Asymptotic Properties of Solution

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Abstract

In this paper , the aim of this research is to find out if the non-trivial solutions for the second order differential equation

$$y''(x) + \{q(x)+r(x)\}\, y(x) = 0$$

wear resemble to those of the equation

$$y''(x) + q(x)\, y(x) = 0 \ .$$

Many theorems and properties are used , to achieve this aim.

1. Introduction

Consider the second order differential equation

$$y''(x) + \{q(x)+r(x)\}\, y(x) = 0 \qquad \ldots..(1\text{-}1)$$

with q(x) , r(x) defined on array $[0 , \infty)$

r(x) vary small in some sense as $x \to \infty$

or $r(x) \to 0$ as $x \to \infty$ ……. (1-2)

or $\int^{\infty} |r(x)|\, dx$ convergent ……..(1-3)

The researcher is interested in finding out what happens to the non – trivial solutions of equation (1.1).

In other words , do the solutions of (1.1) resemble those of

$$y''(x) + q(x) \,.\, y(x) = 0 \qquad \ldots\ldots\ldots.. (1\text{-}4)$$

To prove this , the following theorems and properties are required.

Theorem 1-1

Let $\varnothing_1(x)$, $\varnothing_2(x)$, ……,$\varnothing_n(x)$ be n Solutions of $a_0(x)\, y^{(n)}(x) + \ldots\ldots + a_n(x)y(x)=0$

Then $w(\varnothing_1, \varnothing_2 , \ldots\ldots\ldots , \varnothing_n)(x)$ is

Either no where zero in I (here $\varnothing_1$, …… , $\varnothing_n$) are linearly independent solutions .

Or identically zero in I (here $\varnothing_1$, …… , $\varnothing_n$) are linearly dependent .

Consider the linear homogenous system

$\underline{y}'(x) = A(X)\,\underline{y}(X)$ ..…….. (1-5)

$$\underline{y}'(x) = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{pmatrix}', \qquad A(x) = \begin{pmatrix} a_{ij}(x) \end{pmatrix}_{\substack{1 \le i \le n \\ 1 \le J \le n}}$$

$$\underline{y}(x) = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{pmatrix}$$

and

$y'(x) = [A(x) + B(x)]\, y(x) + b(x)$

$y'(x) = A(x)\, y(x) + [B(x)\, y(x) + b(x)]$ …… (1-6)

Theorem 1-2

Let $\varnothing_J(x)$ $1 \le J \le n$ be linear independent Solutios of (1-5) and let $\psi(x)$ be a solution of (1-6) .

Let $\psi_0(x)$ satisfy $y'(x) = A(x)\, y(x) + [B(x)\, \psi(x) + b(x)]$ Then there are unique constants $c_1, c_2, \ldots\ldots\ldots, c_n$ Such that

$$\psi(x) = c_1 \psi_1(x) + c_2 \psi_2(x) + \ldots\ldots\ldots\ldots + c_n \psi_n(x) + \psi_0(x)$$

Take $\psi_0(x) = \int_a^x \frac{-\varphi_1(x)\,\varphi_2(t) + \varphi_2(x)\,\varphi_1(t) - \alpha(t)\,\psi(t) + b(t)}{w(\varphi_1, \varphi_2)(t) \qquad a_0(t)}\, dt$

Properties 1-3

i) Let $\varnothing_1(x)$ and $\varnothing_2(x)$ be linear independent solution of (1-4) and let w denote the (constant) Value of their wronskian (by th 1-1).

Let $\psi(x)$ be any Solution of (1-1) then there are unique Constant c_1 and c_2 such that

$$\psi(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + \frac{1}{w}\int_a^x \{\varphi_1(x)\varphi_2(t) - \varphi_2(x)\varphi_1(t)\} r(t)\psi(t)\, dt \ldots (1-7)$$

(from thm 1-2)

conversely given any constant c_1 and c_2 there is a $\psi(x)$for which (1-7) holds

ii) Let (1-1) have a solution which has no zero in some interval [b, ∞) and

$$\text{let} \quad \int_b^\infty \frac{1}{\psi^2(t)}\, dt < \infty \quad \ldots\ldots\ldots\ldots (1-8)$$

then a second linear independent solution $\psi_2(x)$ of (1-1) is defined in [b,∞)by

$$\psi_2(x) = \psi_1(x)\int_x^\infty \frac{1}{\psi_1^2(t)}\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (1\text{-}9)$$

