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Compare several methods for estimating parameters for [0,1] Truncated Nadarajah-Haghighi Exponential distribution with simulation and Real Data Application

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Abstract: This work introduces a number of methods for estimating the parameters of the [0,1] Truncated Nadarajah-Haghighi exponential distribution. The percentile method, Cramer von Misses, maximum likelihood, ordinary least square, weighted least square, maximum product spaces, and Anderson darling are a few of these methods. Also, we find the quantile function, survival function, hazard function, cumulative hazard function, and odd function of the distribution, in addition to that, we find the entropy. To find the best method for estimating parameters for the new distribution, a simulation analysis is carried out. Table 3 shows that the new distribution has the highest p-value, indicating a better fit compared to the other distributions. In contrast, the Anderson value (A) and Cramer value (W) and K-S value for the new distribution are the lowest, suggesting a relatively poorer fit compared to the other distributions when fitted. According to Table 5, the new distribution exhibits the lowest values for information criteria such as AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion), and HQIC (Hannan-Quinn Information Criterion). This suggests that the new distribution fits the data better than the other distributions being compared.

Keywords: exponential distribution, hazard function, estimation, simulation, entropy.

مقارنة عدة طرق لتقدير معلمات التوزيع الأسي ناداراجا حقيقي المبتور [٠،١] مع المحاكاة والتطبيق العملي

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المستخلص: قدمنا في هذا العمل عددا من طرق تقدير المعلمات للتوزيع الأسي ناداراجا حقيقي المبتور [٠،١] مثل طريقة الإمكان الأعظم وطريقة المربعات الصغرى الاعتيادية وطريقة المربعات الصغرى المرجحة وطريقة الحد الأقصى الناتج لمقدرات التباعد وطريقة كريمر-فون ميزس وطريقة اندرسون دارلينج طريقة الذيل الأيمن اندرسون دارلينج وطريقة المقدر المئوية. كما وتم إيجاد دوال الكمية المخاطرة والبقاء والدالة الفردية للتوزيع. بالإضافة إلى ذلك استخدمنا المحاكاة لبيان الطريقة الأفضل لتقدير معلمات التوزيع، يوضح الجدول ٣ أن التوزيع الجديد يحتوي على أعلى قيمة p-value في المقابل، فإن قيم Anderson (A و Cramer (W و K-S للتوزيع الجديد هي الأدنى، مما يشير إلى ملاءمة أفضل مقارنة بالتوزيعات الأخرى. وفقًا للجدول ٥، يعرض التوزيع الجديد القيم الأدنى لمعايير المعلومات مثل AIC (معيار معلومات Akaike) و CAIC (معيار معلومات Akaike المتسق) و BIC (معيار المعلومات البايزية) و HQIC (معيار معلومات Hannan-Quinn). يشير هذا إلى أن التوزيع الجديد يناسب البيانات بشكل أفضل من التوزيعات الأخرى التي تتم مقارنتها.

الكلمات المفتاحية: توزيع اسي، دالة مخاطرة، تقدير، محاكاة، دالة الانتروبي.

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1 Introduction

The [0,1] Truncated Nadarajah-Haghighi exponential distribution is a statistical distribution used to describe continuous data that is constrained to the interval [0, 1]. This distribution is an extension of the standard exponential distribution and can be useful in a variety of applications, including finance, engineering, and medicine. Estimating the parameters of the [0,1] Truncated Nadarajah-Haghighi exponential distribution involves finding the values of the parameters that best fit the data. This can be done using OLSE (Ordinary Least Squares), WLSE (Weighted Least Squares), MPSE (Maximum Product of Spacing Estimation), CVM (Cramér-Von Mises Estimation), Anderson–Darling (ADE), Right-Tail Anderson–Darling (RADE) , PCE (Percentile estimation) Analysis, and MLE (Maximum Likelihood estimation). The parameters of the distribution include the scale parameter and the shape parameter, which determine the location and shape of the distribution, respectively. Accurate parameter estimation is crucial for making informed decisions based on data analysis. By fitting the truncated Nadarajah-Haghighi exponential distribution to your data, you can gain valuable insights into the behavior of your system or process, and make more accurate predictions about future outcomes.

In addition, many researcher introduced new extensions for the exponential distribution such as. Khalaf and Khaleel (2022), [0, 1] Truncated exponentiated exponential gompertz distribution. Gupta et al., (2001), Generalized exponential distribution, Nassar (2018c) , A new generalization of the exponentiated Pareto distribution. Bilal et al., (2021). Weibull-exponential, Nadarajah et al., (2006), the beta exponential distribution. Rasekhi et al., (2017), the Modified exponential distribution. On other hand the estimating of the parameters for some distributions related to the exponential distribution are proposed like, Akahira, M. (2017). Statistical estimation for truncated exponential families. Proschan and Sullo (1976. Estimating the parameters of multivariate exponential distribution. Nasiri et al., (2011), Estimation parameters of the weighted exponential distribution. Bilal et al., (2021), Weibull-exponential distribution and its application in monitoring industrial process. Almetwally et al., (2018), Bayesian and maximum likelihood estimation for the Weibull generalized exponential distribution parameters using progressive censoring schemes. Bdair, O. M. (2012). Different methods of estimation for Marshall-Olkin exponential distribution. Nassar et al., (2020), Estimation methods of alpha power exponential distribution with applications to engineering and medical data. Dey et al., (2018). Kumaraswamy distribution: different methods of estimation.

Sections of the paper are organized as follows: Section 1: Introduction with Previous Studies; Section 2 [0, 1] Truncated Nadarajah-Haghighi Exponential distribution, section 3: some statistical properties of the distribution, section 4: different methods to estimate the parameters of the new distribution, section 5: a simulation study, section 6: application with a real dataset, and finally section 7: a conclusion.

2 [0,1] Truncated Nadarajah-Haghighi Exponential distribution

The [0,1] Truncated Nadarajah-Haghighi-G family of distributions introduced and studied by Al-Habib et al., (2023) with the following CDF and PDF:

$$F(x)_{TNH-G} = \frac{1- e^{1-\left(1+bM(x,\boldsymbol{\varphi})\right)^{a}}}{1- e^{1-(1+b)^{a}}} \tag{1}$$

and

$$f(x)_{TNH-G} = \frac{ab\left(1+bM(x,\boldsymbol{\varphi})\right)^{a-1} e^{1-\left(1+bM(x,\boldsymbol{\varphi})\right)^{a}} m(x,\boldsymbol{\varphi})}{1- e^{1-(1+b)^{a}}} \tag{2}$$

where a is a shape parameter and b is scale parameter, and $M(x,\boldsymbol{\varphi}), m(x,\boldsymbol{\varphi})$ are respectively the CDF and PDF for the baseline distribution with vector of parameters $\boldsymbol{\varphi}$.

