

A new subclass of meromorphic univalent functions associated with linear operator L_k involving complex order λ

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Received 2 June 2022, accepted 7 September 2022, published 31 December 2022.

DOI: 10.52113/2/09.02.2022/16-21

Abstract: In this paper, we introduce a new subclass of analytic and meromorphic univalent function associated with linear operator involving complex order in the punctured unit disk. Then we characterized these functions and obtained some of properties of this subclass.

Keywords: Meromorphic function, Linear operator, univalent function, complex order.

2000 Mathematics Subject Classification.30C45.

1. Introduction

There are many researchers who have studied different classes of meromorphic univalent functions and meromorphic p -valent functions involving integral operators like Aouf [1], Atshan[2], Atshan and Kulkarni [3], Ponnusamy [8], Tehranchi and Kulkarni [10] and this operator was studied by Urologaddi and Somanatha [11]. Let RZ denote of the class of all meromorphic functions $f(z)$ defined as the following :

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (1)$$

Which are analytic and univalent in the punctured unit disk.

$$L^1(f(z)) = \frac{1}{z} + \sum_{n=1}^{\infty} (n+1)a_n z^n = \frac{zf'(z)}{z^2},$$

$$U^* = \{z: z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}.$$

Let RZH denote the subclass of RZ consist of the functions defined as:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geq 0. \quad (2)$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by the form

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n = (g * f)(z).$$

following the recent work of Liu and Sairastava [7] for a function belong to the class RZ given by (1) the Linear operator L^k is defined by:

$$L^0(f(z)) = f(z),$$

$$\begin{aligned} L^k f(z) &= L [L^{k-1} f(z)] \\ &= \frac{1}{z} \\ &+ \sum_{n=1}^{\infty} (n+2)^k a_n z^n \\ &= \frac{(z^2 L^{k-1} f(z))'}{z} . k \in N . \quad (3) \end{aligned}$$

It is easily verified from (4), that for $f \in RZ . k \in N$,

$$z[L^k f(z)]' = L^{k+1} f(z) - 2L^k f(z). \quad (5)$$

A function $f(z)$ defined by (2) belongs to the class RZH is said to be in the class RZH $(\alpha . m . \lambda . t . k)$ if and only if satisfies the following condition:

$$\left| \frac{\alpha [L^k f(z)]''' + (1-m)z^{-2}[L^k f(z)]'' + z^{-3}[L^k f(z)] - [2(1-m) - 6\alpha + 1]z^{-4}}{(1-\lambda)t - \alpha z^{-2}[L^k f(z)]' - (1-m)z^{-3}[L^k f(z)] + (1-\alpha - \lambda)z^{-4}} \right| < 1 \quad (6)$$

Where $0 < \alpha \leq 1 . 0 \leq m < 1 . 0 \leq \lambda < 1 . t \in \mathbb{C} / \{0\} . k \in N$.

2. Characterization of the functions

We now investigate the coefficient characterization theorem for the function

$f(z) \in RZH(\alpha . m . \lambda . t . k)$, then by obtaining the coefficient boundes .

Theorem 1 A function $f(z)$ defined by (2) is in the class RZH $(\alpha . m . \lambda . t . k)$ if and only if

$$\sum_{n=1}^{\infty} [[n(n-1)[\alpha(n-2) + 1 - m] + 1] + [\alpha n + 1 - m]] (n+2)^k a_n \leq (1-\lambda)|t|. \quad (7)$$

