Solving Edges Deletion Problem of Generalized Petersen Graphs

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Abstract— *The dependability and effectiveness of a network can be investigated using a variety of graph-theoretic techniques and the network's connectivity determines how reliable it is. Diameter in a network is often used to measure the efficiency of a network if a network experiences issues such as a decline in the communication signal or a breakdown in the communication between its components. This paper examines the increase in diameter of the generalized Petersen graph $GP(n,k)$ after removing a certain number of edges. Finding the exact values of $f(n,t)$ that represents the maximum diameter of an altered generalized Petersen graph $GP(n,k,t)$ obtained after removing t edges from $GP(n,k)$ for $k = 1$ and $t \geq 2$.*

Keywords— ***Edges deletion problem; Graph theory; Generalized Petersen graph; Altered graph; Diameter.***

I. INTRODUCTION

The field of graph theory has expanded quickly in recent years, especially because of how it is widely applied. Graph algorithms are crucial to the design of many types of computer networks [1].

Systems for data transmission and communication must include data security. Its primary responsibility is to safeguard and integrate sensitive data from the source to the recipient [2].

Many researchers have looked into the design of fault-resistant interconnection networks. The connection of graph is the foundation for understanding of fault tolerance. Thus, a fault-tolerant interconnection network guarantees that connections will not be lost even in the case of malfunctioning vertices or edges [3]. The Petersen graph is a known network with a small diameter, a fixed valence, and numerous other ideal characteristics. There have been several network topologies presented that are based on the Petersen graph [4]. In 1898, the Petersen graph was named after the Danish mathematician Julius. That is a minimal counter-example to numerous conjectures in graph theory and one of the most significant finite graphs, constructible in various ways [5]. The Petersen graph has enthralled the interest of several graph theorists throughout the years. Because of how prevalent it was, it seemed like an obvious graph could be used to address a number of problems. In 1950, H. Coxeter proposed a family of graphs generalizing the Petersen graph [6].

The generalized Petersen graph is the most efficient network in terms of size, diameter, and node degree. Several networks based on the generalized Petersen graph have been studied and evolved in the literature because of their ideal and unique characteristics [4].

In this article, we examine how the diameter will increase in the generalized Petersen graph $GP(n,k)$ after removing a certain number of edges. To do so, we determine the exact values of $f(n,t)$ that represent the maximum diameter of an altered graph obtained after removing t edges from $GP(n,k)$ for $k = 1$. The results are obtained with the help of the MATLAB program.

II. PRELIMINARY

Suppose the graph $G = (V,E)$ is an undirected and simple graph, where $E(G)$ denotes the set of edges of size m and $V(G)$ represents the set of vertices of order n. In graph G, a Hamiltonian path is a path that passes through each vertex precisely once [7]. A Hamiltonian path in a graph with n vertices have $n-1$ edges, while a Hamiltonian cycle has n edges [8]. Let u and v be two vertices in G, i.e. $u,v \in V(G)$. The distance between u and v is denoted by $d_G(u,v)$, which is the length of the shortest path connecting u and v in G. If there is no path connecting u and v, $d_G(u,v) = \infty$. The greatest distance in a graph G is known as its diameter, denoted by the symbol d, such that $d = max(d_G(u,v))$ and $u,v \in V(G)$ [9].

III. GENERALIZED PETERSEN GRAPH

The generalized Petersen graph $GP(n,k)$ with n and k being integers $n \geq 3$, $n \neq 2k$ and $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, k is the skip. $GP(n,k)$ is a 3-regular undirected graph and contains

$V(GP(n,k))$ vertex set and $E(GP(n,k))$ edges set that are respectively defined by $V\ (GP(n,k)) = \{u_i, v_i, 1 \le i \le n\}$ and $E\big(GP(n,k)\big) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k}, 1 \le i \le n\}$, where i is an integer and all the indices greater than n will be taken modulo n. The above three forms of edges made by vertex pairs are called outer edges (u_i, u_{i+1}), spoke edges (u_i, v_i), and inner edges (v_i, v_{i+k}) [10].