Now for the exponential distribution then we have the following CDF and PDF successively.

$$M(x,\lambda) = 1 - e^{-\lambda x} \tag{3}$$

$$m(x,\lambda) = \lambda e^{-\lambda x} \tag{4}$$

Then, by inserting Equations (3) and (4) into the equations (1) and (2), we get that, the CDF and PDF of the [0,1] Truncated Nadarajah-Haghighi exponential distribution will be as follows:

$$F(x)_{[0,1]TNHExpo} = \frac{1 - e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}}}{1 - e^{1-(1+b)^{a}}} \qquad x > 0, a, b, \lambda > 0 \tag{5}$$

$$f(x)_{[0,1]TNHExpo} = \frac{ab\lambda\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a-1} e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}} e^{-\lambda x}}{1 - e^{1-(1+b)^{a}}} \quad x > 0, a, b, \lambda > 0 \tag{6}$$

3 Some Statistical Properties

A. The Survival Rate Function

The survival function of [0,1] TNHE distribution can be obtained as follows

$$S(x,a,b,\lambda)_{[0,1]TNHExpo} = \frac{e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}} - e^{1-(1+b)^{a}}}{1 - e^{1-(1+b)^{a}}} \tag{7}$$

B. Hazard Rate Function

Hazard rate function of [0,1] TNHE distribution can be written as follows

$$h(x,a,b,\lambda)_{[0,1]TNHE} = \frac{ab\lambda\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a-1} e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}} e^{-\lambda x}}{e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}} - e^{1-(1+b)^{a}}} \tag{8}$$

C. Reversed Hazard Rate Function

The reversed hazard rate function of the distribution can be written as

$$r(x,a,b,\lambda)_{[0,1]TNHE} = \frac{ab\lambda\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a-1} e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}} e^{-\lambda x}}{1-e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}}} \tag{9}$$

D. Cumulative Hazard Rate Function

The cumulative hazard rate function of [0,1] TNHE can be expressed as

$$H(x,a,b,\lambda)_{[0,1]TNHE} = -ln\left\{\frac{e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}} - e^{1-(1+b)^{a}}}{1-e^{1-(1+b)^{a}}}\right\} \tag{١٠}$$

E. Odd Function

The odd function of [0,1] TNHE distribution can be written as

$$O(x,a,b,\lambda)_{[0,1]TNHE} = \frac{1-e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}}}{e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}} - e^{1-(1+b)^{a}}} \tag{11}$$

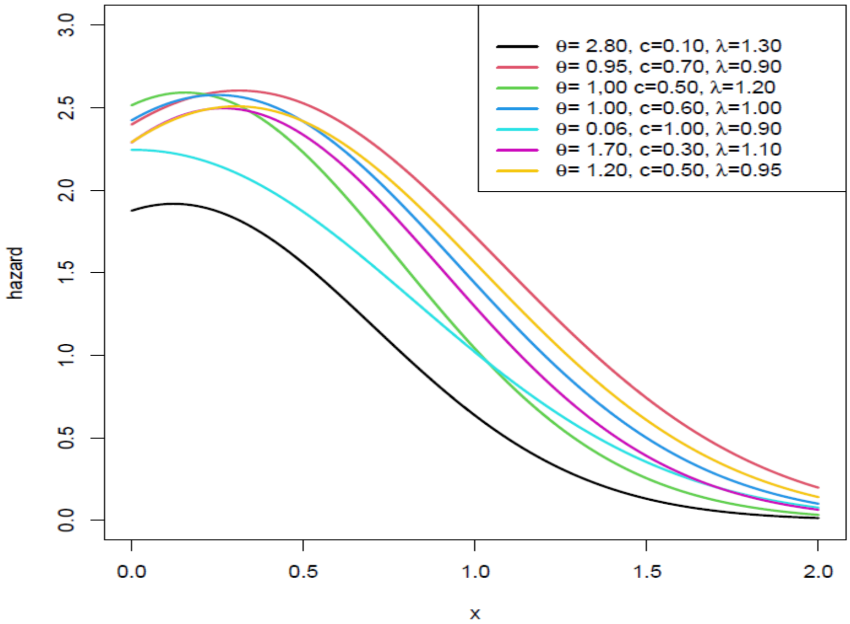

Figure (1): The hazard function of [0,1] TNHE distribution for some selected parameters.

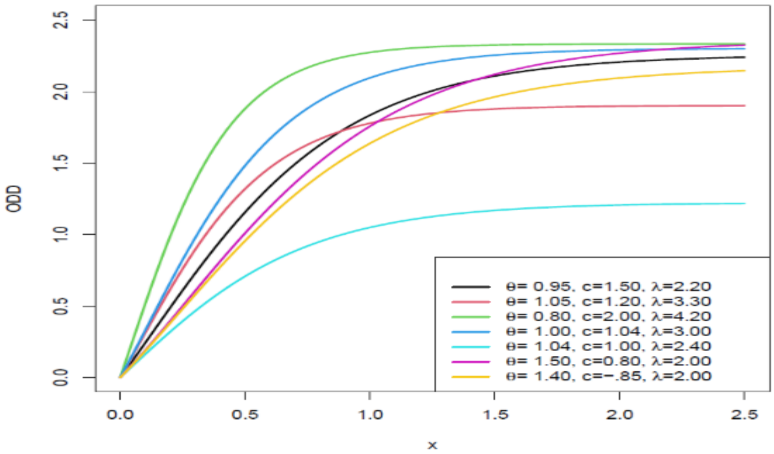

Figure (2): The Odd function of [0,1] TNHE distribution for some selected parameters.

F. Quantile function

The [0,1] TNHE distribution quantile function can be abtained by inverting the CDF which defined in (5) as follow $Q(u) = F^{-1}{}_{[0,1]TNHExpo}(x)$ where $u = F_{[0,1]TNHE}(x)$ so that

$$x = \frac{-1}{\lambda}\left(\ln\left\{1-\frac{1}{b}\left(\left(1-\ln\left\{1-\ u\left(1-e^{1-(1+b)^{a}}\right)\right\}\right)^{\frac{1}{a}}-1\right)\right\}\right) \tag{12}$$

Now according to eq.(3.10) the median of [0.1] TNHE distribution can be obtained by instead $u = 0.5$, then

$$x = \frac{-1}{\lambda}\left(\ln\left\{1-\frac{1}{b}\left(\left(1-\ln\left\{1-\ 0.5\left(1-e^{1-(1+b)^{a}}\right)\right\}\right)^{\frac{1}{a}}-1\right)\right\}\right) \tag{13}$$

G. Survival function stress strength

Let Y and X be the stress and strength random variables, independent of each other, with parameters (a_2, b_2, λ_2) , (a_1, b_1, λ_1), respectively, then

$$SS = P(y < x) = \int_0^\infty f_X(x)F_Y(x)dx$$

$$SS_{[0,1]TNHE} = \int_0^\infty f_X(x)_{[0,1]TNHE}\ F_Y(x)_{[0,1]TNHE}dx$$

$$SS_{[0,1]TNHE} = \int_0^\infty \Omega_{s,n,k,d}\, e^{-\lambda_1 x(k+1)}\ \left(\frac{1-\Omega_{l,c,i}\ e^{-\lambda_2 x}}{1-e^{1-(1+b_2)^{a_2}}}\right)dx$$

$$= \Omega_{s,n,k,d}\left(\frac{1}{1-e^{1-(1+b_2)^{a_2}}}\right)\int_0^\infty\ e^{-\lambda_1 x(k+1)}\ \left(1-\Omega_{l,c,i}\ e^{-\lambda_2 x}\right)dx$$