Proof Let (7) holds true , for $|z| = 1$, we have

$$\begin{aligned} &|\alpha [L^k f(z)]''' + (1-m)z^{-2}[L^k f(z)]'' \\ &+ z^{-3}[L^k f(z)] - [2(1-m) - 6\alpha + 1]z^{-4}| \\ &- |(1-\lambda)t - \alpha z^{-2}[L^k f(z)]' \\ &- (1-m)z^{-3}[L^k f(z)] + (1-\alpha - m)z^{-4}| \\ &= \left| \alpha [-6z^{-4} + \sum_{n=1}^{\infty} n(n-1)(n-2)(n+2)^k a_n z^{n-3} \right. \\ &+ (1-m)z^{-1}[2z^{-3} \\ &+ \sum_{n=1}^{\infty} n(n-1)(n+2)^k a_n z^{n-2}] \\ &+ z^{-3}[z^{-1} \\ &+ \sum_{n=1}^{\infty} (n-2)^k a_n z^n] \\ &\left. - [2(1-m) + 6\alpha + 1]z^{-4} \right| \\ &- \left| (1-\lambda)t - \alpha z^{-2}[-z^{-2} \right. \\ &+ \sum_{n=1}^{\infty} n(n+2)^k a_n z^{n-1}] - (1-m)z^{-3}[z^{-1} \\ &+ \sum_{n=1}^{\infty} (n+2)^k a_n z^n] \\ &\left. + (1-\alpha)z^{-4} \right| \\ &= | -6\alpha z^{-4} + \sum_{n=1}^{\infty} \alpha n(n-1)(n-2)(n+2)^k a_n z^{n-3} + 2(1-m)z^{-4} + \sum_{n=1}^{\infty} (1-m)n(n-1)(n+2)^k a_n z^{n-3} + z^{-4} + \sum_{n=1}^{\infty} n(n+2)^k a_n z^{n-3} - [2(1-m) + 6\alpha + 1] - (1-\lambda)t + \alpha z^{-4} - \sum_{n=1}^{\infty} \alpha n(n+2)^k a_n z^{n-3} - (1-m)z^{-4} - \sum_{n=1}^{\infty} (1-m)(n+2)^k a_n z^{n-3} + (1-\alpha - m)z^{-4} | \end{aligned}$$

$$\begin{aligned}
 &= \left| \sum_{n=1}^{\infty} \alpha n(n-1)(n-2)(n+2)^k a_n z^{n-3} + \sum_{n=1}^{\infty} (1-m)n(n-1)(n+2)^k a_n z^{n-3} + \sum_{n=1}^{\infty} (n+2)^k a_n z^{n-3} \right| \\
 &\quad - \left| (1-\lambda)t - \sum_{n=1}^{\infty} \alpha n(n+2)^k a_n z^{n-3} - \sum_{n=1}^{\infty} (1-m)(n+2)^k a_n z^{n-3} \right| \\
 &= \left| \sum_{n=1}^{\infty} [\alpha n(n-1)(n-2) + (1-m)n(n-1) + 1](n+2)^k a_n z^{n-3} \right| - \left| (1-\lambda)t - \sum_{n=1}^{\infty} [\alpha n + (1-m)](n+2)^k a_n z^{n-3} \right| \\
 &= \left| \sum_{n=1}^{\infty} [n(n-1)[\alpha(n-2) + (1-m)] + 1](n+2)^k a_n z^{n-3} \right| - \left| (1-\lambda)t - \sum_{n=1}^{\infty} [\alpha n + 1 - m](n+2)^k a_n z^{n-3} \right| \leq \sum_{n=1}^{\infty} [n(n-1)[\alpha(n-2) + 1 - m] + 1](n+2)^k a_n |z|^{n-3} \\
 &\quad - (1-\lambda)|t| + \sum_{n=1}^{\infty} [\alpha n + 1 - m](n+2)^k a_n |z|^{n-3}
 \end{aligned}$$

For $|z|=1$, we have

$$= \sum_{n=1}^{\infty} [(n(n-1)[\alpha(n-2) + 1 - m] + 1) + [\alpha n + 1 - m]](n+2)^k a_n - (1-\lambda)|t|$$

So, by hypothesis (7), we have $f(z)$ belongs to the class $RZH(\alpha, m, \lambda, t, k)$.

