

The cyclic decomposition of the group $D_{2^n} \times C_3$

التجزئة الدائرية للزمرة $D_{2^n} \times C_3$

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Abstract

The group of all $\mathbb{Z}$ -valued characters of a finite group G over the group of induced unit characters from all cyclic subgroups of G forms a finite abelian group, called **Artin Cokernel of G** , denoted by $AC(G)$. The problem of finding the cyclic decomposition of Artin cokernel $D_{2^n} \times C_3$ has been considered in this paper, the cyclic decomposition of $D_{2^n} \times C_3$ is :

$$AC(D_{2^n} \times C_3) = \bigoplus_{i=1}^{2n} C_2.$$

Also we give the general form of rational character and Artin's characters tables of $D_{2^n} \times C_3$ group.

المستخلص :

إن زمرة كل الشواخص العمومية ذات القيم الصحيحة للزمرة المنتهية G على زمرة الشواخص المحتثة من الشواخص الأحادية للزمر الجزئية الدائرية من الزمرة G تكون زمرة أبيلية منتهية و تسمى النواة المشترك – آرتن للزمرة G ويرمز لها بالرمز $(AC(G))$.

إن مسألة إيجاد التجزئة الدائرية للزمرة القسمة $AC(G)$ قد اعتبرت في هذه الرسالة للزمرة $D_{2^n} \times C_3$, حيث وجدنا إن

التجزئة الدائرية للزمرة $D_{2^n} \times C_3$:

$$AC(D_{2^n} \times C_3) = \bigoplus_{i=1}^{2n} C_2$$

للزمرة $D_{2^n} \times C_3$ وكذلك وجدنا الصيغة العامة لجداول الشواخص النسبية وجداول شواخص آرتن

Introduction:

The problem of determining the cyclic decomposition of $AC(G)$ seem to be untouched. we use the concepts of invariant matrix in linear algebra to find the cyclic decomposition of $AC(G)$, G is considered to be the group $D_{2^n} \times C_3$. In 1968 T.Y Lam [8] defined $AC(G)$ and he studied $AC(G)$, when G is a cyclic group.

In 2000 H.R. Yassin [4] studied the cyclic decomposition of $AC(G)$ when G is an elementary abelian group . In 2006 A.S. Abed [2] found $Ar(C_n)$ when C_n is the cyclic group of order n .

In this paper ,we find the rational valued character table and the artin's characters table of the direct product group $D_{2^n} \times C_3$, where D_{2^n} is the dihedral group of order 2^{n+1} and C_3 is the cyclic group of order 3 ,we also find the cyclic decomposition of the factor group $AC(D_{2^n} \times C_3)$.

1. Some Basic Concepts:

In this section, we give basic concepts, notations and theorems about matrix representation, characters and Artin characters.

Definition (1.1) [3]

The general linear group $GL(n,F)$ is a multiplicative group of all non-singular $n \times n$ matrices over the field F .

Definition (1.2) [3]

A matrix representation of the group G is a homomorphism of G into $GL(n,F)$, n is the degree of matrix representation T . In particular, T is called a unit representation (principal) if $T(g) = 1$, for all $g \in G$.

Definition (1.3) [3]

The trace of an $n \times n$ matrices A is the sum of the main diagonal elements, denoted by $\text{tr}(A)$.

Definition (1.4) [3]

Let T be a matrix representation of degree n of a finite group G over the field F . The character χ of degree n of T is the mapping $\chi: G \rightarrow F$ defined by $\chi(g) = \text{tr}(T(g))$ for all $g \in G$. In particular, the character of the principal representation if $(\chi(g)=1, \text{for all } g \in G)$ is called the principal character.

Definition (1.5) [3]

Two elements g and h in a group G are said to be conjugate if $h = xgx^{-1}$ for some $x \in G$. The relation of conjugacy is an equivalence relation on G . The equivalence classes determined by this relation are referred to be as the conjugate classes, denoted by CL_g , $g \in G$ is the conjugate class of the element g .

Definition (1.6) [3]

The centralizer of x in G is the subgroup $C_G(x) = \{a \in G: a x a^{-1} = x\}$.