In this original definition, $GP(n,k)$ is a trivalent graph of order $2n$ and size $3n$. It can be seen that when $n = 2k$ the resulting graph is not cubic. And because of the obvious isomorphism $GP(n,k) \cong GP(n, n - k)$. In $GP(n,k)$, there exist one outer cycle and one or more inner cycles [11] [12].

IV. EDGES DELETION AND DIAMETERS IN A GENERALIZED PETERSEN GRAPH

Suppose that G is a graph and e is any edge in G. $G' = G - e$ is a spanning subgraph with $V(G') = V(G)$ and $E(G') = E(G - e)$. Note that the altered graph G' is the graph that results from deleting e from G [13].

Given $GP(n,k)$ is a generalized Petersen graph with the diameter d. In [12], Loudiki and Kchikech find the exact value for the diameter of $GP(n,k)$ for almost all cases of n and k as explained in Table 1 where $\beta = \frac{n}{k}$, θ is the remainder of dividing n by k, $a = \frac{k}{\theta}$, b is the remainder of dividing k by θ, $q_0 = \lfloor\frac{\beta+\theta}{2}\rfloor$, $q_1 = \lfloor\frac{\theta-b+(a+1)\beta+1}{2}\rfloor$, $q_2 = \lfloor\frac{\theta+b+(a-1)\beta+1}{2}\rfloor$, $q_3 = \lfloor\frac{b+a\beta+1}{2}\rfloor$, $e_1 = min\{max\{q_1, q_3\}, max\{q_0, q_2\}\}$, and $\gamma \in \{1,2\}$ [14].

The diameter of the graph obtained by removing t edges is represented by d', where $d' \ge d$ [15]. Figure (1) denotes the generalized Petersen graph $GP(n,k)$ with $n \ge 3$ and $k = 1$.