$$= \Omega_{s,n,k,d}\left(\frac{1}{1-e^{1-(1+b_2)^{a_2}}}\right)\left[\int_0^\infty\ \left(e^{-\lambda_1 x(k+1)}-\Omega_{l,c,i}\, e^{-\lambda_1 x(k+1)}\, e^{-\lambda_2 x}\right)dx\right]$$

$$= \Omega_{s,n,k,d}\left(\frac{1}{1-e^{1-(1+b_2)^{a_2}}}\right)\left[\int_0^\infty\ \left(e^{-\lambda_1 x(k+1)}\right)dx\ -\int_0^\infty \Omega_{l,c,i}\, e^{-\lambda_1 (m+1)x}\ e^{-\lambda_2 x}\ dx\right]$$

$$= \Omega_{s,n,k,d}\left(\frac{1}{1-e^{1-(1+b_2)^{a_2}}}\right)\left[\int_0^\infty\ \left(e^{-\lambda_1 x(k+1)}\right)dx\ -\Omega_{l,c,i}\int_0^\infty\ e^{-\lambda_2-\lambda_1 (m+1)x}\ \ dx\right]$$

$$= \Omega_{s,n,k,d}\left(\frac{1}{1-e^{1-(1+b_2)^{a_2}}}\right)\left(\frac{1}{\lambda_1(k+1)}-\left(\Omega_{l,c,i}\right)\ \frac{e^{-\lambda_1}}{\lambda_1(k+1)}\right)$$

4 Estimation Methods

In this section, we will use eight methods to estimate the parameters of the [0,1] TNHE distribution, which are the ordinary least squares and weighted least squares methods, the maximum product spacing method, the Cramér–von Mises method, the Anderson-Darling and Right-Tail Anderson-Darling methods, and the maximum likelihood method.

A. Ordinary Least - Squares (OLS)

The ordinary least-squares estimates of distribution can be found by minimize the equation

$$V(a,b,\lambda) = \sum_{i=1}^{n}\left(\frac{1-\ e^{1-(1+bM(x_i,\xi))^{a}}}{1-\ e^{1-(1+b)^{a}}} - \frac{i}{n+1}\right)^{2}$$

Then for [0,1] TNHE distribution the ordinary least-squares estimates of distribution can be found by minimize the equation

$$V(a,b,\lambda) = \textstyle\sum_{i=1}^{n}\left(\frac{1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^{a}}}{1-e^{1-(1+b)^{a}}} - \frac{i}{n+1}\right)^{2}$$

They can also be found by solving the non-linear equations

$$\sum_{i=1}^{n}\left(\frac{1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^{a}}}{1-e^{1-(1+b)^{a}}} - \frac{i}{n+1}\right)\Delta_s(x_i) = 0 \quad ,s = 1,2,3$$

Where$\Delta_1(x_i) = \Delta_1(x_i \backslash a,b,\lambda)$, $\Delta_2(x_i) = \Delta_2(x_i \backslash a,b,\lambda)$, $\Delta_3(x_i) = \Delta_3(x_i \backslash a,b,\lambda)$ are defined as follows

$$\Delta_1(x_i) = \frac{\partial}{\partial a}\left(\frac{1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^{a}}}{1-e^{1-(1+b)^{a}}}\right)$$

$$= \frac{1}{\left[1-e^{1-[1+b]^{a}}\right]^{2}}\left\{\begin{array}{c}\left(1-e^{1-(1+b)^{a}}\right)\times\left(e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^{a}}\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^{a}\right) \\ \times\, log\left(1+b\left(1-e^{-\lambda x_i}\right)\right) - \left(1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^{a}}\right) \\ \times\left(e^{1-(1+b)^{a}}\log(1+b)\,(1+b)^{a}\right)\end{array}\right\} \quad (14)$$

and

$$\Delta_2(x_i) = \frac{\partial}{\partial b}\left(\frac{1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^{a}}}{1-e^{1-(1+b)^{a}}}\right)$$

$$= \quad (15)$$

$$\frac{1}{\left(1-e^{1-(1+b)^a}\right)^2}\begin{Bmatrix}\left(1-e^{1-(1+b)^a}\right)\times\left(a\left(1-e^{-\lambda x_i}\right)e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^a}\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^{a-}\right.\\ -\left(1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^a}\right)\times\left(a(1+b)^{a-1}e^{1-(1+b)^a}\right)\end{Bmatrix}$$

and

$$\Delta_3(x_i)=\frac{\partial}{\partial\lambda}\left(\frac{1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^a}}{1-e^{1-(1+b)^a}}\right)$$

$$=\frac{1}{1-e^{1-(1+b)^a}}\left\{ab\,x_i\,e^{-\lambda x_i}\left(1+b(1-e^{-\lambda x_i})\right)^{a-1}e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^a-\lambda x_i}\right\} \qquad (16)$$

B. Weighted least-squares (WLS)

The weighted least-squares estimates of [0.1] TNHE distribution can be found by minimize the equation

$$W(a,b,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(\frac{1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^a}}{1-e^{1-(1+b)^a}}-\frac{i}{n+1}\right)^2$$

This can be found by solving the non-linear system of equation.

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(\frac{1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^a}}{1-e^{1-(1+b)^a}}-\frac{i}{n+1}\right)\Delta_s(x_i)=0 \quad ,s=1,2,3$$

where $\Delta_1(x_i)=\Delta_1(x_i\backslash a,b,\lambda)$,$\Delta_2(x_i)=\Delta_2(x_i\backslash a,b,\lambda)$, $\Delta_3(x_i)=\Delta_3(x_i\backslash a,b,\lambda)$ as in equations (14), (15), (16).

C. Maximum Product Spacing (MPS)

If $x_1<\cdots<x_n$ be a random sample of size n we can describe the uniform spacing of the [0,1] TNHE distribution as:

$$D_i(a,b,\lambda)=F(x_i)-F(x_{i-1});i=1,\dots,n+1$$

Where

$$F\left(x_{(0)}\right)=0,F\left(x_{(n+1)}\right)=1$$

And

$$\sum_{i=1}^{n+1}D_i=1.$$

The MPSE of [0,1] TNHE distribution parameters can be obtained by maximizing.

$$G(a,b,\lambda)=\sum_{i=1}^{n+1}\log\{D_i\}$$

This can be determined from the non-linear equations.

$$\sum_{i=0}^{n+1}\frac{1}{D_i}\left(\Delta_s(x_i)-\Delta_s(x_{i-1})\right)=0\,,s=1,2,3$$

where $\Delta_1(x_i) = \Delta_1(x_i \backslash a,b,\lambda)$,$\Delta_2(x_i) = \Delta_2(x_i \backslash a,b,\lambda)$, $\Delta_3(x_i) = \Delta_3(x_i \backslash a,b,\lambda)$ as in equations (14), (15), (16).