Definition (1.7) [3]

Let H be a subgroup of G and ϕ be a character of H , the induced character on G is given by

$$\phi \uparrow^G (g) = \frac{1}{|H|} \sum_{x \in G} \phi^\circ(xgx^{-1}) \quad \text{where } g \in G \text{ and } \phi^\circ \text{ is defined by } \phi^\circ(h) = \begin{cases} \phi(h) & \text{if } h \in H \\ 0 & \text{if } h \notin H \end{cases}.$$

Theorem (1.8) [4]

Let H be a cyclic subgroup of G and $h_1, h_2, h_3, \dots, h_m$ are chosen representatives for the m -conjugate classes of H in $CL_g, g \in G$, then

$$\phi \uparrow^G (g) = \frac{|C_G(g)|}{|C_H(g)|} \sum_{i=1}^m \phi(h_i) \quad \text{if } h_i \in H \cap CL(g)$$

$$\phi \uparrow^G (g) = 0 \quad \text{if } H \cap CL(g) = \phi$$

Definition (1.9) [4]

Let G be a finite group, any character induced from the principal character of a cyclic subgroup of G is called Artin character of G .

Definition (1.10) [5]

Two elements of G are said to be Γ -conjugate if the cyclic subgroups they generate are conjugate in G , this defines an equivalence relation on <

Proposition (1.11): [8]

The number of all distinct Artin valued characters of a finite group G equal to the number of all distinct Γ -classes on G .

Definition (1.12): [2]

The complete information about Artin valued characters of a finite group G is displayed in a table called the Artin characters table of G .denoted by $Ar(G)$ which is $l \times l$ matrix whose columns are Γ -classes and rows are the values of all Artin characters of G , where l is the number of Γ -classes.

Definition (1.13): [3]

A rational valued character θ of G is a character whose values are in $\mathbb{Z}$, that is $\theta(g) \in \mathbb{Z}$, for all $g \in G$.

Proposition (1.14): [6]

The number of all distinct rational valued characters of a finite group G is equal to the number of all distinct Γ -classes on G .

Definition (1.15): [6]

The complete information about rational valued characters of a finite group G is displayed in a table called the rational valued characters table of G .denoted it by $\equiv^*(G)$ which is $l \times l$ matrix whose columns are Γ -classes and rows are the values of all rational valued characters of G , where l is the number of Γ -classes.

Theorem [Artin] (1.16): [5]

Every rational valued character of G can be written as a linear combination of Artin characters with coefficient rational numbers .

Theorem (1.17):[5]

Let $T_1: G_1 \rightarrow GL(n, K)$ and $T_2: G_2 \rightarrow GL(m, K)$ are two irreducible representations of the group G_1 and G_2 with characters χ_1 and χ_2 respectively , then $T_1 \otimes T_2$ is irreducible representation of the group $G_1 \times G_2$ with the character $\chi_1 \chi_2$.

Proposition (1.18):[6]

The rational valued characters $\theta_i = \sum_{\sigma \in Gal(Q(\chi_i)/Q)} \sigma(\chi_i)$ form basis for $\bar{R}(G)$,

where χ_i are the irreducible characters of G and their numbers are equal to the number of all distinct Γ - classes of G .

2.The factor Group AC(G):-

The definition of the group $AC(G)$ was introduced by T.Y Lam [8] in 1967. The applications of the factor group $AC(G)$ not only in the mathematics but also in physics and chemistry .In this section we shall study $AC(G)$, dihedral group D_n and $\equiv^*(D_n)$.

Definition (2.1): [8]

Let $\bar{R}(G)$ be the group of $\mathbb{Z}$ -valued generalized characters of G under the operation pointwise addition and $T(G)$ is the normal subgroup of $\bar{R}(G)$ generated by Artin characters. The abelian group $\bar{R}(G)/T(G)$ is called **Artin cokernel of G** , denoted by $AC(G)$.

Definition (2.2): [6]

Let M be a matrix with entries in a principal ideal domain R . A k -minor of M is the determinant of $k \times k$ sub matrix preserving row and column order.

