

The Number of Idempotent Elements in Symmetric Semigroup T_n

عدد العناصر المتساوية القوى في شبه الزمرة التناظرية T_n

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Abstract:

The main purpose of this paper to find the number of the idempotent elements in sym T_n Semigroup .

The problem of finding the number of idempotent elements in the symmetric semigroup is solved by using the partitions of an N -set ;we have found that :

$$U_n = \sum_{b_1+2b_2+\dots+nb_n=n} P(b_1, \dots, b_n) \Pi \text{ parts}$$

Such that $P(b_1, \dots, b_n)$ is the number of partition for a given partition and by *parts* we mean the parts for the given partition .Some other results concerning U_n have been established .

الخلاصة :

الهدف الرئيسي من هذا البحث هو ايجاد عدد العناصر المتساوية القوى في شبه الزمرة التناظرية T_n . مشكلة ايجاد عدد العناصر المتساوية القوى في شبه الزمرة التناظرية تم حلها باستخدام التجزئة على المجموعة التي تحوي n من العناصر حيث وجدنا ان

$$U_n = \sum_{b_1+2b_2+\dots+nb_n=n} P(b_1, \dots, b_n) \Pi \text{ parts}$$

حيث ان $P(b_1, \dots, b_n)$ هو العدد الكلي للتجزئة المعطاة للمجموعة N التي تحتوي على n من العناصر ونعني بـ *parts* اجزاء التجزئة المعطاة . ثم برهنا بعض النتائج الأخرى المتعلقة بـ U_n .
تم بعد ذلك تقديم طرق أخرى لحل المشكلة أعلاه وباستخدام العلاقة المرتدة ودالة التوليد

1.Introduction :

Let T_n be the semigroup of all mappings of the set $N=\{1, \dots, n\}$ into itself under the operation of composition of mappings ,inside this semigroup we have the group S_n of all one to one and onto mappings of the set N onto itself. We have in T_n idempotent elements that is mappings f with the property $f \circ f = f$, it is our aim in this work is to find the number of idempotent elements in T_n for $n \in N$.

Let U_n be the number of idempotent elements in T_n .It is shown that:

$$U_n = \sum_{b_1+2b_2+\dots+nb_n=n} P(b_1, \dots, b_n) \Pi \text{ parts}$$

Then we prove that if n is even number then U_n is odd number and if n is odd number then U_n is even number .

2.Defintions and Notations:

Definition 2.1 [1],[2] :

Let X be a finite set and T_X be a set of all mapping from X into X ,then T_X with binary operation the composition of mapping form a semigroup called the *Symmetric Semigroup* denoted by T_X .If $|X|=n$, then we can identify X with the set $\{1, \dots, n\}$ and write T_n .

Definition 2.2 [1],[2],[3] :

Let $(S, *)$ be a semigroup and let $y \in S$, then we say that y is an idempotent element in S if $y*y=y$.

Definition 2.3 [4],[5],[6] :

Partition of a set is decomposition of the set into cells such that every element of the set exactly in one of the cells.

The number of Partitions of the integer n into exactly m classes is denoted by P_n^m . Hence the total number of Partitions of n into m or fewer parts: $P_n^1 + \dots + P_n^m$.

If $m=n$, this number is denoted by $P(n)$, the number of all Partitions of the integer n .

Example 2.1:

The partition of	are	whence
2	2,11	$P_2^1=1=P_2^2$,
3	3,21,111	$P_3^1=P_3^2=P_3^3=1$,

Thus, $P(2)=2, P(3)=3$.

If $\partial=(\partial_1, \dots, \partial_p)$ is a partition of n , ∂ may also can be written as $1^{r_1} 2^{r_2} \dots n^{r_n}$. Where r_i is the number of parts equal to i in ∂ , i varying in $\{1, 2, \dots\}$. For example, all the following denote the same partition of 17:

$4+3+3+2+2+2+1$, 4332221 (condensed as $4 3^2 2^3 1$;

in the other relation, it is $(1 2^3 3^2 4)$, and we say that ∂ is a partition of type $1^{r_1} 2^{r_2} \dots n^{r_n}$.

Theorem 2.1 [7] :

The number of mapping $f: N \rightarrow R$ is :

$$|Map(N, R)| = r^n, \text{ such that } |N| = n, |R| = r.$$

Proposition 2.1 [3] :

Denote $Per(b_1, \dots, b_n)$ the number of permutation of an n -set N of type $1^{b_1} \dots n^{b_n}$ and by $P(b_1, \dots, b_n)$ the number of Partitions of N of type $1^{b_1} \dots n^{b_n}$. Then

$$i) Per(b_1, \dots, b_n) = \begin{cases} \frac{n!}{b_1! \dots b_n! 2^{b_2} \dots n^{b_n}} & \text{if } n = \sum_{i=1}^n i b_i \\ 0 & \text{otherwise} \end{cases}$$

$$ii) P(b_1, \dots, b_n) = \begin{cases} \frac{n!}{b_1! \dots b_n! (2!)^{b_2} \dots (n!)^{b_n}} & \text{if } n = \sum_{i=1}^n i b_i \\ 0 & \text{otherwise} \end{cases}.$$

3.The Main result

The subset S_n of T_n consisting of all one to one and onto mappings form a subsemigroup and infact it is a group of all permutations of n letters laying inside T_n which is called *Symmetric group*.