D. Cramér–Von Mises (CVM)

In Cramér-von Mises it is obtained on the basis of the difference between the estimates of the cumulative distribution and the empirical distribution function. The CVM of the [0,1] TNHE distribution parameters are obtained by minimizing.

$$CVM(a,b,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left(\frac{1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^a}}{1-e^{1-(1+b)^a}} - \frac{2i-1}{2n}\right)^2$$

Which also follow by solving the non-linear equation.

$$\sum_{i=1}^{n}\left(\frac{1-e^{1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^a}}{1-e^{1-(1+b)^a}} - \frac{2i-1}{2n}\right)\Delta_s(x_i) = 0\,,\ \ s = 1,2,3$$

where $\Delta_1(x_i) = \Delta_1(x_i \backslash a,b,\lambda)$,$\Delta_2(x_i) = \Delta_2(x_i \backslash a,b,\lambda)$, $\Delta_3(x_i) = \Delta_3(x_i \backslash a,b,\lambda)$ as in equations (14), (15), (16).

E. Anderson- Darling (AD)

The AD of the parameters of the [0,1] TNHE distribution are obtained by minimizing

$$AD(a,b,\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\log\left\{\frac{1-e^{1-(1+b(1-e^{-\lambda x}))^a}}{1-e^{1-(1+b)^a}}\right\} + \log\left\{\frac{e^{1-(1+b(1-e^{-\lambda x}))^a} - e^{1-(1+b)^a}}{1-e^{1-(1+b)^a}}\right\}\right)$$

This can be found by solving the non-linear equation:

$$AD(a,b,\lambda) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\frac{\Delta_s(x_i)}{F(x_i)} - \frac{\Delta_i(x_{n+1-i})}{S(x_{n+1-i})}\right) = 0\,, s = 1,2,3$$

where $\Delta_1(x_i) = \Delta_1(x_i \backslash a,b,\lambda)$,$\Delta_2(x_i) = \Delta_2(x_i \backslash a,b,\lambda)$,

$\Delta_3(x_i) = \Delta_3(x_i \backslash a,b,\lambda)$as defined in (14), (15), (16).

F. Right Anderson-Darling (RAD)

The Right-Tail Anderson-Darling of the [0,1] TNHE distribution parameters are obtained by minimizing.

$$RAD(\Psi) = \frac{n}{2} - 2\sum_{i=1}^{n} \frac{1 - e^{1-\left(1+b\left(1-e^{-\lambda x}\right)\right)^{a}}}{1 - e^{1-(1+b)^{a}}} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left\{\frac{e^{1-\left(1+b\left(1-e^{-\lambda x_{n+1-i}}\right)\right)^{a}} - e^{1-(1+b)^{a}}}{1 - e^{1-(1+b)^{a}}}\right\}$$

Additionally, they can be acquired from the non-linear equations.

$$-2\sum_{i=1}^{n}\Delta_s(x_i) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\ \frac{\Delta_S(x_{n+1-i})}{S(x_{n+1-i})} = 0\ , \quad s = 1,2,3$$

where$\Delta_1(x_i) = \Delta_1(x_i \backslash a, b, \lambda)$,$\Delta_2(x_i) = \Delta_2(x_i \backslash a, b, \lambda)$,

$\Delta_3(x_i) = \Delta_3(x_i \backslash a, b, \lambda)$ as in equations (14), (15), (16).

G. Percentile Method (PC)

The percentile (PCE) method can be used to estimate the parameters of a distribution. Given that $q_i = \frac{i}{n+1}$ is an unbiased estimator of $G\left(x_{(i)}|a,b,\lambda\right)$ then PCE of the parameters of [0,1]TNHE distribution can be obtained by minimizing.

$$PC(a,b,\lambda) = \sum_{i=1}^{n}\left(x_{(i)} + \frac{1}{\lambda}\left(\ln\left\{1 - \frac{1}{b}\left(\left(1 - \ln\left\{1 - u_i\left(1 - e^{1-(1+b)^{a}}\right)\right\}\right)^{\frac{1}{a}} - 1\right)\right\}\right)\right)^{2}$$

With respect to a, b, λ.

H. Maximum Likelihood Method (MLE)

Suppose that $x_1, x_2, \dots, x_n$ is random sample of size n from the [0,1]TNHE distribution. Then the corresponding log-likelihood function is given by:

$$L(\varphi \backslash X) = \frac{(ab\lambda)^n\ e^{-\lambda\sum_{i=1}^{n}x_i}\ \sum_{i=1}^{n}\left(\left(1 + b\left(1 - e^{-\lambda x_i}\right)\right)\right)^{a-1}\ e^{\sum_{i=1}^{n}\left(1-\left(1+b\left(1-e^{-\lambda x_i}\right)\right)^{a}\right)}}{\left(1 - e^{1-(1+b)^{a}}\right)^{n}}$$

Let $l = logL(\varphi \backslash X)$ be the natural logarithm probability function.

$$l = nlog(ab\lambda) - \lambda\sum_{i=1}^{n}x_i + (a-1)\sum_{i=1}^{n}log\left\{\left(1 + b\left(1 - e^{-\lambda x_i}\right)\right)\right\} + \sum_{i=1}^{n}\left(1 - \left(1 + b\left(1 - e^{-\lambda x_i}\right)\right)^{a}\right) - nlog\left\{1 - e^{1-(1+b)^{a}}\right\}$$

$$l = nlog(a) + nlog(b) + nlog(\lambda) - \lambda \sum_{i=1}^{n} x_i + \left(a \sum_{i=1}^{n} log\{(1 + b(1 - e^{-\lambda x_i}))\} - \sum_{i=1}^{n} log\{(1 + b(1 - e^{-\lambda x_i}))\} \right) + \sum_{i=1}^{n} 1 - \sum_{i=1}^{n} (1 + b(1 - e^{-\lambda x_i}))^{a}) - nlog(1 - e^{1-(1+b)^a})$$

$$l = nlog(a) + nlog(b) + nlog(\lambda) - \lambda \sum_{i=1}^{n} x_i + \left(a \sum_{i=1}^{n} log\{(1 + b(1 - e^{-\lambda x_i}))\} - \sum_{i=1}^{n} log\{(1 + b(1 - e^{-\lambda x_i}))\} \right) + n - \sum_{i=1}^{n} (1 + b(1 - e^{-\lambda x_i}))^{a}) - nlog\{1 - e^{1-(1+b)^a}\}$$

Now by take the first partial derivative for log likelihood function with respect to the parameters (a, b, λ) we have get:

$$\frac{\partial(l)}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} log\{(1 + b(1 - e^{-\lambda x_i}))\} + \left(1 + b(1 - e^{-\lambda x_i})\right)^a log\{(1 + b(1 - e^{-\lambda x_i}))^a\} - \frac{n\,(1 + b)^a \; log\,\{1 + b\} e^{1-(1+b)^a}}{1 - e^{1-(1+b)^a}}$$

$$\frac{\partial(l)}{\partial b} = \frac{n}{b} + (a-1) \sum_{i=1}^{n} \frac{1 - e^{-\lambda x_i}}{1 + b(1 - e^{-\lambda x_i})} + \sum_{i=1}^{n} -\left(a\left(1 - e^{-\lambda x_i}\right) \left(1 + b\left(1 - e^{-\lambda x_i}\right)\right)^{a-1} \right) - \frac{na(1 + b)^{a-1} \;\; e^{1-(1+b)^a}}{1 - e^{1-(1+b)^a}}$$

$$\frac{\partial(l)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (a-1) \sum_{i=1}^{n} \frac{b\, x_i\, e^{-\lambda x_i}}{1 + b(1 - e^{-\lambda x_i})} + \sum_{i=1}^{n} -ab\, x_i\, e^{-\lambda x_i} \left(1 + b(1 - e^{-\lambda x_i})\right)^{a-1}$$

Now and by setting the above equations $\frac{\partial(l)}{\partial a}, \frac{\partial(l)}{\partial b}, \frac{\partial(l)}{\partial b}$ to zero and solving it numerically using iterative methods such as Newton-Raphson type algorithms we can get the estimators of the parameters.