Definition (2.3): [6]

A k -th determinant divisor of M is the greatest common divisor (g.c.d) of all the k -minors of M , this is denoted by $D_k(M)$.

Theorem (2.4): [6]

Let M be an $k \times k$ matrix with entries in a principal ideal domain R , then there exists matrices P and W such that:

- 1 - P and W are invertible.
- 2 - $P M W = D$.
- 3 - D is a diagonal matrix.
- 4 - If we denote D_{jj} by d_j then there exists a natural number m ; $0 \leq m \leq k$

such that $j > m$ implies $d_j = 0$ and $j \leq m$ implies $d_j \neq 0$ and $1 \leq j \leq m$ implies d_j / d_{j-1}

Definition (2.5): [6]

Let M be matrix with entries in a principal ideal domain R , equivalent to matrix $D = \text{diag} \{d_1, d_2, \dots, d_m, 0, 0, \dots, 0\}$ such that d_j / d_{j-1} for $1 \leq j < m$, we call D **the invariant factor matrix of M** and $d_1, d_2, \dots, d_m$ the invariant factors of M .

Remark(2.6) :

According to the Artin theorem (1.16) there exists an invertible matrix $M^{-1}(G)$ with entries in the set of rational numbers such that: $\equiv^*(G) = M^{-1}(G) \cdot \text{Ar}(G)$

and this implies, $M(G) = \text{Ar}(G) \cdot (\equiv^*(G))$

by theorem (2.4) there exist two matrices $P(G)$, $W(G)$ such that $P(G) \cdot M(G) \cdot W(G) =$

$\text{diag} \{d_1, d_2, \dots, d_l\} = D(G)$, where $d_j = \frac{D_j(M(G))}{D_{j-1}(M(G))}$ and l is the number of Γ -classes.

Theorem (2.7): [4]

$$AC(G) = \bigoplus_{j=1}^l C_{d_j} \text{ where } d_j = \frac{D_j(M(G))}{D_{j-1}(M(G))}$$

Definition(2.8)[9]

The group of all symmetries of the regular polygon with n sides, including both rotations and reflections, is called **dihedral group** and denoted by D_n .

The set of rotations generated by r - counterclockwise rotation with angle $2\pi/n$ of order n , and the set of reflections are of order 2 and every element s^j generates

Remark (2.10):[5]

In this work we consider the direct product group $D_{2^n} \times C_3$, where C_3 is a cyclic group of the order 3 consisting of elements $\{1, r', r'^2\}$ with $(r')^3=1$.

The order of the group $|D_{2^n} \times C_3| = |D_{2^n}| \cdot |C_3| = 2 \cdot 2^n \cdot 3 = 3 \cdot 2^{n+1}$

Proposition (2.11): [6]

The rational valued characters table of the cyclic group $(\equiv^* C_p)$, where p is a prime number can be given as follows :

$(\equiv^* C_p) =$	Γ -classes	$[1]$	$[r]$
	θ_1	$p-1$	-1
	θ_2	1	1

Proposition (2.12): [6]

The rational valued characters table of the cyclic group C_{2^n} of the rank $n+1$ which is denoted by $(\equiv^* (C_{2^n}))$, is given as follows:

Γ -classes	$[1]$	$[r^{2^{n-1}}]$	$[r^{2^{n-2}}]$	$\dots$	$[r^2]$	$[r]$
θ_1	2^{n-1}	-2^{n-1}	0	$\dots$	0	0
θ_2	2^{n-2}	2^{n-2}	-2^{n-2}	$\dots$	0	0
θ_3	2^{n-3}	2^{n-3}	2^{n-3}	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
θ_{n-1}	1	1	1	$\dots$	-2	0
θ_n	1	1	1	$\dots$	1	-1
θ_{n+1}	1	1	1	$\dots$	1	1

Theorem(2.13):[4]

The rational valued character table of the dihedral group D_{2^n} is equal to $\equiv^* (D_{2^n}) =$

	Γ -classes of C_{2^n}	$[s]$	$[sr]$
θ_1	$\equiv^* (C_{2^n})$	0	0
θ_2		0	0
$\vdots$		$\vdots$	$\vdots$
θ_n		-1	1
θ_{n+1}		1	1
	$1 \ 1 \ \dots \ 1$		
θ_{n+2}	$1 \ 1 \ \dots \ 1$	-1	-1
$\$			