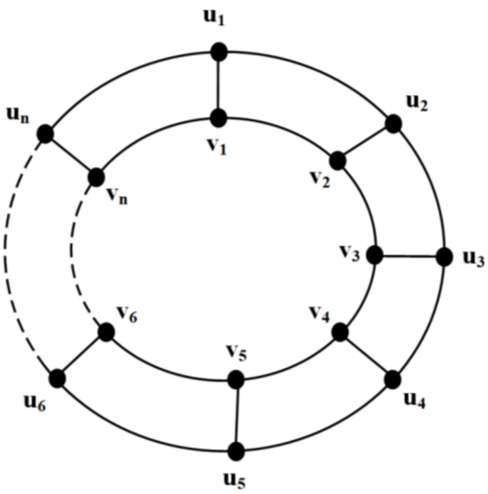

Fig. 1. Generalized Petersen graph $GP(n,k)$

TABLE I. EXACT VALUES FOR THE DIAMETER OF $GP(n,k)$

			Diameter $GP(n,k)$
$\theta = 0$			$= \left\lfloor\frac{\beta+k+3}{2}\right\rfloor$
$\beta > \theta > 0$	**n even**	**K odd**	$= \left\lceil\frac{\beta}{2}\right\rceil + \frac{k-1}{2} - \left(min\left\lceil\frac{\theta}{2}\right\rceil, \left\lceil\frac{k-\theta+1}{2}\right\rceil - 3\right).$
		K even	$= \begin{cases} \left\lceil\frac{\beta}{2}\right\rceil + \frac{k-\theta}{2} + 2\ \ if\ \theta \le \left\lceil\frac{k-2}{2}\right\rceil, \\ \left\lceil\frac{\beta}{2}\right\rceil + \frac{\theta}{2} + 2\ \ otherwise. \end{cases}$
	n odd	**K odd**	$= \left\lceil\frac{\beta}{2}\right\rceil + \frac{k-1}{2} - \left(min\left\lceil\frac{\theta+1}{2}\right\rceil, \left\lceil\frac{k-\theta+2}{2}\right\rceil - 3\right).$
		K even	$= \begin{cases} \left\lceil\frac{\beta}{2}\right\rceil + \frac{k+2}{2}\ if\ \theta \in \{1, k-1\}, \\ \left\lceil\frac{\beta}{2}\right\rceil + \frac{k-\theta+5}{2}\ \ if\ 3 \le \theta \le \left\lceil\frac{k}{2}\right\rceil - 1, \\ \left\lceil\frac{\beta}{2}\right\rceil + \frac{\theta+3}{2}\ otherwise. \end{cases}$
	$\beta \le \theta$ and $b \le a\beta + 1$		$= \begin{cases} q_1 - 1 + \gamma\ if\ (\theta+b)(a\beta - \beta + 1) \equiv 1(mod2), \\ \qquad and\ q_1 = q_2 \\ e_{1+}\gamma\ \ otherwise. \end{cases}$
All n and k			$\le \min\{\max\{\ \beta+1, \theta-2, k-\theta-1\}, \left\lfloor\frac{n+2}{4}\right\rfloor, \left\lfloor\frac{\lfloor\frac{n}{2}\rfloor}{k}\right\rfloor + \left\lceil\frac{k}{2}\right\rceil\} + 2$

V. DELETING ONE EDGE

Ekinci and Gauci discussed the deletion of one edge e from $Gp(n,1)$ leaves the diameter of the graph $Gp(n,1) - \mathrm{e} = Gp(n,1)$ except when $n = 3$ and e is a spoke, in which case diameter of $(D(GP[3,1] - e) = 3$ instead of 2 [16].

Thus, in the sequel, we will focus on how the diameter will increase after deleting t edges, $t \ge 2$ from $GP(n,k)$ with $n \ge 3, k = 1$.

VI. MAIN RESULTS

An altered graph $Gp(n,k,t)$ can be obtained from a $Gp(n,k)$ with $n \ge 3, k = 1$ and $t \ge 2$, after deleting t edges from $Gp(n,k)$, and then calculate $f(n,t)$ with $t \ge 2$ and $n \ge 3$ for the obtained altered graph.

Lemma 4.1:

The maximum number of edges whose removal from generalized Petersen graph $GP(n,1)$, $n \ge 3$ is given by $t = n + 1$, keeping the output graph connected.

Proof: Let $GP(n,1)$ be a generalized Petersen graph with $2n$ vertices, and $3n$ edges. From Figure (1), $GP(n,1)$ has a Hamiltonian path denoted by

$$HP\big(GP(n,1)\big) = u_1, u_2, \dots, u_{n-1}, u_n, v_n, v_{n-1}, v_{n-2}, \dots, v_1$$

that has $2n - 1$ edges. This implies that the diameter of $HP\big(GP(n,1)\big)$ is $2n - 1$. Thus, the maximum number of edges that can be removed from the $GP(n,1)$ graph while the graph remains connected is given by $3n - (2n - 1) = n + 1$. Therefore, Lemma (1) is proved.

Remark 4.2:

From Lemma (4.1), it can be concluded that $f(n,t) = 2n - 1$, $n \geq 3$ and $t = \text{n} + 1$. In the sequel, we will focus on finding $f(n,t)$ with $n \geq 3$ and $2 \leq t < \text{n} + 1$.

Theorem (4.2):

If $n \geq 3$ and $t \in \{2,3,4\}$, then the maximain diameter $f(n,t)$ for the generalized Peterson graph $GP(n,1)$ are $n, n+1, n+2$ respectively.

Proof : Let $GP(n,k)$ be a generalized Peterson graph with $n \geq 3, k = 1.$ From Figure 1, we note that it consists of two cycles C_1 and C_2, where $C_1 = \{u_1, u_2, u_3, \dots, u_n\}$ and $C_2 = \{v_1, v_2, v_3, \dots, v_n\}$ such that any vertex in C_1 is adjacent to two vertices in C_1 and one vertex in C_2 , and any vertex in C_2 is adjacent to two vertices in C_2 and one vertex in C_1.