5 Simulation

In this section, the success of the estimators for [0,1] TNHE distribution parameters is examined for different sample sizes n. It is generated from N =5000 samples of size n = 30, 50, 80, 120, and 200 from the [0,1] TNHE distribution based on the true parameter values $a = \{0.7, 3\}, b = \{3, 1.75\}, \boldsymbol{\lambda} = \{1.6, 0.4\}$. The R software is used to acquire all estimations. In addition, for comparisons between the methods, the average of absolute value of biases |BIAS|, the average of mean square error MREs and the average of mean relative error MSEs.

$$|\mathrm{Bais}\tau| = \frac{1}{N}\sum_{i=1}^{N} |\hat{\tau} - \tau|,\ \mathrm{MSE} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\tau} - \tau)^2,\ \text{and MRE} = \frac{1}{N}\sum_{i=1}^{N} |\hat{\tau} - \tau|/\tau\ .$$

Table 1: simulated results of the [0,1] TNHE distribution for
$\tau = (a = 0.75, b = 3, \lambda = 1.6)^T$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	MQE
30	|BIAS|	$\hat{a}$	0.69258{6}	0.68590{5}	0.70680{8}	0.58180{2}	0.69412{7}	0.68222{4}	0.59767{3}	0.54988{1}
		$\hat{b}$	1.94093{6}	1.83557{4}	2.16870{7}	2.96560{8}	1.54080{2}	1.84367{5}	1.69743{3}	1.33454{1}
		$\hat{\lambda}$	0.46341{2}	0.44190{1}	0.63370{8}	0.54066{6}	0.47490{5}	0.47247{3}	0.47366{4}	0.55723{7}
	MSE	$\hat{a}$	0.71568{7}	0.66981{4}	0.78556{8}	0.57594{2}	0.67631{5}	0.68578{6}	0.56482{1}	0.66575{3}
		$\hat{b}$	8.81785{6}	7.99164{4}	10.62214{7}	21.90099{8}	5.85224{2}	8.54910{5}	6.68414{3}	3.15446{1}
		$\hat{\alpha}$	0.32170{2}	0.30200{1}	0.67415{8}	0.41531{6}	0.34403{5}	0.33340{3}	0.33631{4}	0.42213{7}
	MRE	$\hat{a}$	0.92344{6}	0.91453{5}	0.94240{8}	0.77573{2}	0.92549{7}	0.90963{4}	0.79689{3}	0.73317{1}
		$\hat{b}$	0.64698{6}	0.61186{4}	0.72290{7}	0.98853{8}	0.51360{2}	0.61456{5}	0.56581{3}	0.44485{1}
		$\hat{\lambda}$	0.28963{2}	0.27619{1}	0.39606{8}	0.33791{6}	0.29681{5}	0.29529{3}	0.29604{4}	0.34827{7}
	$\sum$ Ranks		43{6}	29{2.5}	69{8}	48{7}	40{5}	38{4}	28{1}	29{2.5}
50	|BIAS|	$\hat{a}$	0.61838{6}	0.63157{8}	0.60555{4}	0.51760{2}	0.62673{7}	0.61119{5}	0.53850{3}	0.43949{1}
		$\hat{b}$	1.69499{6}	1.60738{4}	1.92184{7}	2.50463{8}	1.38629{2}	1.62454{5}	1.56380{3}	1.18798{1}
		$\hat{\lambda}$	0.41704{2}	0.39010{1}	0.56133{8}	0.50772{6}	0.42270{3}	0.42885{4}	0.43263{5}	0.53917{7}
	MSE	$\hat{a}$	0.57843{8}	0.57588{6}	0.57672{7}	0.45202{2}	0.56002{4}	0.56248{5}	0.46267{3}	0.40891{1}
		$\hat{b}$	6.65090{6}	5.90590{4}	8.10134{7}	15.13042{8}	4.45637{2}	6.32955{5}	5.52342{3}	2.43347{1}
		$\hat{\lambda}$	0.26130{2}	0.23571{1}	0.51917{8}	0.36454{6}	0.27230{3}	0.27674{4}	0.28331{5}	0.39513{7}
	MRE	$\hat{a}$	0.82450{6}	0.84209{8}	0.80740{4}	0.69013{2}	0.83564{7}	0.81493{5}	0.71800{3}	0.58598{1}
		$\hat{b}$	0.56500{6}	0.53579{4}	0.64061{7}	0.83488{8}	0.46210{2}	0.54151{5}	0.52127{3}	0.39599{1}
		$\hat{\lambda}$	0.26065{2}	0.24381{1}	0.35083{8}	0.31733{6}	0.26419{3}	0.26803{4}	0.27039{5}	0.33698{7}
	$\sum$ Ranks		44{6}	37{4}	60{8}	48{7}	33{2.5}	42{5}	33{2.5}	27{1}
80	|BIAS|	$\hat{a}$	0.54897{5}	0.56121{8}	0.54227{4}	0.44685{2}	0.55877{7}	0.54972{6}	0.47730{3}	0.36613{1}
		$\hat{b}$	1.40998{6}	1.34817{4}	1.62896{7}	2.03626{8}	1.18962{2}	1.39187{5}	1.34770{3}	1.07483{1}
		$\hat{\lambda}$	0.38199{2}	0.36090{1}	0.49163{7}	0.47762{6}	0.38889{4}	0.38563{3}	0.40036{5}	0.52681{8}
	MSE	$\hat{a}$	0.46417{5}	0.46702{6}	0.47448{8}	0.35408{2}	0.45828{4}	0.47059{7}	0.37701{3}	0.25935{1}
		$\hat{b}$	4.35259{6}	4.16562{4}	5.47655{7}	9.66311{8}	3.12432{2}	4.25224{5}	3.84818{3}	1.92278{1}
		$\hat{\lambda}$	0.21815{2}	0.20072{1}	0.38857{8}	0.31226{6}	0.22862{4}	0.22055{3}	0.23712{5}	0.36811{7}
	MRE	$\hat{a}$	0.73196{5}	0.74828{8}	0.72302{4}	0.59579{2}	0.74503{7}	0.73296{6}	0.63641{3}	0.48817{1}
		$\hat{b}$	0.46999{6}	0.44939{4}	0.54299{7}	0.67875{8}	0.39654{2}	0.46396{5}	0.44923{3}	0.35828{1}
		$\hat{\lambda}$	0.23874{2}	0.22556{1}	0.30727{7}	0.29851{6}	0.24305{4}	0.24102{3}	0.25022{5}	0.32926{8}
	$\sum$ Ranks		39{5}	37{4}	59{8}	48{7}	36{3}	43{6}	33{2}	29{1}
120	|BIAS|	$\hat{a}$	0.50008{6}	0.50594{8}	0.47585{4}	0.40761{2}	0.50276{7}	0.48910{5}	0.43055{3}	0.34736{1}
		$\hat{b}$	1.23991{6}	1.15549{3}	1.44947{7}	1.72738{8}	1.04433{2}	1.22517{5}	1.20630{4}	0.94215{1}
		$\hat{\lambda}$	0.35413{3}	0.32928{1}	0.43311{6}	0.44365{7}	0.34895{2}	0.35778{4}	0.36863{5}	0.50423{8}
	MSE	$\hat{a}$	0.39689{8}	0.38584{7}	0.36616{4}	0.29417{2}	0.38016{6}	0.37973{5}	0.30814{3}	0.21296{1}
		$\hat{b}$	3.32050{6}	2.88655{4}	4.12917{7}	6.06715{8}	2.33056{2}	3.19514{5}	2.81924{3}	1.49895{1}
		$\hat{\lambda}$	0.18671{3}	0.16414{1}	0.29106{7}	0.26918{6}	0.18317{2}	0.19112{4}	0.19831{5}	0.33455{8}
	MRE	$\hat{a}$	0.66677{6}	0.67459{8}	0.63447{4}	0.54348{2}	0.67035{7}	0.65213{5}	0.57406{3}	0.46314{1}
		$\hat{b}$	0.41330{6}	0.38516{3}	0.48316{7}	0.57579{8}	0.34811{2}	0.40839{5}	0.40210{4}	0.31405{1}
		$\hat{\lambda}$	0.22133{3}	0.20580{1}	0.27070{6}	0.27728{7}	0.21809{2}	0.22361{4}	0.23039{5}	0.31515{8}
	$\sum$ Ranks		47{6}	36{4}	52{8}	50{7}	32{2}	42{5}	35{3}	30{1}
200	|BIAS|	$\hat{a}$	0.42840{6}	0.43320{7}	0.40433{4}	0.34839{2}	0.43403{8}	0.42412{5}	0.37631{3}	0.31861{1}
		$\hat{b}$	1.03233{5}	0.96879{3}	1.17512{7}	1.41937{8}	0.90549{2}	1.01729{4}	1.03314{6}	0.79574{1}
		$\hat{\lambda}$	0.31634{3}	0.30121{1}	0.36932{6}	0.40399{7}	0.31128{2}	0.31704{4}	0.33012{5}	0.47163{8}
	MSE	$\hat{a}$	0.29806{8}	0.29186{6}	0.27272{4}	0.21265{2}	0.28802{5}	0.29467{7}	0.23861{3}	0.16245{1}
		$\hat{b}$	2.13065{6}	1.95815{3}	2.63168{7}	3.85600{8}	1.70160{2}	2.08986{5}	2.06688{4}	1.09110{1}
		$\hat{\lambda}$	0.14506{4}	0.13330{1}	0.20493{6}	0.22384{7}	0.14190{2}	0.14374{3}	0.15942{5}	0.29321{8}
	MRE	$\hat{a}$	0.57120{6}	0.57760{7}	0.53910{4}	0.46452{2}	0.57871{8}	0.56549{5}	0.50175{3}	0.42481{1}
		$\hat{b}$	0.34411{5}	0.32293{3}	0.39171{7}	0.47312{8}	0.30183{2}	0.33910{4}	0.34438{6}	0.26525{1}