Conversely, assume that $f(z)$ is defined by (2) belongs to the class $RZH(\alpha, m, \lambda, t, k)$, from (6), we have

$$\begin{aligned}
 &\left| \frac{\alpha [L^k f(z)]''' + (1-m)z^{-2}[L^k f(z)]'' + z^{-3}[L^k f(z)] - [2(1-m) - 6\alpha + 1]z^{-4}}{(1-\lambda)t - \alpha z^{-2}[L^k f(z)]' - (1-m)z^{-3}[L^k f(z)] + (1-\alpha-\lambda)z^{-4}} \right| \\
 &= \left| \frac{\sum_{n=1}^{\infty} [n(n-1)[\alpha(n-2) + (1-m)] + 1](n+2)^k a_n z^{n-3}}{(1-\lambda)t - \sum_{n=1}^{\infty} [\alpha n + 1 - m](n+2)^k a_n z^{n-3}} \right|
 \end{aligned}$$

Since $|z| \leq |z|$, therefor, we have

$$\text{Re} \left\{ \frac{\sum_{n=1}^{\infty} [n(n-1)[\alpha(n-2) + (1-m)] + 1](n+2)^k a_n z^{n-3}}{(1-\lambda)|t| - \sum_{n=1}^{\infty} [\alpha n + 1 - m](n+2)^k a_n z^{n-3}} \right\} < 1. \quad (8)$$

Now, letting $z \rightarrow 1$, thought real values in (8), at once obtain (7) and theorem is completely proved.

In the next theorem, we concentrated on getting the growth and distortion theorem for the $f(z)$ to belong in the class $(\alpha, m, \lambda, t, k)$.

Theoreme2 Let $f(z)$ belongs to the class $RZH(\alpha, m, \lambda, t, k)$

$$\frac{1}{|z|} - \frac{(1-\lambda)|t||z|}{(2+\alpha-m)3^k} \leq |f(z)| \leq \frac{1}{|z|} - \frac{(1-\lambda)|t||z|}{(2+\alpha-m)3^k} \quad (9)$$

This for $0 < |z| < 1$

Proof Let $f \in RZH(\alpha, m, \lambda, t, k)$, then

$$\begin{aligned}
 |f(z)| &= \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| \leq \left| \frac{1}{z} \right| + \sum_{n=1}^{\infty} a_n |z| \\
 &\leq \left| \frac{1}{z} \right| + |z| \sum_{n=1}^{\infty} a_n
 \end{aligned}$$

So, by using Theorem 1, we have

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\lambda)|t|}{(2+\alpha-m)3^k},$$

thus

$$|f(z)| \leq \frac{1}{|z|} + \frac{(1-\lambda)|t||z|}{(2+\alpha-m)3^k}.$$

Similarly, we have

$$|f(z)| \geq \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n |z|^n \geq \frac{1}{|z|} - |z| \sum_{n=1}^{\infty} a_n.$$

thus

$$|f(z)| \geq \frac{1}{|z|} - \frac{(1-\lambda)|t||z|}{(2+\alpha-m)3^k}$$

Theorem 3

Let $f \in RZH(\alpha, m, \lambda, t, k)$. Then

$$\begin{aligned} \frac{1}{|z|^2} + \frac{(1-\lambda)|t|}{(2-m)3^k} &\leq |f'(z)| \\ &\leq \frac{1}{|z|^2} + \frac{(1-\lambda)|t|}{(2-m)3^k}. \end{aligned}$$

Proof Let $f \in RZH(\alpha, m, \lambda, t, k)$ we have $|f'(z)| \leq \frac{1}{|z|^2} + \sum_{n=1}^{\infty} n a_n$, by using theorem 1 we have

$$\sum_{n=1}^{\infty} n a_n \leq \frac{(1-\lambda)|t|}{(2-m)3^k}.$$

So

$$|f'(z)| \leq \frac{1}{|z|^2} + \frac{(1-\lambda)|t|}{(2-m)3^k}.$$

Similarly we have

$$|f'(z)| \leq \frac{1}{|z|^2} + \sum_{n=1}^{\infty} n a_n \geq \frac{1}{|z|^2} - \frac{(1-\lambda)|t|}{(2-m)3^k}.$$

Thus we complete the proof.

3. The inclusion relation

Next, we determine the inclusion relationship involving (n, s) -neighborhoods.