Now inside T_n we have idempotent elements for example T_3 :

The set of all mapping is $\{f_1, f_2, \dots, f_{27}\}$ where:

$$\begin{aligned}
 f_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}, f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}, \\
 f_6 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}, f_7 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, f_8 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}, f_9 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}, f_{10} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \\
 f_{11} &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, f_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, f_{13} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}, f_{14} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, f_{15} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}, \\
 f_{16} &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}, f_{17} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{pmatrix}, f_{18} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, f_{19} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}, f_{20} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}, \\
 f_{21} &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}, f_{22} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, f_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, f_{24} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, f_{25} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \\
 f_{26} &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, f_{27} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.
 \end{aligned}$$

The idempotent elements for T_3 are :

$$\begin{aligned}
 f_{22} &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}, \\
 f_{12} &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, f_{16} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}, f_{18} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, f_{20} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}, f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}
 \end{aligned}$$

In this section we are interested finding the number of the idempotent elements in the T_n semigroup ,a few example show that :

N	Number of idempotent
1	1
2	3
3	10
4	41
5	196
6	1057
.	.
.	.
.	.

For example let us take T_4 we can make partition for it as follows :

1111 , 2 11 , 2 2 , 1 3 , 4 ;

211 connecting two elements $\{1,2,3,4\}$ =partitioning the set $\{1,2,3,4\}$ into cell containing two elements and two cells each containing one element= number of partition of 4 of type $1^2 2^1$.

Remark: Let $f \in T_n$,now to be f an idempotent element in T_n we have either $f(i)=i$ for some $i \in N$ or if $f(i)=j$,then we must have $f(j)=j$,since otherwise if $f(j)=k$,where $k \neq j$ then $(f \circ f)(i)=f(f(i))=f(j)=k \neq f(i)$ whence $f^2 \neq f$, so the set N can be partitioned into subsets either singleton subsets $\{i\}$ if $f(i)=i$ or to subsets S_k consisting k elements in the other cases.

For example in T_3

$$\begin{aligned}
 f_{22} &= \{\{1\}, \{2\}, \{3\}\}, f_1 = \{1,2,3\}, f_2 = \{1,2,3\}, f_4 = \{1,2,3\}, f_5 = \{\{1,2\}, \{3\}\}, \\
 f_{12} &= \{\{1,3\}, \{2\}\}, f_{16} = \{\{1,3\}, \{2\}\}, f_{18} = \{2,3\}, \{1\}\}, f_{20} = \{\{2,3\}, \{1\}\}, f_4 = \{\{1,2\}, \{3\}\}
 \end{aligned}$$

consider the three partitions of 3 which are:

$$1+1+1=1^3$$

$$1+2=12$$

$$3=3$$

So we have the number of idempotent elements in T_3 equal to

$$\begin{aligned} & \frac{3!}{3!} \times 1 + \frac{3!}{1!2!} \times 1 \times 2 + \frac{3!}{3!} \times 3 \\ & = 1 + 6 + 3 = 10 \end{aligned}$$

Theorem 3.1: The number of idempotent elements in T_n semigroup is

$$U_n = \sum_{b_1+2b_2+\dots+nb_n=n} P(b_1, \dots, b_n) \Pi \text{ parts}$$

Proof: The idempotent elements in T_n are mappings sending the elements of the partitioned subsets $S_1, S_2, \dots, S_k$ into one of the elements in the subset S_i such that S_k is a subset of the partition contains k elements, so we have for each partition of n of type $1^{b_1} \dots n^{b_n}$ idempotent elements as many as the number of $|S_1| \dots |S_k|$ which implies that as many as the product of the parts of the partition and since by proposition 2.1. the number of partition of type $1^{b_1} \dots n^{b_n}$ is $p(b_1, \dots, b_n)$, therefore :

$$U_n = \sum_{b_1+2b_2+\dots+nb_n=n} P(b_1, \dots, b_n) \Pi \text{ parts}$$

4. Another Way to Find The Number of Idempotent Elements of Symmetric Semigroup.

Definition 4.1 [8] :

For a sequence of number $(a_0, \dots, a_n, \dots)$ an equation relating a number a_n to some of its predecessors in the sequence, for any n , is called a recurrence relation.