		$\hat{\lambda}$	*0.19771{3}*	*0.18825{1}*	*0.23083{6}*	*0.25249{7}*	*0.19455{2}*	*0.19815{4}*	*0.20632{5}*	*0.29477{8}*
	$\sum$ **Ranks**		*46{6}*	*32{2}*	*51{7.5}*	*51{7.5}*	*33{3}*	*41{5}*	*40{4}*	*30{1}*

Table 1. Showed that most of the estimators revealed the property of consistency, with the average of absolute value of biases |BIAS|, the average of mean square error MREs and the average of mean relative error MSEs decreasing as the sample size increased for all parameter combinations.

Table 2: simulated results of the [0,1] TNHE distribution for
$\boldsymbol{\tau = (a = 3, b = 1.75, \alpha = 0.4)^T}$

n	Est.	Est. Par.	WLSE	OLSE	MLE	MPSE	CVME	ADE	RADE	MQE
30	**|BIAS|**	$\hat{a}$	0.77914{5}	0.71513{3}	1.16753{8}	0.97334{7}	0.73152{4}	0.79651{6}	0.67067{2}	0.66014{1}
		$\hat{b}$	0.33671{5}	0.31753{3}	0.43242{7}	0.45862{8}	0.30366{1}	0.33908{6}	0.30603{2}	0.32414{4}
		$\hat{\lambda}$	0.18017{5}	0.17298{4}	0.19083{6}	0.27923{8}	0.15970{1}	0.19175{7}	0.17176{3}	0.16025{2}
	MSE	$\hat{a}$	0.98316{5}	0.82784{3}	2.41236{8}	1.61658{7}	0.87560{4}	1.02843{6}	0.75217{2}	0.71264{1}
		$\hat{b}$	0.22318{6}	0.20222{4}	0.33418{7}	0.39615{8}	0.17913{1}	0.21907{5}	0.18987{2}	0.19300{3}
		$\hat{\lambda}$	0.07385{5}	0.06864{4}	0.08200{7}	0.17209{8}	0.05828{2}	0.08138{6}	0.06051{3}	0.04625{1}
	MRE	$\hat{a}$	0.25971{5}	0.23838{3}	0.38918{8}	0.32445{7}	0.24384{4}	0.26550{6}	0.22356{2}	0.22005{1}
		$\hat{b}$	0.19241{5}	0.18144{3}	0.24710{7}	0.26207{8}	0.17352{1}	0.19376{6}	0.17488{2}	0.18522{4}
		$\hat{\lambda}$	0.45042{5}	0.43246{4}	0.47708{6}	0.69808{8}	0.39925{1}	0.47936{7}	0.42939{3}	0.40063{2}
	$\sum$ **Ranks**		46{5}	31{4}	64{7}	69{8}	19{1.5}	55{6}	21{3}	19{1.5}
50	**|BIAS|**	$\hat{a}$	0.69428{5}	0.63133{3}	0.95776{8}	0.75462{7}	0.63841{4}	0.70500{6}	0.58601{2}	0.57802{1}
		$\hat{b}$	0.29743{4}	0.27948{3}	0.38593{7}	0.41064{8}	0.26963{2}	0.29749{5}	0.26512{1}	0.30247{6}
		$\hat{\lambda}$	0.14376{4}	0.13753{3}	0.15660{7}	0.21658{8}	0.13028{1}	0.14883{6}	0.13618{2}	0.14399{5}
	MSE	$\hat{a}$	0.78606{5}	0.64377{3}	1.71574{8}	1.01923{7}	0.66463{4}	0.80770{6}	0.56848{2}	0.53703{1}
		$\hat{b}$	0.16898{6}	0.15437{3}	0.26227{7}	0.28864{8}	0.13953{2}	0.16537{5}	0.13386{1}	0.15934{4}
		$\hat{\lambda}$	0.03809{5}	0.03587{4}	0.05027{7}	0.09682{8}	0.03162{1}	0.04292{6}	0.03570{3}	0.03493{2}
	MRE	$\hat{a}$	0.23143{5}	0.21044{3}	0.31925{8}	0.25154{7}	0.21280{4}	0.23500{6}	0.19534{2}	0.19267{1}
		$\hat{b}$	0.16996{4}	0.15970{3}	0.22053{7}	0.23465{8}	0.15408{2}	0.16999{5}	0.15150{1}	0.17284{6}
		$\hat{\lambda}$	0.35941{4}	0.34383{3}	0.39151{7}	0.54144{8}	0.32571{1}	0.37208{6}	0.34044{2}	0.35998{5}
	$\sum$ **Ranks**		42{5}	28{3}	66{7}	69{8}	21{2}	51{6}	16{1}	31{4}
80	**|BIAS|**	$\hat{a}$	0.62117{6}	0.56731{3}	0.74262{8}	0.62183{7}	0.58709{4}	0.61794{5}	0.51935{1}	0.52804{2}
		$\hat{b}$	0.26759{6}	0.24770{3}	0.34393{7}	0.35785{8}	0.23930{2}	0.26597{4}	0.22942{1}	0.26719{5}
		$\hat{\lambda}$	0.11822{4}	0.11466{3}	0.12934{6}	0.16721{8}	0.10942{1}	0.12048{5}	0.11193{2}	0.13150{7}
	MSE	$\hat{a}$	0.63412{6}	0.52087{3}	1.06414{8}	0.71062{7}	0.56865{4}	0.62538{5}	0.44744{1}	0.45053{2}
		$\hat{b}$	0.13664{6}	0.11675{3}	0.19866{7}	0.20709{8}	0.10566{2}	0.13430{5}	0.09968{1}	0.12517{4}
		$\hat{\lambda}$	0.02321{3}	0.02338{4}	0.02893{7}	0.04902{8}	0.01969{1}	0.02543{5}	0.02188{2}	0.02870{6}
	MRE	$\hat{a}$	0.20706{6}	0.18910{3}	0.24754{8}	0.20728{7}	0.19570{4}	0.20598{5}	0.17312{1}	0.17601{2}