Theorem (2.14): [2] [3]

The general form of Artin character of C_2^n is given by table:

$Ar(C_2^n)=$

Γ - classes	$[1]$	$[r^{2^{n-1}}]$	$[r^{2^{n-2}}]$	...	$[r]$
φ_1	2^n	0	0	...	0
φ_2	2^{n-1}	2^{n-1}	0	...	0
φ_3	2^{n-2}	2^{n-2}	2^{n-2}	...	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
φ_n	n	n	n	...	0
φ_{n+1}	1	1	1	...	1

And the general form of Artin characters table of C_p when p is a prime number is given by:

$Ar(C_p)=$

Γ - classes	$[1]$	$[r]$
$ CL_\alpha $	1	1
$ C_{C_5}(CL_\alpha) $	P	P
φ'_1	P	0
φ'_2	1	1

Proposition (2.15): [7]

$$P(C_{2^n}) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \text{ which is } (n+1) \times (n+1) \text{ square matrix .}$$

$$W(C_{2^n}) = I_{n+1} \text{ where } I_{n+1} \text{ is an identity matrix and } D(C_{2^n}) = \{1, 1, \dots, 1\}$$

Remark :(2.16) :

We can write $M(C_{2^n})$ as the following :

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

which is $(n+1) \times (n+1)$ square matrix .

Theorem(217.):[1]

The Artin's character table of the dihedral group D_2^n when n is an even number is given as follows :

	$[1]$	$\left[\begin{smallmatrix} n \\ r^2 \end{smallmatrix} \right]$	$\Gamma - \text{Classes of } C_n$		$[s]$	$[sr]$
$ CL_\alpha $	1	1	2 2 ...	2	$2^n / 2$	$2^n / 2$
$ C_{D_n}(CL_\alpha) $	2^{n+1}	2^{n+1}	$2^n 2^n \dots$	2^n	2^2	2^2
$\text{Ar}(D_2^n) = \Phi_1$	$2 \cdot \text{Ar}(C_2^n)$				0	0
$\vdots$					$\vdots$	$\vdots$
Φ_l					0	0
Φ_{l+1}	2^n	0	...	0	0	2
Φ_{l+2}	2^n	0	...	0	2	0

Where l is the number of Γ -classes of C_2^n and $\Phi_j, 1 \leq j \leq l+2$ are the Artin's characters of the group D_2^n .

Proposition (2.18):[1]

$$M(D_{2^n}) = \begin{bmatrix} 2M(C_{2^n}) & \begin{matrix} 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{matrix} \\ 0 & 0 & \dots & 0 & \begin{matrix} 1 & 1 & 1 & 1 \end{matrix} \\ 0 & 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 & 1 & 0 & 1 \\ \underbrace{1 & 1 & \dots & 1}_{n-1 \times n-1} & 1 & 1 & 0 & 0 \end{bmatrix}$$

Which is $(n+3) \times (n+3)$ square matrix .

Proposition (2.19):[1]

The matrices $P(D_{2^n})$ and $W(D_{2^n})$ are taking the forms :

$$P(D_{2^n}) = \begin{bmatrix} & & & 0 & 0 \\ & & & 0 & 0 \\ & & & \vdots & \vdots \\ P(C_{2^n}) & & & 0 & 0 \\ & & & -1 & 1 \\ & & & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix} \quad \text{and } W(D_{2^n}) = \begin{bmatrix} & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & \vdots & \vdots & \vdots \\ I_n & & & & & \\ & & & 0 & 0 & 0 \\ -1 & -1 & \dots & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & \dots & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & \dots & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Where I_n is an identity matrix of the order n , $P(D_{2^n})$ and $W(D_{2^n})$ are $(n+3) \times (n+3)$ square matrices.

3.The Main Results

In this section we give the general forms of rational valued character table , Artin's characters table and the cyclic decomposition of the factor group of the group $D_{2^n} \times C_3$.