Assume that $Gp(n,k,t)$ denotes the altered generalized Peterson graph that gotten by removing t edges from $GP(n,k)$. Let's now prove some special classes regarding the values of t, t = 2,3,4.

Case 1: $t = 2$

Let e_1 and e_2 be two edges in $GP(n,1)$ such that e_1 is an outer edge, $e_1 \in C_1$ and e_2 is an inner edge, $e_2 \in C_2$. Assume that e_1 and e_2 are facing one another on location in $GP(n,1)$, where $e_1 = (u_1, u_n)$, $e_2 = (v_1, v_n)$.

Hence, after deleting e_1 and e_2 from $GP(n,1)$, we get the graph in Figure (2).

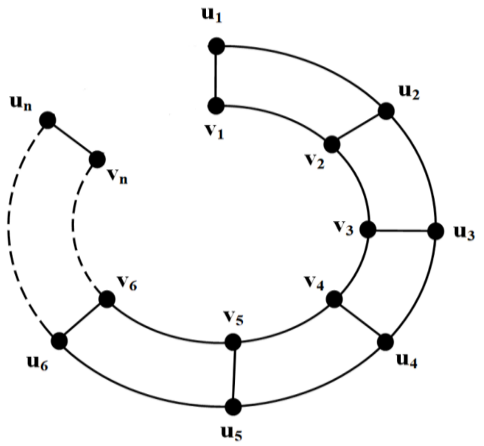

Fig. 2. Altered Generalized Petersen graph $GP(n,1,2)$

From Figure 2, we note that $d = d(u_1, v_n) = d(v_1, u_n)$, where $d(x_i, y_j)$ denotes the distance between any two vertices $x_i, y_j \in V(GP(n,1))$ $\forall$ $i,j \in \{1,2,3, \dots n\}$.

Then we get that $d > d(x_i, y_j)$ $\forall$ $i,j \in \{1,2,3, \dots n\}$ such that

$$d = max\ d(ui, vj) \forall\ i,j\ \{1,2,3, \dots n\}.$$

In fact, from Figure (2) we get that: $d = n$.

Next, d presents the diameter of the connected graphs $Gp(n,1,2)$ obtained upon removing $t = 2$ edges from $Gp(n,1)$. By considering $f(n,2)$, which is the maximum diameter of the connected graph $Gp(n,1,2)$ obtained upon removing $t = 2$ edges from $Gp(n,1)$.

Hince $f(n,2) = n$.

Which can be written in the form $f(n,2) = n, n \geq 3$.

Upon removing two edges from Figure (1), other graphs that match Figure (2) in maximum diameter $f(n,2)$ are generated. We selected Figure (2) because it clearly shows the deletion process for all $t \geq 2$.

Case 2: $t = 3$

Suppose that $e_3 = (u_1, u_2)$, $e_3 \in Gp(n,1,2)$ as shown in Figure (2) (that was obtained by deleting $e_1 = (u_1, u_n)$, $e_2 = (v_1, v_n)$ from Figure (1)). Then after deleting e_3 from Figure (2), we get the graph in Figure (3).

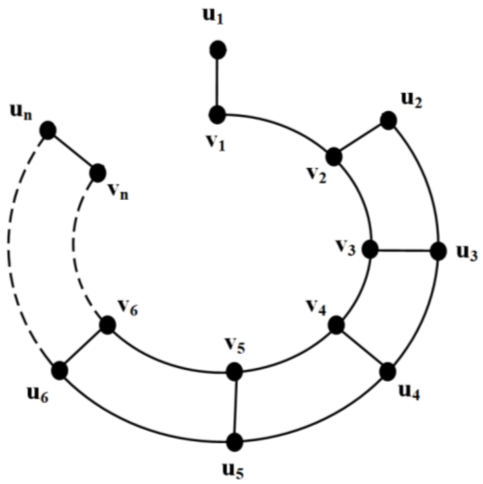

Fig. 3. Altered Generalized Petersen graph $GP(n,1,3)$

From Figure (3) we note that $d' = d(u_1, u_n) > d(x_i, y_j)$, where $d(x_i, y_j)$ denotes the distance between any two vertices x_i and y_j $\forall$ $i,j\ \{1,2,3, \dots n\}$ such that $x_i \neq u_1$ and $y_j \neq u_n$. Then we get that $d' > d(x_i, y_j)$ $\forall$ $i,j \in \{1,2,3, \dots n\}$ such that