Now

$$\psi'_2(x) = \psi'_1(x)\int_x^\infty \frac{1}{\psi_1^2(t)}\, dt + \psi_1(x)\left(-\frac{1}{\psi_1^2}\right)$$

$$\psi_2'' = \psi_1'' \int_x^\infty \frac{1}{\psi_1^2(t)}\,dt + \psi_1' \left(\frac{1}{\psi_1^2}\right) + \frac{\psi_1'}{\psi_1^2}$$

$$\psi_2'' = \psi_1'' \int_x^\infty \frac{1}{\psi_1^2(t)}\,dt$$

$$= -(q+r)\,\psi_1 \int_x^\infty \frac{1}{\psi_1^2(t)}\,dt$$

$$\psi_2'' = -(q+r)\,\psi_2$$

Gronwall theorem 1-3

Let u(x) and v(x) be real-valued continuous and non-negative in [a,b] and suppose that

$$u(x) \le c + \int_a^x u(t)\, v(t)\, dt \qquad \ldots\ldots\ldots\ldots (1-10)$$

Where $c \ge 0$ is a constant , then

$$u(x) \le c \exp \int_a^x v(t)\, dt \qquad \ldots\ldots\ldots\ldots (1-11)$$

proof

Let $w(x) = c + \int_a^x u(t)\, v(t)\, dt$ then (1-10) is $u(x) \le w(x)$ and

$w'(x) = u(x)\, v(x) \le w(x)\, v(x)$ divide the in equality by w, get

$$\frac{w'(x)}{w(x)} \le v(x)$$

Integrate over [a,x]

$$\text{Log} \frac{w(x)}{w(a)} \le \int_a^x v(t)\, dt$$

$$\frac{w(x)}{w(a)} \le \exp \left(\int_a^x v(t)\, dt\right)$$

$$w(x) \le c \exp \left(\int_a^x v(t)\, dt\right)$$

Hence $u(x) \le c \exp \left(\int_a^x v(t)\, dt\right)$

2- Bounded Solutions

theorem 2-1

Let all Solutions of (1-4) be bounded in [0,∞) , let (1-3) hold then all Solutions of (1-1) are also bounded in [0,∞).

proof

Let $\varnothing_1(x)$ and $\varnothing_2(x)$ be linear independent solutions of (1-4).

Let $\psi(x)$ be any Solution of (1-1) then (1-7) holds, there is a constant K Such that

$$\left|\varphi_1(x)\right| \le K \ , \quad \left|\varphi_2(x)\right| \le K \quad \text{for} \quad 0 \le x < \infty$$

then (1-7) gives

$$\left|\psi(x)\right| \le \left|c_1\right| K + \left|c_2\right| K + \frac{1}{\left|w\right|} \int_a^x 2K^2 \left|r(t)\right| \left|\psi(t)\right| dt$$

$$u(x) = \left|\psi(x)\right| \ , c = \left(\left|c_1\right| + \left|c_2\right|\right)K \ , v(t) = \frac{2K^2}{\left|w\right|}\left|r(t)\right|$$

$$\left|\psi(x)\right| \le \left(\left|c_1\right| + \left|c_2\right|\right)K \exp\left(\frac{2K^2}{\left|w\right|}\int_a^x \left|r(t)\right| dt\right)$$

$$\le \left(\left|c_1\right| + \left|c_2\right|\right)K \exp\left(\frac{2K^2}{\left|w\right|}\int_a^x \left|r(t)\right| dt\right)$$

2.2- Definition

let f(x) , g(x) be given functions , g(x) continuous.

We say f(x) = O (g(x)) as $x \to \infty$

If there is a constant k such that $|f(x)| \le k|g(x)|$ x large enough

We say f(x) = o (g(x)) as $x \to \infty$

If $\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = 0$

Finally , we say that f(x) ~ g(x) as $x \to \infty$

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

Or $\frac{f(x)}{g(x)} = 1 +$ some thing tend to zero say o(1)

Or f(x) = g(x) + o(g(x))