Theorem(1.3) :-

The rational valued character table of the group $D_{2^n} \times C_3$ is equal to the tensor product of the rational valued characters table of D_{2^n} and the rational valued characters table of C_3 that is

$$\equiv^*(D_{2^n} \times C_3) = \equiv^*(D_{2^n}) \otimes \equiv^*(C_3)$$

Proof :-

We denote by χ_i to the irreducible characters of D_{2^n} and θ_i , $1 < i < n+3$ to the rational valued characters of D_{2^n} ,

Since the character table of C_3 is equal to

$$\equiv C_3 =$$

CL_α	$[g'_1]$]	$[g'_2]$	$[g'_3]$
χ'_1	1	1	1
χ'_2	1	ω	ω^2
χ'_3	1	ω^2	ω

$\omega = e^{2\pi i/n}$; $i = 1, 2, 3$ and by proposition(2.11) ,the rational valued character table of C_3 is equal to

$$\equiv^*(C_3) =$$

Γ - Clases	$[g'_1]$	$[g'_2]$
θ'_1	2	-1
θ'_2	1	1

From definition of $D_{2^n} \times C_3$ and by theorem (1.17) we have each element g_{hk} in $D_{2^n} \times C_3$ can be written as follows $g_{hk} = g_h \cdot g'_k$ where $g_h \in D_{2^n}$, $h = 1, 2, 3, \dots, n+1$ and $g'_k \in C_3$, $k = 1, 2$ and each irreducible character χ_{ij} of $D_{2^n} \times C_3$ can be written as follows

$$\chi_{ij} = \chi_i \cdot \chi'_j$$

where χ_i is an irreducible character of D_{2^n} , $i = 1, 2, \dots, n+3$ and χ'_j is the irreducible character of C_3 , $j = 1, 2$.

$$\text{then } \chi_{ij}(g_{hk}) = \begin{cases} 2\chi_i(g_h) & \text{if } j=1 \text{ and } k=1 \\ -\chi_i(g_h) & \text{if } j=2 \text{ and } k=1 \\ \chi_i(g_h) & \text{if } j=1, 2 \text{ and } k=2 \end{cases}$$

denote by θ_{ij} to the rational valued characters of $D_{2^n} \times C_3$.

From Proposition (2.18)

$$[I] \theta_{i1} = \sum_{\sigma \in \text{Gal}(Q(\chi_{i2})/Q)} \sigma(\chi_{i2})$$

$$\text{then } \theta_{i1}(g_{hk}) = \sum_{\sigma \in \text{Gal}(Q(\chi_{i2}(g_{hk}))/Q)} \sigma(\chi_{i2}(g_{hk}))$$

(a) If $k=1$.

$$\theta_{i1}(g_{hk}) = \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_h))/Q)} \sigma(2\chi_i(g_h)) = 2\theta_i(g_h) = \theta_i(g_h) \cdot 2 = \theta_i(g_h) \cdot \theta'_1(g'_k)$$

(b) $k=2$

$$\begin{aligned} \theta_{i1}(g_{hk}) &= \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_h))/Q)} \sigma(\chi_i(g_h)) = \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_h))/Q)} \sigma(\chi_i(g_h)) \\ &= \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_h))/Q)} \sigma(\chi_i(g_h)) \cdot (1) = \theta_i(g_h) \cdot \theta'_1(g'_k). \end{aligned}$$

$$[II] \theta_{i2} = \sum_{\sigma \in \text{Gal}(Q(\chi_{i1})/Q)} \sigma(\chi_{i1})$$

$$\text{then } \theta_{i2}(g_{hk}) = \sum_{\sigma \in \text{Gal}(Q(\chi_{i1}(g_{hk}))/Q)} \sigma(\chi_{i1}(g_{hk}))$$

(a) If $k=1$.

$$\theta_{i2}(g_{hk}) = \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_h))/Q)} \sigma(\chi_i(g_h)) = \theta_i(g_h) = \theta_i(g_h) \cdot 1 = \theta_i(g_h) \cdot \theta'_2(g'_k)$$

(b) $k=2$

$$\theta_{i2}(g_{hk}) = \sum_{\sigma \in \text{Gal}(Q(\chi_i(g_h))/Q)} \sigma(-\chi_i(g_h)) = -\theta_i(g_h) = \theta_i(g_h) \cdot -1 = \theta_i(g_h) \cdot \theta'_2(g'_k)$$

From [I] and [II] we have $\theta_{ij} = \theta_i \cdot \theta'_j$

Then $\equiv^*(D_{2^n} \times C_3) = \equiv^*(D_{2^n}) \otimes \equiv^*(C_3)$.