$$d' = max\ d(x_i, y_j) \forall\ i,j\ \{1,2,3, \dots n\}$$

In fact, from Figure (3) we get that: $d' = n + 1$

Next, d' presents the diameter of the connected graphs $Gp(n,1,3)$ obtained upon removing $t = 3$ edges from $Gp(n,1)$. By considering $f(n,3)$, which is the maximum diameter of the connected graph $Gp(n,1,3)$ obtained upon removing $t = 3$ edges from $Gp(n,1)$.

Hince $f(n,3) = n + 1$.

Which can be written in the form $f(n,3) = n + 1, n \geq 3$.

Case 3: $t = 4$

Suppose that $e_4 = (v_2, v_3)$, $e_4 \in Gp(n,1,3)$ as shown in Figure 3 (that was obtained by deleting $e_1 = (u_1, u_n)$, $e_2 = (v_1, v_n)$, $e_3 = (u_1, v_2)$ from Figure (1)). Then after deleting e_4 from Figure (3), we get the graph in Figure (4).

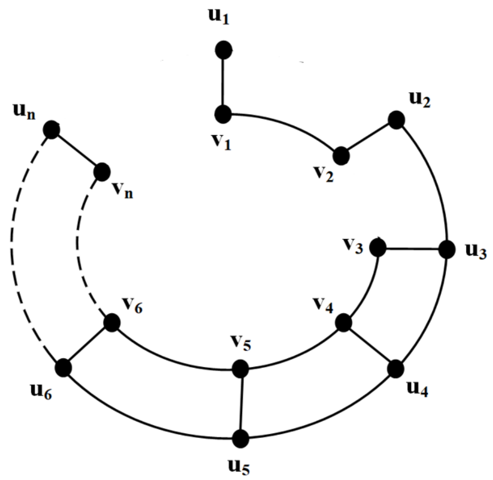

Fig. 4. Altered Generalized Petersen graph $GP(n,1,4)$

From Figure (4) we note that $d'' = d(u_1, v_n) > d(x_i, y_j)$, where $d(x_i, y_j)$ denotes the distance between any two vertices x_i and y_j $\forall\, i, j\ \{1,2,3, \dots n\}$ such that $x_i \neq u_1$ and $y_j \neq v_n$. Then we get that $d'' > d(x_i, y_j)$ $\forall\, i, j \in \{1,2,3, \dots n\}$ such that

$$d'' = max\ d(x_i, y_j) \forall\, i, j\ \{1,2,3, \dots n\}.$$

In fact, from Figure (4) we get that: $\mathrm{t}d'' = n + 2.$

Next, d'' presents the diameter of the connected graphs $Gp(n,1,4)$ obtained upon removing $t = 4$ edges from $Gp(n,1)$. By considering $f(n,4)$, which is the maximum diameter of the connected graph $Gp(n,1,4)$ obtained upon removing $t = 4$ edges from $Gp(n,1)$.

Hince $f(n,4) = n + 2.$

Which can be written in the form $f(n,4) = n + 2\,, n \geq 3.$

Therefore, Theorem (4.2) is proved.

Continuing in the same way in Theorem (4.2) to delete $t \geq 5$, we obtain general formula for $f(n,t)$ given in the Table 2.

In fact, Table 2 illustrates the general formulas for calculating the maximum diameter $f(n,t)$ of the connected graph $Gp(n,1,t)$ obtained upon removing $t \geq 2$ edges from $Gp(n,1)$, $n \geq 3$.