Following the earlier works on neighborhoods of analytic function by Goodman [5], Ruscheweyh [9] and Atshan and Kulkarni [4], but for meromorphic function studied by Lin and Srivastava, we define the (n, s) -neighborhoods of function $f \in RZH$, by

$$N_{n,\delta}(f) = \{g \in RZH : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad (11)$$

and

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, 0 \leq \delta < 1\}. \quad (12)$$

Definition 1 A function $g \in RZH$ is said to be in the class $RZH(\alpha, m, \lambda, t, k, \delta)$ if there exists a function $f \in RZH(\alpha, m, \lambda, t, k)$ such that

$$\left| \frac{g(z)}{f(z)} - 1 \right| < 1 - \sigma, \quad z \in U, 0 \leq \sigma < 1. \quad (13)$$

Theorem 4 Let $f \in RZH(\alpha, m, \lambda, t, k)$ and $\sigma = 1$

$$- \frac{\delta(2 + \alpha - m)3^k}{(2 + \alpha - m)3^k - (1 - \lambda)(t)}. \quad (14)$$

Then

$$N_{n,\delta}(f) \subset RZH(\alpha, m, \lambda, t, k).$$

Proof Let $g \in N_{n,\delta}(f)$, then we have from (11) that

$$\sum_{n=1}^{\infty} n |a_n - b_n| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, n \in N.$$

Also since $g \in RZH(\alpha, m, \lambda, t, k)$, we have from theorem (1)

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\lambda)|t|}{(2-m)3^k}.$$

so that

$$\begin{aligned} \left| \frac{g(z)}{f(z)} - 1 \right| &= \left| \frac{\sum_{n=1}^{\infty} (a_n - b_n) z^n}{\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n} \right| \\ &\leq \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} a_n} \end{aligned}$$

$$\leq \frac{\delta(2+\alpha-m)3^k}{(2+\alpha-m)3^k-(1-\lambda)|t|} = 1 - \delta.$$

Thus, by Definition 1, $g \in RZH$ for δ , is given by (13).

This complete the proof. In the next, we considered integral transform of functions in the class $RZH(\alpha, m, \lambda, t, k)$.

Theorem 5 Let the function f given by (2) be in the class $RZH(\alpha, m, \lambda, t, k)$, then the integral operator

$$f(z) = c \int_0^1 u^c f(uz) du, \quad 0 < u \leq 1, \quad 0 < c < \infty. \quad (15)$$

in the class $RZH(\alpha, m, \lambda, t, k)$

Proof Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ in the class $RZH(\alpha, m, \lambda, t, k)$.

Then

$$\begin{aligned} f(z) &= c \int_0^1 u^c f(uz) du \\ &= c \int_0^1 u^c f(uz) du \\ &= c \int_0^1 \left[\frac{u^{c-1}}{z} + \sum_{n=1}^{\infty} a_n u^{n-c} \right] du \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{c}{n+c+1} \right) a_n z^n. \end{aligned}$$

It is easy to show that

$$\sum_{n=1}^{\infty} \frac{c \left[[n(n-1)][\alpha(n-2) + (1-m)] + 1 \right] + [\alpha n + 1 - m](n-2)^k}{(c+n+1)(1-\lambda)|t|} a_n \leq 1. \quad (16)$$

Since

$$f \in RZH(\alpha, m, \lambda, t, k),$$

$$\sum_{n=1}^{\infty} \frac{[n(n-1)][\alpha(n-2) + (1-m)] + 1 + [\alpha n + 1 - m](n-2)^k}{(1-\lambda)|t|} a_n \leq 1.$$

note that (15) is satisfied

$$\begin{aligned} &\frac{c \left[[n(n-1)][\alpha(n-2) + (1-m)] + 1 \right] + [\alpha n + 1 - m](n-2)^k}{(c+n+1)(1-\lambda)|t|} \\ &\leq \frac{[n(n-1)][\alpha(n-2) + (1-m)] + 1 + [\alpha n + 1 - m](n-2)^k}{(1-\lambda)|t|}. \end{aligned}$$

Since $\frac{c}{c+n+1} < 1$ for all $n \in \mathbb{N}$. Hence, we obtained the required result.

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