Example(3.2) :

To find the matrix $\equiv^*(D_{2^3} \times C_3)$

From theorem (2.13) and proposition (2.11), we have

$$\equiv^*(D_{2^3}) = \begin{bmatrix} 4 & -4 & 0 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \end{bmatrix} \text{ and } C_3 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

By theorem (1.3) we get $\equiv^*(D_{2^3} \times C_3) = \equiv^*(D_{2^3}) \otimes \equiv^*(C_3)$

$$= \begin{bmatrix} 8 & -8 & 0 & 0 & 0 & 0 & -4 & 4 & 0 & 0 & 0 & 0 \\ 4 & 4 & -4 & 0 & 0 & 0 & -2 & -2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & -2 & -2 & 2 & -1 & -1 & -1 & 1 & 1 & -1 \\ 2 & 2 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ 2 & 2 & 2 & 2 & -2 & -2 & -1 & -1 & -1 & -1 & 1 & 1 \\ 2 & 2 & 2 & -2 & 2 & -2 & -1 & -1 & -1 & 1 & -1 & 1 \\ 4 & -4 & 0 & 0 & 0 & 0 & 4 & -4 & 0 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 0 & 0 & 2 & 2 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

Theorem(3.3):

The artin character table of the group $D_{2^n} \times C_3$ is equal to the tensor product of the artin characters table of D_{2^n} and the artin characters table of C_3 that is $Ar(D_{2^n} \times C_3) = Ar(D_{2^n}) \otimes Ar(C_3)$

Proof :-

We denote by χ_i to the irreducible characters of D_{2^n} and θ_i , $1 < i < n + 3$ to the rational valued characters of D_{2^n} , Since the character table of C_3 equal to

$$\equiv C_3 =$$

CL_α	$[g'_1]$	$[g'_2]$	$[g'_3]$
ϕ'_1	1	1	1
ϕ'_2	1	ω	ω^2
ϕ'_3	1	ω^2	ω

$\omega = e^{2\pi i/n}$; $i = 1, 2, 3$ and by proposition(2.14), the artin character table of C_3 is equal to

$$Ar(C_3) =$$

Γ - Clases	$[g'_1]$	$[g'_2]$
θ'_1	3	0
θ'_2	1	1

From definition of $D_{2^n} \times C_3$ and by theorem (1.17) we have each element g_{hk} in $D_{2^n} \times C_3$ can be written as follows $g_{hk} = g_h \cdot g'_k$ where $g_h \in D_{2^n}$, $h = 1, 2, 3, \dots, n + 3$ and $g'_k \in C_3$, $k = 1, 2$ and each irreducible character ϕ_{ij} of $D_{2^n} \times C_3$ can be written as follows

$$\phi_{ij} = \phi_i \cdot \phi'_j$$

where ϕ_i is an irreducible character of D_{2^n} , $i = 1, 2, \dots, n + 3$ and ϕ'_j is the irreducible character of C_3 , $j = 1, 2$.

then
$$\phi_{ij}(g_{hk}) = \begin{cases} 3\phi_i(g_h) & \text{if } j=1 \text{ and } k=1 \\ 0 & \text{if } j=2 \text{ and } k=1 \\ \phi_i(g_h) & \text{if } j=1,2 \text{ and } k=2 \end{cases}$$

denote by $\Phi_{(i,j)}$ to the artin characters of $D_{2^n} \times C_3$.