3- Asymptotic formula

If q(x) is constant and = K the equation (1-1) becomes

$$y'' + \{K + r(x)\}\ y = 0 \qquad \ldots\ldots\ldots\ldots(3\text{-}1)$$

and equation (1-4) becomes

$$y'' + Ky = 0 \qquad \ldots\ldots\ldots\ldots(3\text{-}2)$$

The Solution of equation (3-2) is

$$y = e^{\pm i x \sqrt{K}}$$

theorem 3-1

Let $K \neq 0$ and let (1-3) hold then (3-1) has Solutions $\psi_1(x)$ and $\psi_2(x)$ Such that

$$\psi_1(x) \sim e^{i x \sqrt{K}}\ ,\ \psi_2(x) \sim e^{-i x \sqrt{K}}$$

$$\psi'_1(x) \sim i\sqrt{K}\, e^{i x \sqrt{k}}\ ,\ \psi'_2(x) \sim -i\sqrt{K}\, e^{-i x \sqrt{k}}$$

theorem 3-2

in (3-1) Let k=0 and let

$$\int^{\infty} x \left| r(x) \right| dx < \infty \qquad \ldots\ldots\ldots\ldots(3-3)$$

i.e $x \left| r(x) \right|$ small , $\left| r(x) \right| = \dfrac{\text{Small}}{x}$

then (3-1) has Solution $\psi_1(x)$ and $\psi_2(x)$ such that

$$\left.\begin{aligned} \psi_1(x) &\sim x \quad ,\ \psi'_1(x) \sim 1 \\ \psi_2(x) &\sim 1 \quad ,\ \psi'_2(x) \sim o(x)^{-1} \end{aligned}\right\} \quad \ldots\ldots(*)$$

Note :-

When $x \longrightarrow \infty$ and $|f(x)| \leq k|g(x)|$ x large

$f(x)= O(g(x))$

$\frac{f(x)}{g(x)} \rightarrow 0$ if $|f(x)| \leq \in |g(x)|$ $(x \geq X(\in))$ large enaugh

$f(x)= O(g(x))$

$f(x)= O(1)$ f is bounded

$f(x) = o(1)$ $f(x) \rightarrow 0$

$f_1f_2 = O(g_1g_2) = o(g_1g_2)$

if $f_1(x)$ is $O(g(x))$, $f_2(x)$ is $O(g(x))$ $\Rightarrow f_1(x) - f_2(x)$ is $O(g(x))$

$f(x) \sim g(x)$ means $\frac{f(x)}{g(x)} \rightarrow 1$ as $x \rightarrow \infty$

or $\frac{f(x)}{g(x)} = 1$ + some thing tend to zero say o(1)

$f(x)= g(x) + o(g(x))$

if we have $f_1 \sim g$, $f_2 \sim g$ $\Rightarrow f_1 - f_2 = g + o(g) - g - o(g)$

$= o(g)$

Proof the theorem

$\varnothing_1(x) = x$, $\varnothing_2(x) = 1$ are Solutions of (3.2) with K=0. Hence by (1-7) with $c_1 = 1$ and $c_2 = 0$ there is a Solution $\psi(x)$of (3-1) Such that

$$\psi(x) = x - \int_a^x (x-t)\, r(t)\, \psi(x)\, dt \quad \ldots\ldots\ldots\ldots\ldots (3\text{-}4)$$

here a is appoint in $[0,\infty)$ which will be fixed later.

$$\frac{\psi(x)}{x} = 1 - \int_a^x (1 - \frac{t}{x})\, t\, r(t)\, \frac{\psi(t)}{t}\, dt$$

$$\left|\frac{\psi(x)}{x}\right| \le 1 + \int_a^x (1 - \frac{t}{x})\, t\, |r(t)| \left|\frac{\psi(t)}{t}\right| dt$$

$$\left|\frac{\psi(x)}{x}\right| \le 1 + \int_a^x t\, |r(t)| \left|\frac{\psi(t)}{t}\right| dt$$

hence by Gronwall theorem

$$\left|\frac{\psi(x)}{x}\right| \le \exp\left(\int_a^x t\, |r(t)|\, dt\right)$$

Hence

$$\left|\frac{\psi(x)}{x}\right| \le \exp\left(\int_0^\infty t\, |r(t)|\, dt\right)$$

Hence

$$|\, \psi(x)\, | \le k\, x \quad \text{where}$$

$$k = \exp\left(\int_0^\infty t\, |\, r(t)\, |\, dt\right) \ldots\ldots\ldots\ldots\ldots\ldots (3-5)$$

Differentiate (3-4)

$$\psi'(x) = 1 - \int_a^x r(t)\,\psi(t)\,dt$$

N.B $\dfrac{d}{dx}\int^{x} k(x,t)\,dt = \int^{x} \dfrac{\partial k(x,t)}{\partial x}\,dx + k(x,x)$

In general $\dfrac{d}{dx}\int^{\varphi(x)} k(x,t)\,dt = \int^{\varphi(x)} \dfrac{\partial k(x,t)}{\partial x}\,dt + \varphi'(x)\,k(x,x)$

Hence $\psi'(x) \to \ell$ a finite limit, as $x \to \infty$

Here $\ell = 1 - \int_a^{\infty} r(t)\,\psi(t)\,dt \ldots\ldots\ldots\ldots\ldots (3-6)$

The infinite integral being convergent Since

$$|r(t)\,\psi(t)| \le k \int_a^x t|r(t)|\,dt \quad \text{by } (3-5)$$

Also

$$\left|\int_a^{\infty} r(t)\,\psi(t)\,dt\right| \le k \int_a^{\infty} t|r(t)|\,dt < 1$$

by taking a large enough to making the right hand equal 1.