		$\hat{b}$	0.15291{6}	0.14154{3}	0.19653{7}	0.20449{8}	0.13674{2}	0.15198{4}	0.13109{1}	0.15268{5}
		$\hat{\lambda}$	0.29555{4}	0.28666{3}	0.32335{6}	0.41802{8}	0.27355{1}	0.30120{5}	0.27982{2}	0.32876{7}
	$\sum$ **Ranks**		47{6}	28{3}	64{7}	69{8}	21{2}	43{5}	12{1}	40{4}
120	**|BIAS|**	$\hat{a}$	0.53705{7}	0.50831{4}	0.61978{8}	0.50401{3}	0.52942{5}	0.53650{6}	0.46419{2}	0.45801{1}
		$\hat{b}$	0.23603{5}	0.22536{3}	0.29988{7}	0.32296{8}	0.22290{2}	0.23230{4}	0.20560{1}	0.24001{6}
		$\hat{\lambda}$	0.10496{4}	0.10199{3}	0.11246{6}	0.14274{8}	0.09988{2}	0.10599{5}	0.09765{1}	0.11955{7}
	MSE	$\hat{a}$	0.48605{7}	0.42744{3}	0.76006{8}	0.47879{5}	0.46014{4}	0.48070{6}	0.37016{2}	0.34690{1}
		$\hat{b}$	0.10667{6}	0.09786{3}	0.15152{7}	0.16642{8}	0.09462{2}	0.09975{4}	0.07978{1}	0.10072{5}
		$\hat{\lambda}$	0.01790{4}	0.01733{3}	0.02015{6}	0.03475{8}	0.01623{2}	0.01882{5}	0.01605{1}	0.02340{7}
	MRE	$\hat{a}$	0.17902{7}	0.16944{4}	0.20659{8}	0.16800{3}	0.17647{5}	0.17883{6}	0.15473{2}	0.15267{1}
		$\hat{b}$	0.13487{5}	0.12878{3}	0.17136{7}	0.18455{8}	0.12737{2}	0.13275{4}	0.11748{1}	0.13715{6}
		$\hat{\lambda}$	0.26240{4}	0.25498{3}	0.28115{6}	0.35684{8}	0.24970{2}	0.26498{5}	0.24412{1}	0.29888{7}
	$\sum$ **Ranks**		49{6}	29{3}	63{8}	59{7}	26{2}	45{5}	12{1}	41{4}
200	**|BIAS|**	$\hat{a}$	0.46311{6}	0.44864{4}	0.46654{7}	0.40147{2}	0.45916{5}	0.46771{8}	0.40359{3}	0.37830{1}
		$\hat{b}$	0.20933{6}	0.19421{3}	0.25717{7}	0.27960{8}	0.19403{2}	0.19952{4}	0.17206{1}	0.20910{5}
		$\hat{\lambda}$	0.08884{4}	0.08720{3}	0.09268{6}	0.11399{8}	0.08518{2}	0.08916{5}	0.08270{1}	0.10192{7}
	MSE	$\hat{a}$	0.36172{6}	0.33572{4}	0.44223{8}	0.30496{3}	0.36116{5}	0.36774{7}	0.27904{2}	0.23780{1}

	$\hat{b}$	0.08352{6}	0.07207{3}	0.10732{7}	0.12166{8}	0.06839{2}	0.07289{4}	0.05295{1}	0.07302{5}
	$\hat{\lambda}$	0.01222{3}	0.01223{4}	0.01345{6}	0.02061{8}	0.01127{2}	0.01284{5}	0.01116{1}	0.01687{7}
MRE	$\hat{a}$	0.15437{6}	0.14955{4}	0.15551{7}	0.13382{2}	0.15305{5}	0.15590{8}	0.13453{3}	0.12610{1}
	$\hat{b}$	0.11962{6}	0.11097{3}	0.14695{7}	0.15977{8}	0.11088{2}	0.11401{4}	0.09832{1}	0.11948{5}
	$\hat{\lambda}$	0.22211{4}	0.21801{3}	0.23169{6}	0.28496{8}	0.21294{2}	0.22289{5}	0.20674{1}	0.25480{7}
$\sum$ Ranks		47{5}	31{3}	61{8}	55{7}	27{2}	50{6}	14{1}	39{4}

Table 2. Demonstrated that the majority of the estimators exhibited consistency, as evidenced by the decline in the average absolute value of biases (|BIAS|), mean square error (MREs), and mean relative error (MSEs) with increasing sample sizes for all parameter combinations.

6 Application

In this section, we fit the [0,1]TNHE distribution a two real data set to demonstrate that the proposed distribution fits well when compared to competing distributions. *R* Statistical Software used to calculate all the results. In order to obtain the best results, we used the following statistical criteria such as AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion), and HQIC (Hannan-Quinn Information Criterion). for the proposed distribution compared to other distributions, such as beta exponential (BeE), Kumaraswamy exponential (KuE), Exponential Generalized exponential(EGE), Weibull exponential (WeE), Gompertez exponential (GoE), Marshal-Olkin exponential (MoE) and Exponential (E). The real data used represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients.