If H is a cyclic subgroup of the group $D_{2^n} \times C_3$, we use theorem(1.8) to find each

$$\Phi_{(i,j)}(g_{hk}) = \frac{|C_{D_{2^n} \times C_3}(g_{hk})|}{|C_H(g_{hk})|} \sum_{t=1}^m \phi(x_t) \text{ if } H \cap CL(g) = \emptyset$$

and $\Phi_{(i,j)}(g) = 0$ if $H \cap CL(g) = \phi$ where $1 \leq i \leq l, 1 \leq j \leq 2$

$$[I] \Phi_{(i,2)}(g_{hk}) = \frac{|C_{D_{2^n} \times C_3}(g_{hk})|}{|C_H(g_{hk})|} \sum_{t=1}^m \phi_{ij}(x_t) \text{ then}$$

$$\Phi_{(i,2)}(g_{hk}) = \frac{|C_{D_{2^n}}| \cdot |C_3|}{|C_H(g_h)| \cdot |C_H(g'_k)|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_2(x_t)$$

(a) If $k=1$ then $\Phi_{(i,j)}(g) = 0$ (since $H \cap CL(g) = \phi$)

$$\Phi_{(i,2)}(g_{hk}) = \frac{|C_{D_{2^n}}| \cdot |C_3|}{|C_H(g_h)| \cdot |C_H(g'_1)|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_2(x_t) = \frac{|C_{D_{2^n}}|}{|C_H(g_h)|} \sum_{t=1}^{m_1} \phi_i(x_t) \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}$$

$$= \Phi_i(g_h) \cdot 0 = \Phi_i(g_h) \cdot \Phi_2(g'_1)$$

(b) If $k=2$ then $\Phi_{(i,j)}(g) = 3$ (since $H \cap CL(g) = \{(g,1'),(g,r'),(g,r'_2)\}$)

$$\Phi_{(i,2)}(g_{hk}) = \frac{|C_{D_{2^n}}| \cdot |C_3|}{|C_H(g_h)| \cdot |C_H(g'_2)|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_2(x_t) = \frac{|C_{D_{2^n}}|}{|C_H(g_h)|} \sum_{t=1}^{m_1} \phi_i(x_t) \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix} \quad (3)$$

$$= \Phi_i(g_h) \cdot 0 = \Phi_i(g_h) \cdot \Phi_2(g'_2)$$

$$[II] \Phi_{(i,1)}(g_{hk}) = \frac{|C_{D_{2^n}}| \cdot |C_3|}{|C_H(g_h)| \cdot |C_H(g'_k)|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_1(x_t)$$

(a) If $k=1$ then $\Phi_{(i,j)}(g) = 1$ (since $H \cap CL(g) = \{(g,1')\}$)

$$\Phi_{(i,1)}(g_{hk}) = \frac{|C_{D_{2^n}}| \cdot |C_3|}{|C_H(g_h)| \cdot |C_H(g'_1)|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_1(x_t) = \frac{|C_{D_{2^n}}|}{|C_H(g_h)|} \sum_{t=1}^{m_1} \phi_i(x_t) \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$= \Phi_i(g_h) \cdot 1 = \Phi_i(g_h) \cdot \Phi_1(g'_1)$$

(b

$$\Phi_{(i,1)}(g_{hk}) = \frac{|C_{D_{2^n}}| \cdot |C_3|}{|C_H(g_h)| \cdot |C_H(g'_2)|} \sum_{t=1}^{m_1} \phi_i(x_t) \sum_{t=1}^{m_2} \phi'_1(x_t) = \frac{|C_{D_{2^n}}|}{|C_H(g_h)|} \sum_{t=1}^{m_1} \phi_i(x_t) \left(\frac{3}{3}\right) \quad (1)$$

$$= \Phi_i(g_h) \cdot 1 = \Phi_i(g_h) \cdot \Phi_1(g'_2)$$

From [I] and [II] we have

$$\Phi_{ij} = \Phi_i \cdot \Phi_j$$

Then $\text{Ar}(D_{2^n} \times C_3) = \text{Ar}(D_{2^n}) \otimes \text{Ar}(C_3)$.

