$A=\begin{bmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 2 & 1 & 4 & 1\\ 6 & 8 & 7 & 2 & 1\\ 9 & -4 & 0 & 0 & 1\\ -8 & -6 & 4 & 1 & 7\end{bmatrix}$$

Where reduction value t=4.

Sol.

$$|A|=\begin{vmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 2 & 1 & 4 & 1\\ 9 & 8 & 7 & 2 & 1\\ 9 & -4 & 0 & 0 & 1\\ -8 & -6 & 4 & 1 & 7\end{vmatrix}=\left(1/|AA|_{t-1xt-1}\right)^{N-t}\begin{vmatrix}D_{11}D_{12}\\ D_{21}D_{22}\end{vmatrix}_{N-(t-1)xN-(t-1)}$$

$$|AA|=\begin{vmatrix}1 & 2 & 3\\ 1 & 2 & 1\\ 6 & 8 & 7\end{vmatrix}=\begin{vmatrix}2 & 1\\ 6 & 7\end{vmatrix}-2\begin{vmatrix}1 & 1\\ 6 & 7\end{vmatrix}+3\begin{vmatrix}1 & 2\\ 6 & 8\end{vmatrix}=6-2+(-12)=-8$$

Now each $D_{11}, D_{12}, D_{21}, D_{22}$ evaluate by the previous technique as follows:

$$D_{11}=|B|=\begin{vmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 1 & 4\\ 6 & 8 & 7 & 2\\ 9 & -4 & 0 & 0\end{vmatrix}=\left(1/|BB|_{t1-1xt1-1}\right)^{N1-t1}\begin{vmatrix}B_{11}B_{12}\\ B_{21}B_{22}\end{vmatrix}_{N1-(t1-1)xN1-(t1-1)}$$

Here N1=4, t1=3. $|BB|=\begin{vmatrix}1 & 2\\ 1 & 2\end{vmatrix}=0$ Then we must change the column

$$D_{11}=|B|=\begin{vmatrix}1 & 3 & 2 & 4\\ 1 & 1 & 2 & 4\\ 6 & 7 & 8 & 2\\ 9 & 0 & -4 & 0\end{vmatrix}=\left(1/|B'B'|_{t1-1xt1-1}\right)^{N1-t1}\begin{vmatrix}B'_{11}B'_{12}\\ B'_{21}B'_{22}\end{vmatrix}_{N1-(t1-1)xN1-(t1-1)}$$

$$|B'B'| = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 1 - 3 = -2 \quad ,$$

$$|B'_{11}| = \begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 2 \\ 6 & 7 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 6 & 8 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 6 & 7 \end{vmatrix} = -6 + 12 + 2 = 8$$

$$|B'_{12}| = \begin{vmatrix} 1 & 3 & 4 \\ 1 & 1 & 4 \\ 6 & 7 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 6 & 7 \end{vmatrix} = -26 + 66 + 4 = 44$$

$$|B'_{21}| = \begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 2 \\ 9 & 0 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & -4 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 9 & -4 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 9 & 0 \end{vmatrix} = -4 + 66 - 18 = 44$$

$$|B'_{22}| = \begin{vmatrix} 1 & 3 & 4 \\ 1 & 1 & 4 \\ 9 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ 9 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 9 & 0 \end{vmatrix} = 0 + 108 - 36 = 72$$

$$D_{11} = |B| = (1/-2)\begin{vmatrix} 8 & 44 \\ 44 & 72 \end{vmatrix} = -680$$

By the same way we can find D_{12}, D_{21}, D_{22} , so

$D_{12} = -388 \qquad D_{21} = 176 \qquad D_{22} = 212$

Finally

$$|A| = (1/-8)\begin{vmatrix} -680 & -388 \\ 176 & 212 \end{vmatrix} = 9484.$$

2-Solve the following system of linear equations by general new formula with reduction value t=4.

$$\begin{aligned} 4X_1 - X_2 &= 100 \\ -X_1 + 4X_2 - X_3 &= 0 \\ -X_2 + 4X_3 - X_4 &= 0 \\ -X_3 + 4X_4 - X_5 &= 0 \\ -X_4 + 4X_5 &= 200. \end{aligned}$$

<u>Sol.</u>

$$A = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix} \quad , \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \\ 200 \end{bmatrix}$$

$$|A| = \left(1/\begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} \right)^{5-4} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2x2}$$

$$\begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} = 4\begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & 4 \end{vmatrix} = 60 - 4 = 56$$

Each of $a_{11}, a_{12}, a_{21}, a_{22}$ are the determinant of order 4x4 which can evaluate by previous technique [6].

$$a_{11} = \begin{vmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{vmatrix} = \left(1/\begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} \right) \begin{Vmatrix} \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} & \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & -1 & -1 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & 0 & -1 \end{vmatrix} & \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix} \end{Vmatrix} = (1/15)\begin{vmatrix} 56 & -15 \\ -15 & 60 \end{vmatrix} = 209$$

$$a_{12} = \left(1/\begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} \right) \begin{Vmatrix} \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} & \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & -1 & 0 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & 0 & -1 \end{vmatrix} & \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & -1 \end{vmatrix} \end{Vmatrix} = (1/15)\begin{vmatrix} 56 & 0 \\ -15 & -15 \end{vmatrix} = -56$$

$$a_{21} = \left(1/\begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} \right) \begin{Vmatrix} \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} & \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & -1 & -1 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & -1 \end{vmatrix} \end{Vmatrix} = (1/15)\begin{vmatrix} 56 & -15 \\ 0 & -15 \end{vmatrix} = -56$$