If a is chosen suitably then $l \neq 0$

Define $\psi_1(x) = \dfrac{\psi(x)}{\ell}$ then $\psi_1'(x) \to 1$ as $x \to \infty$.

to get the result for $\psi_1(x)$ we have

$$1 - \in < \psi_1'(x) < 1 + \in \quad (x \ge X(\in)) \text{ large enough}$$

$$(1-\in)(x-X) < \psi_1(x) - \psi_1(X) < (1+\in)(x-X)$$

$$(1-\in)(x-\frac{X}{x}) < \frac{\psi_1(x)}{x} - \frac{\psi_1(X)}{x} < (1+\in)(1-\frac{X}{x})$$

Small for x large enough

$1-2\in < \frac{\psi_1(x)}{x} < 1+2\in \qquad (x > X_1(\in))$

$\frac{\psi_1(x)}{x} \to 1$ as $x \to \infty$

by (1-9) $\psi_2(x) = \psi_1(x)\int_x^\infty \frac{1}{\psi^2{}_1(t)}\,dt$

$$= x\int_x^\infty \frac{1}{t^2}\,dt$$

$$= x \cdot \frac{1}{x} = 1$$

4-Discussion

under the condition $\int^{\infty} |r(x)|\,dx$ the theorem does not hold as

in

$y'' - c(c-1)\,x^{-2}\,y = 0 \qquad a < x < \infty$

and $r(x) = \dfrac{c(c-1)}{x^2}$

Hence $\int^{\infty} x|r(x)|dx$ diverges

In Euler Equation if we take $y = x^{\lambda}$ then

$\lambda(\lambda-1)\,x^{\lambda-2} - c(c-1)\,x^{\lambda-2} = 0$

$\therefore\ \lambda(\lambda-1) - c(c-1) = 0$

$\lambda^2 - \lambda - c(c-1) = 0$

either $\lambda = c$ or $\lambda = 1-c$

Solutions are $\psi_1(x) = x^c$, $\psi_2(x) = x^{1-c}$

If $c = \frac{1}{4}$ the Solutions $\psi_1(x) = x^{\frac{1}{4}}$, $\psi_1(x) = x^{\frac{3}{4}}$

Satisfy Euler equation

If $c = \frac{1}{2}$ the Solutions $\psi_1(x) = x^{\frac{1}{2}}$, $\psi_2(x) = x^{\frac{-1}{2}}$

do not satisfy Euler equation.

4- Conclusion

The condition (3-3)in this theorem is of course stronger than (1-3) and , in fact the theorem does not hold under the condition (1-3) To se this.

Consider the Euler equation

$y'' - c(c-1)\, x^{-2}\, y = 0 \qquad , c \neq \frac{1}{2}$

Here $r(x) = -c(c-1)\, x^{-2}$ So that (1-3) hold , the equation has two Solutions

$\psi_1(x) = x^c$ and $\psi_2(x) = x^{c-1}$ and , if $c \neq 0$ and $c \neq 1$ no linear combination of them Satisfies equation (*).

الخواص المحاذية لحل المعادلة التفاضلية

الباحث : رشيد عبد المجيد عبد الستار

المستخلص

أذا تأملنا المعادلة التفاضلية من المرتبة الثانية

$$y''(x) + \{q(x)+r(x)\}\, y(x) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

حيث $r(x)$ تكون صغيرة في بعض الأحيان عندما تكون x كبيرة جداً أي أن $x\to\infty$.

او بمعنى اخر $r(x)\to 0$ عندما $x\to\infty$ او أن $\int_{\infty}^{\infty}|r(x)|dx$ يكون متقارب.

السؤال هو هل أن حل المعادلة (١) يشابه حل المعادلة التفاضلية

$$y''(x) + q(x)\, y(x) = 0$$

ان الاجابة على هذا السؤال يتطلب ذكر بعض النظريات والخواص للمعادلة التفاضلية من المرتبة الثانية والتي هي موضوع هذا البحث.