TABLE II. ILLUSTRATES THE GENERAL FORMULAS FOR $f(n,t)$.

t	**Edges proposed for deletion**	(u_1, v_n) **or** (u_1, u_n) **paths**	$f(n,t)$
2	$(u_n, u_1), (v_n, v_1)$	$u_1, u_2, u_3, u_4, \dots, u_n, v_n$	n
3	$(u_n, u_1), (v_n, v_1), (u_1, u_2)$	$u_1, v_1, v_2, u_2, u_3, u_4, \dots, u_n$	$n+1$
4	$(u_n, u_1), (v_n, v_1), (u_1, u_2), (v_2, v_3)$	$u_1, v_1, v_2, u_2, u_3, u_4, \dots, u_n, v_n$	$n+2$
5	$(u_n, u_1), (v_n, v_1), (u_1, u_2), (v_2, v_3), (u_3, u_4)$	$u_1, v_1, v_2, u_2, u_3, v_3, v_4, u_4, u_5, u_6 \dots, u_n$	$n+3$
6	$(u_n, u_1), (v_n, v_1), (u_1, u_2), (v_2, v_3), (u_3, u_4), (v_4, v_5)$	$u_1, v_1, v_2, u_2, u_3, v_3, v_4, u_4, u_5, u_6, u_7 \dots, u_n, v_n$	$n+4$
:	:	:	:
$t = n+1$	$(u_n, u_1), (v_n, v_1), (u_1, u_2), (v_2, v_3), (u_3, u_4), (u_4, u_5), \dots, (v_{n-1}, v_n)$	$u_1, v_1, v_2, u_2, u_3, v_3, v_4, u_4, u_5, v_5, v_6, u_6, \dots, v_{n-1}, u_{n-1}, u_n, v_n$	$n+t-2$

Using the procedure outlined in Table 2 to determine the precise values for $f(n,t)$ for $GP(n,1)$. We may derive the following relationship for the values of $f(n,t)$.

$$f(n,t) = f(n,t-1) + 1$$

Theorem 4.3:

If $2 \leq t \leq n+1$, $n \geq 3$, then $f(n,t) = \mathrm{n} + \mathrm{t} - 2$.

Proof: **(i)** if t is odd, the edges deleting are:

$(u_n, u_1), (v_n, v_1), (u_1, u_2), (v_2, v_3), \dots, (u_{\mathrm{t}-2}, u_{\mathrm{t}-1})$.

And $f(n,t) = d(u_1, u_\mathrm{n}) = n + t - 2,$ ... (1)

where (u_1, u_n)-path is:

$u_1, v_1, v_2, u_2, u_3, v_3, v_4, u_4, u_5, v_5, v_6, u_6, \dots, v_{n-1}, u_{n-1}, u_n$.

(ii) if t is even, the edges deleting are:

$(u_n, u_1), (v_n, v_1), (u_1, u_2), (v_2, v_3), \dots, (v_{t-2}, v_{t-1})$.

And $f(n,t) = d(u_1, v_\mathrm{n}) = n + t - 2,$ ... (2)

where (u_1, v_n)-path is:

$u_1, v_1, v_2, u_2, u_3, v_3, v_4, u_4, u_5, v_5, v_6, u_6 \dots, v_{n-1}, u_{n-1}, u_n, v_n$

From equations (1) and (2)

So, $f(n,t) = n + t - 2$. Therefore, Theorem (4.3) is proved.

VII. CONCLUSION

In this work, we solve the edge deletion problem in the generalized Petersen graph $GP(n,k)$ after removing a certain number of edges. So, we are finding out how the diameter will increase in the generalized Petersen graph $GP(n,k)$ upon edges being removed. In fact, we determine the exact values of $f(n,t), with\ t \geq 2\ and\ n \geq 3$, that represents the maximum diameter of an altered generalized Petersen graph $GP(n,k,t)$ obtained after removing t edges from $GP(n,k)$. And the results were obtained with the help of the MATLAB program.

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