0.08 2.09 3.48 4.87 6.94 8.66 13.11 23.63 0.20 2.23 3.52 4.98 6.97 9.02 13.29 0.40 2.26 3.57 5.06 7.09 9.22 13.80 25.74 0.50 2.46 3.64 5.09 7.26 9.47 14.24 25.82 0.51 2.54 3.70 5.17 7.28 9.74 14.76 26.31 0.81 2.62 3.82 5.32 7.32 10.06 14.77 32.15 2.64 3.88 5.32 7.39 10.34 14.83 34.26 0.90 2.69 4.18 5.34 7.59 10.66 15.96 36.66 1.05 2.69 4.23 5.41 7.62 10.75 16.62 43.01 1.19 2.75 4.26 5.41 7.63 17.12 46.12 1.26 2.83 4.33 7.66 11.25 17.14 79.05 1.35 2.87 5.62 7.87 11.64 17.36 1.40 3.02 4.34 5.71 7.93 11.79 18.10 1.46 4.40 5.85 8.26 11.98 19.13 1.76 3.25 4.50 6.25 8.37 12.02 2.02 3.31 4.51 6.54 8.53 12.03 20.28 2.02 3.36 6.76 12.07 21.73 2.07 3.36 6.93 8.65 12.63 22.69 5.49

Table 3. The K-S value with its corresponding p-value and W value of the data set

Distribution	W	A	K-S	p-value
[0, 1]TNHE	0.0366	0.2653	0.0408	0.9829
BeE	0.0680	0.4627	0.0534	0.8572
KuE	0.0657	0.4461	0.0496	0.9103
EGE	0.0692	0.4700	0.0548	0.8365
WeE	0.0616	0.4190	0.0520	0.8779
GoE	0.0442	0.3081	0.0497	0.9087
MoE	0.0644	0.4368	0.0525	0.8709
E	0.0678	0.4617	0.0598	0.7491

Table 3. Shows that the new distribution has the highest p-value, indicating a better fit compared to the other distributions. In contrast, the Anderson value (A) and Cramer value (W) and K-S value for the new distribution are the lowest, suggesting a relatively poorer fit compared to the other distributions when fitted.

Table 4. Comparing the statistical criteria and estimation parameters of the [0,1] TNHE distribution with other distributions that are similar to it.

Distribution	MLEs	- l	AIC	CAIC	BIC	HQIC
[0,1] TNHE	$\hat{a}$ = 2.1986 $\hat{b}$ = 1.1129 $\hat{\lambda}$ = 0.0505	411.5697	806.538	806.7315	815.0941	810.0144

BeE	$\hat{a}$ = 0.9155 $\hat{b}$ = 2.9399 $\hat{\lambda}$ = 0.0368	412.3829	810.8642	811.0577	819.4203	814.3406
KuE	$\hat{a}$ = 0.9191 $\hat{b}$ = 4.5135 $\hat{\lambda}$ = 0.0227	412.4691	810.4805	810.674	819.0366	813.9569
EGE	$\hat{a}$ = 2.0183 $\hat{b}$ = 0.9223 $\hat{\lambda}$ = 0.0548	413.0776	811.0216	811.2152	819.5777	814.498
WeE	$\hat{a}$ = 0.9229 $\hat{b}$ = 0.1015 $\hat{\lambda}$ = 0.0123	414.0869	809.9406	810.1341	818.4967	813.417
GoE	$\hat{a}$ = 0.9987 $\hat{b}$ = 0.1178 $\hat{\lambda}$ = 0.1364	413.7588	807.3747	807.5682	815.9308	810.8511
MoE	$\hat{a}$ = 0.8662 $\hat{b}$ = 0.1084	414.3278	808.3763	808.4723	814.0803	810.6939
E	$\hat{\lambda}$ = 0.1168	414.3419	807.5198	807.5515	810.3718	808.6786

According to **Table 4**, the new distribution exhibits the lowest values for information criteria such as AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion), and HQIC (Hannan-Quinn Information Criterion). This suggests that the new distribution fits the data better than the other distributions being compared.

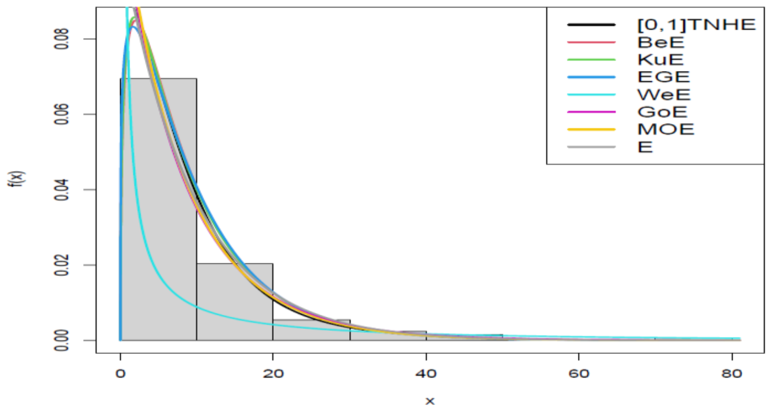

Figure 3. Estimated densities of distribution for second dataset.

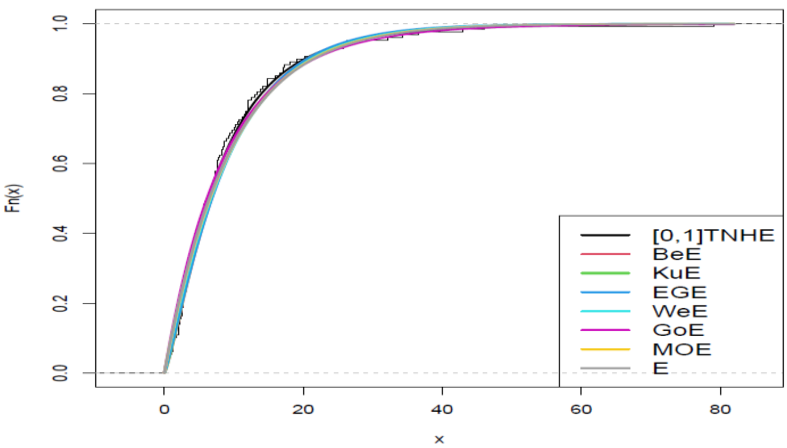

Figure 4. Estimated CDF of distribution for second dataset.

The empirically fitted plots of the PDF, CDF for the dataset presented in Figure 3 and Figure 4, respectively. These plots confirm the close fit of the [0,1] TNHE distribution to the dataset. The numerical results in this section are calculated using the statistical software R software statistical program.

Conclusion

In this research, the [0,1] Truncated Nadarajah-Haghighi exponential distribution is introduced, and its quantile function, survival function, hazard function, and entropy are proposed. The study employed eight parameter estimation methods, including OLSE (Ordinary Least Squares), WLSE (Weighted Least Squares), MPSE (Maximum Product of Spacing Estimation), CVM (Cramér-Von Mises), Anderson–Darling (ADE), Right-Tail Anderson–Darling (RADE) , PCE (Percentile estaimation) Analysis, and MLE (Maximum Likelihood estimation).to estimate the parameters of the new distribution. Compared to other distributions like the exponential distribution and the Weibull exponential distribution, which have been derived from real-life data applications, the three-parameter distribution proposed in this paper demonstrates greater flexibility.

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