Example(3.4):

To find the matrix $\text{Ar}(D_{2^3} \times C_3)$ from theorems (2.17) and (2.14)

$$\text{Ar}(D_{2^3}) = \begin{bmatrix} 16 & 0 & 0 & 0 & 0 & 0 \\ 8 & 8 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 \\ 8 & 0 & 0 & 0 & 2 & 0 \\ 8 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \text{Ar}(C_3) = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$$

Which is $(3+3) \times (3+3)$ square matrix .

Using theorem (3.3) , we have

$$\text{Ar}(D_{2^3} \times C_3) = \text{Ar}(D_{2^3}) \otimes \text{Ar}(C_3) = \begin{bmatrix} 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 12 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 8 & 8 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ 8 & 0 & 0 & 0 & 2 & 0 & 8 & 0 & 0 & 0 & 2 & 0 \\ 8 & 0 & 0 & 0 & 0 & 2 & 8 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Proposition (3.5):

$$M(D_{2^3} \times C_3) = \left[\begin{array}{c|c} M(D_{2^n}) & M(D_{2^n}) \\ \hline 0 & M(D_{2^n}) \end{array} \right]$$

which is $[2(n+3)] \times [2(n+3)]$ square matrix .

$$M(D_{2^n} \times C_3) = Ar(D_{2^n} \times C_3). (\equiv^* (D_{2^n} \times C_3))^{-1} = \left[\begin{array}{c|c} M(D_{2^n}) & M(D_{2^n}) \\ \hline 0 & M(D_{2^n}) \end{array} \right]$$

Example (3.6):

To find $(M(D_{2^n} \times C_3))$ we must

$$M(D_{2^3}) = \left[\begin{array}{cccccccc} 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array} \right]$$

Which is $2(3+1) \times 2(3+1)$ square matrix .

Then by proposition (3.5) we have

$$M(D_{2^n} \times C_3) = \left[\begin{array}{c|c} M(D_{2^3}) & M(D_{2^3}) \\ \hline 0 & M(D_{2^3}) \end{array} \right] = \left[\begin{array}{cccccccccccc} 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{array} \right]$$

Theorem (3.7):

$$D(D_{2^n} \otimes C_3) = \text{diag} \{ \overbrace{2, 2, \dots, 2}^{2n\text{-times}}, 1, 1, 1, 1 \}$$

Proof : the matrices $P(D_{2^n} \otimes C_3)$ and $W(D_{2^n} \otimes C_3)$ are taking the forms :

$$P(D_{2^n} \otimes C_3) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \otimes P(D_{2^n})$$

which is $[2(n+3)] \times [2(n+3)]$ square matrix.

$$P(D_{2^n} \otimes C_3) \cdot M(D_{2^n} \otimes C_3) \cdot W(D_{2^n} \otimes C_3) = \text{diag} \{ \overbrace{2, 2, \dots, 2}^{2n\text{-times}}, 1, 1, 1, 1, 1 \} = D(D_{2^n} \otimes C_3)$$

which is $[2(n+3)] \times [2(n+3)]$ square matrix.

Now we can find the invariant factors matrix of the group $D_{2^n} \otimes C_3$, $D(D_{2^n} \otimes C_3)$ and the cyclic decomposition of the factor group $AC(D_{2^n} \otimes C_3)$ by using theorem(3.7).

Corollary (3.8):

The cyclic decomposition of $AC(D_{2^n} \otimes C_3)$ is : $AC(D_{2^n} \otimes C_3) = \bigoplus_{i=1}^{2n} C_2$

Proof:

By theorem (3.7) , we have

$$D(D_{2^n} \otimes C_3) = \text{diag} \{ \overbrace{2, 2, 2, \dots, 2}^{2n}, 1, 1, 1, 1, 1 \}$$

Then by theorem (2.7) we have

$$AC(D_{2^n} \otimes C_3) = \bigoplus_{i=1}^{2n} C_2 .$$

Example(3.9)

$$1-AC(D_{2^5} \otimes C_3) = \bigoplus_{i=1}^{2(5)} C_2 = \bigoplus_{i=1}^{10} C_2$$

$$2-AC(D_{2^3} \otimes C_3) = \bigoplus_{i=1}^{2(3)} C_2 = \bigoplus_{i=1}^6 C_2$$

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