$$a_{22}=\left(1/\begin{vmatrix}4 & -1\\ -1 & 4\end{vmatrix}\right)\begin{vmatrix}\begin{vmatrix}4 & -1 & 0\\ -1 & 4 & -1\\ 0 & -1 & 4\end{vmatrix} & \begin{vmatrix}4 & -1 & 0\\ -1 & 4 & 0\\ 0 & -1 & 0\end{vmatrix}\\ \begin{vmatrix}4 & -1 & 0\\ -1 & 4 & -1\\ 0 & 0 & 0\end{vmatrix} & \begin{vmatrix}4 & -1 & 0\\ -1 & 4 & 0\\ 0 & 0 & 4\end{vmatrix}\end{vmatrix}=(1/15)\begin{vmatrix}56 & 0\\ 0 & 60\end{vmatrix}=224$$

$$|A|=(1/56)\begin{vmatrix}209 & -56\\ -56 & 224\end{vmatrix}=780$$

$X_1=|A1|/|A|$, $|A|$ is calculated =780.

$$|A1|=\begin{vmatrix}100 & -1 & 0 & 0 & 0\\ 0 & 4 & -1 & 0 & 0\\ 0 & -1 & 4 & -1 & 0\\ 0 & 0 & -1 & 4 & -1\end{vmatrix}=\left(1/\begin{vmatrix}100 & -1 & 0\\ 0 & 4 & -1\\ 0 & -1 & 4\end{vmatrix}\right)\begin{vmatrix}a'_{11} & a'_{12}\\ a'_{21} & a'_{22}\end{vmatrix}_{2x2}$$

Each of $a'_{11},a'_{12},a'_{21},a'_{22}$ are the determinant of order 4x4.

$$\begin{vmatrix}100 & -1 & 0\\ 0 & 4 & -1\\ 0 & -1 & 4\end{vmatrix}=1500$$

$$a'_{11}=\begin{vmatrix}100 & -1 & 0 & 0\\ 0 & 4 & -1 & 0\\ 0 & -1 & 4 & -1\\ 0 & 0 & -1 & 4\end{vmatrix}=\left(1/\begin{vmatrix}100 & -1\\ 0 & 4\end{vmatrix}\right)\begin{vmatrix}\begin{vmatrix}100 & -1 & 0\\ 0 & 4 & -1\\ 0 & -1 & 4\end{vmatrix} & \begin{vmatrix}100 & -1 & 0\\ 0 & 4 & 0\\ 0 & -1 & -1\end{vmatrix}\\ \begin{vmatrix}100 & -1 & 0\\ 0 & 4 & -1\\ 0 & 0 & -1\end{vmatrix} & \begin{vmatrix}100 & -1 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{vmatrix}\end{vmatrix}=(1/400)\begin{vmatrix}1500 & -400\\ -400 & 1600\end{vmatrix}=5600$$

$$a'_{12}=\begin{vmatrix}100 & -1 & 0 & 0\\ 0 & 4 & -1 & 0\\ 0 & -1 & 4 & 0\\ 0 & 0 & -1 & -1\end{vmatrix}=\left(1/\begin{vmatrix}100 & -1\\ 0 & 4\end{vmatrix}\right)\begin{vmatrix}\begin{vmatrix}100 & -1 & 0\\ 0 & 4 & -1\\ 0 & -1 & 4\end{vmatrix} & \begin{vmatrix}100 & -1 & 0\\ 0 & 4 & 0\\ 0 & -1 & 0\end{vmatrix}\\ \begin{vmatrix}100 & -1 & 0\\ 0 & 4 & -1\\ 0 & 0 & -1\end{vmatrix} & \begin{vmatrix}100 & -1 & 0\\ 0 & 4 & 0\\ 0 & 0 & -1\end{vmatrix}\end{vmatrix}=(1/400)\begin{vmatrix}1500 & 0\\ -400 & -400\end{vmatrix}=-1500$$

$$a'_{21} = \begin{vmatrix} 100 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 200 & 0 & 0 & -1 \end{vmatrix} = \left(1/\begin{vmatrix} 100 & -1 \\ 0 & 4 \end{vmatrix}\right) \begin{Vmatrix} \begin{vmatrix} 100 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} & \begin{vmatrix} 100 & -1 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & -1 \end{vmatrix} \\ \begin{vmatrix} 100 & -1 & 0 \\ 0 & 4 & -1 \\ 200 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 100 & -1 & 0 \\ 0 & 4 & 0 \\ 200 & 0 & -1 \end{vmatrix} \end{Vmatrix} = (1/400)\begin{vmatrix} 1500 & -400 \\ 200 & -400 \end{vmatrix} = -1300$$

$$a'_{22} = \begin{vmatrix} 100 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ 200 & 0 & 0 & 4 \end{vmatrix} = \left(1/\begin{vmatrix} 100 & -1 \\ 0 & 4 \end{vmatrix}\right) \begin{Vmatrix} \begin{vmatrix} 100 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} & \begin{vmatrix} 100 & -1 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 0 \end{vmatrix} \\ \begin{vmatrix} 100 & -1 & 0 \\ 0 & 4 & -1 \\ 200 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 100 & -1 & 0 \\ 0 & 4 & 0 \\ 200 & 0 & 4 \end{vmatrix} \end{Vmatrix} = (1/400)\begin{vmatrix} 1500 & 0 \\ 200 & 1600 \end{vmatrix} = 6000$$

$$|A1| = (1/1500)\begin{vmatrix} 5600 & -1500 \\ -1300 & 6000 \end{vmatrix} = 21100$$

$$|X1| = |A1|/|A| = 21100/780 = 27.051$$

By similar manner we can evaluate X_2, X_3, X_4, X_5