Example 4.1:

The sequence $\{a_n\} = \{1, 2, 3, 5, 8, \dots\}$ which is called Fibonacci sequence given by the boundary conditions $a_1=1, a_2=2$

and the recurrence relation $a_n = a_{n-1} + a_{n-2}, n \geq 3$.

Solving this recurrence relation means obtaining a formula for the n th term a_n as a function of n .

By using recurrence relation we have the number of idempotent elements [8], given by :

$$U_{m+1} = \sum_{j=0}^m \binom{m}{j} (j+1) U_{m-j} \text{ with boundary condition : } U_0=1, U_1=1.$$

For example:

$$\begin{aligned} U_{3+1} &= U_4 = \sum_{j=0}^3 \binom{3}{j} (j+1) U_{3-j} = \binom{3}{0} 1 \cdot U_3 + \binom{3}{1} 2 \cdot U_2 + \binom{3}{2} 3 \cdot U_1 + \binom{3}{3} 4 \cdot U_0 \\ &= 10 + 3 \cdot 2 \cdot 3 + 3 \cdot 3 \cdot 1 + 1 \cdot 4 \cdot 1 \\ &= 41 \end{aligned}$$

Definition 4.2 [8] :

Let $(a_0, \dots, a_r, \dots)$ be a sequence of numbers,

the function $F(x) = a_0 M_0(x) + \dots + a_r M_r(x) + \dots$; is called the ordinary generating function of the sequence $(a_0, \dots, a_r, \dots)$, where $a_0 M_0(x), \dots, a_r M_r(x), \dots$ is a sequence of function of x that are used as indicators. If $M_x = x^r$, in this case for the sequence $(a_0, \dots, a_r, \dots)$, we have

$$F(x) = a_0 + a_1 x + \dots + a_r x^r + \dots$$

By using generating functions we have the number of idempotent elements in [8] given by :

$$\sum_{n=0}^{\infty} U_n \frac{Z^n}{n!} = \exp(Ze^Z)$$

$$\sum_{n=0}^{\infty} U_n \frac{Z^n}{n!} = U_0 + U_1 \frac{Z^1}{1!} + U_2 \frac{Z^2}{2!} + U_3 \frac{Z^3}{3!} + \dots$$

$$\exp(Ze^Z) = (1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + \dots)(1 + Z^2 + \frac{Z^4}{2!} + \frac{Z^6}{3!} + \dots)(1 + \frac{Z^3}{2!} + \frac{Z^6}{(2!)^2 3!} + \dots)$$

So

$$U_1 Z = Z \text{ then } U_1 = 1;$$

$$U_2 \frac{Z^2}{2} = (1 + \frac{1}{2}) Z^2 \text{ then } U_2 = 3;$$

$$\frac{1}{3!} U_3 Z^3 = (\frac{1}{2!} + 1 + \frac{1}{3!}) Z^3 = \frac{10}{6} Z^3 \text{ then } U_3 = 10.$$

We will use the formula of U_n that is given by using recurrence relation to determined that U_n is even or odd .

Proposition 4.1:

If n is even in T_n then the number of idempotent is odd and if n is odd then the number of idempotent is even .

Proof: since $U_n = U_{m+1} = \sum_{j=0}^m \binom{m}{j} (j+1) U_{m-j}$ [8], (1)

if $m+1=2$ then $U_2 = 3$ note that it is odd, if $m+1=3$ then $U_3 = 10$ is even ,suppose that the statement of Proposition true for each positive integer less than $m+1$,we want to prove that it is true for $m+1$,note that in (1)each term contains one of U_{m-j} which is less than U_{m+1} and so the statement is true.

Now, let $m+1$ is even , if j is even then $m-j$ is odd contrary, therefore if $\binom{m}{j}$ is even so

$$\binom{m}{j} (j+1) U_{m-j} \text{ is even and if it is odd then since } \binom{m}{r} = \binom{m}{m-r}, 1 \leq r \leq m, [5] \quad (2)$$

we have $\binom{m}{j}$ twice , first product by $U_{m-j} (j+1)$ and second by $U_j (m-j+1)$ which are both even except $\binom{m}{m-j} m U_1$ which is always odd, the sum of terms above is odd, add the terms contain $\binom{m}{j}$ even .Hence, U_{m+1} is odd.

Let $m+1$ is odd , if j is even(odd)then $m-j$ is even (odd) therefore if $\binom{m}{j}$ is even so $\binom{m}{j} (j+1) U_{m-j}$ is even and if it is odd then by(2) we have $\binom{m}{j}$ twice , first product by $U_{m-j} (j+1)$ and second by $U_j (m-j+1)$ which are both odd(even),the number of this terms is even by(2), so the sum of it will be even , add the terms $\binom{m}{m/2} (m/2 + 1) U_{m/2}$ which is even where it is clear that $\binom{m}{m/2}$ always even when m even and $m/2$ positive integer .Hence, U_{m+1} is even.

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