

Some Types of Normalities of Fuzzy KS-Semigroups

بعض أنواع النظمية في أشباه الزمر الضبابية KS

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Abstract:

In this paper, we introduce new types of normalities of fuzzy KS-Semigroups, namely normal fuzzy subks- semigroups , fuzzy normal subks- semigroup of KS- semigroup and fuzzy normal subks- semigroup of fuzzy subKS- semigroup .Also,we study some of their properties .

الخلاصة :

في هذا البحث قدمنا أنواع جديدة من النظمية في أشباه الزمر الضبابية KS سميت شبه الزمرة الجزئية KS الضبابية النظمية وشبه الزمرة الجزئية النظمية الضبابية من شبه الزمرة KS وشبه الزمرة الجزئية النظمية الضبابية من شبه الزمرة الجزئية الضبابية كما درسنا وبرهنا بعض من خواصها

1.1Introduction:

The notation of BCK algebra introduced by Y.Imai and K.Ise'ki [1] in 1966 . In the same year, K. Ise'ki [2]introduced the notation of BCI algebra which is a generalization of BCK algebra. In 2006 ,Kyung Ho Kim [3] introduced a new class of algebraic structure called KS semigroup .In 2009 Jocelyns S. Paradero Vilea and Mila Cawi [4] characterized ideals of KS- Semigroups and prove some properties .In 2007 , D.R. Prince Williams and Husain Shamshad[5] fuzzify KS semigroup and called it fuzzy KS Semigroups and introduced the notations of fuzzy subKS- Semigroups, fuzzy KS ideal ,fuzzy KS P ideal and investigated some of their related properties. In this paper, we define three new types of normalities of fuzzy KS-Semigroups and study their properties and prove some interested propositions .

2.Preliminary

This section contains some basic concepts that we needed it in this paper.

Definition (2.1)[5]

An algebraic system $(X, *, 0)$ is called a **BCK algebra** if it satisfies the following conditions:

- 1) $((x * y) * (x * z)) * (z * y) = 0$,
- 2) $(x * (x * y)) * y = 0$,
- 3) $x * x = 0$,
- 4) $0 * x = 0$
- 5) if $x * y = 0$ and $y * x = 0$ then $x = y$, $\forall x, y, z \in X$.

Remarks (2.2) [6]

Let X be a BCK algebra :

- A partial ordering " $\leq$ " on X can be defined by $x \leq y$ if and only if $x * y = 0$.
- A BCK-algebra X has the following properties:
 - 1) $x * 0 = x$.
 - 2) if $x * y = 0$ and $y * z = 0$ imply $x * z = 0$.
 - 3) if $x * y = 0$ implies $(x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0$.
 - 4) $(x * y) * z = (x * z) * y$.

- 5) $(x*y)*x=0$.
- 6) $x*(x*(x*y))=x*y$.
- 7) if $(x*y)*z=0$ implies $(x*z)*y=0$.
- 8) $[(x*z)*(y*z)]*(x*y)=0$.
- 9) $[((x*z)*z)*(y*z)]*[(x*y)*z]=0$.

Definition(2.3)[7]

A non empty subset I of a BCK –algebra X is called an **ideal** of X if the following conditions hold :

- 1) $0 \in I$,
- 2) if $x*y \in I$ and $y \in I$ imply $x \in I$, $\forall x, y \in X$.

Definition(2.4) [8]

A non empty subset I of a BCK –algebra X is called a **p- ideal** of X if the following condition hold :

- 1) $0 \in I$,
- 2) if $(x*y)*z \in I$ and $y*z \in I \Rightarrow x*z \in I$, $\forall x, y, z \in X$

Definition (2.5)[4]

A **Semigroup** is an ordered pair $(X, \cdot)$, where X is a non empty set and " $\cdot$ " is an associative binary operation on X .

Definition (2.6)[8]

Let X be a semigroup and x an element of X . An element e of X is a **left identity** of x if $e \cdot x = x$

, a **right identity** of x if $x \cdot e = x$, an **identity** of x if $x = e \cdot x = x \cdot e$.

Definition (2.7)[9]

A non-empty subset T of a semigroup X is a **sub semigroup** of X if it is closed under the operation of X , i.e $\forall a, b \in T$ then $a \cdot b \in T$.

Definition (2.8)[10]

A **semigroup homomorphism** is any mapping $f : X \rightarrow T$ where X and T are semigroups which satisfies $f(xy) = f(x)f(y)$ for all $x, y \in X$.

Definition (2.9)[5]

A **KS-semigroup** is a non-empty set X with two binary operation " $*$ " and " $\cdot$ ", and a constant 0 satisfies the following axioms:

1. $(X, *, 0)$ is a BCK-algebra.
2. $(X, \cdot)$ is a semigroup,
3. $x.(y * z) = (x.y) * (x.z)$ and $(x * y).z = (x.z) * (y.z)$, for all $x, y, z \in X$.

Remarks (2.10)

▪ Throughout this paper X denotes the KS- semigroups unless otherwise specified . We shall write the multiplication $x \cdot y$ by xy .

▪ $x0 = 0x = 0 \quad \forall x \in X$ where X is a KS- semigroup,[5] .

Example (2.11)[5]

Let $X = \{0, 1, 2, 3, 4\}$ be a set with binary operations " $*$ " and " $\cdot$ " defined by the following tables:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	0	0

.	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	2
3	0	0	0	1	2
4	0	1	2	3	4

X is a KS- semigroup .

Definition (2.12)[5]

A non empty subset S of X with binary operation * and . is called **sub KS-semigroup** of X if it satisfies the following condition :

- 1) $x * y \in S \quad \forall \quad x, y \in S$.
- 2) $xy \in S \quad \forall \quad x, y \in S$

Definition (2.13) [3]

A **strong KS-semigroup** is a KS-semigroup X satisfying : $x * y = x * xy \quad \forall \quad x, y \in X$

Lemma(2.14) [3]

Let X be a strong KS-semigroup then :

- 1) $xy * y = 0 \quad \forall \quad x, y \in X$.
- 2) $x * y = 0 \leftrightarrow x * xy = 0$, $\forall \quad x, y \in X$.

Definition (2.15) [11] Let X be a non-empty set a fuzzy subset μ of X is a function $\mu : X \rightarrow [0, 1]$.

Definition (2.16) [3]

Let X and Y be KS-semigroups a mapping $f : X \rightarrow Y$ is called a **KS-semigroup homomorphism** (briefly homomorphism) if $f(x * y) = f(x) * f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in X$.

Let $f : X \rightarrow Y$ KS-semigroup homomorphism . then the set $\{x \in X : f(x) = 0\}$ is called the **kernel of f** , and denote by **ker f** . Moreover, the set $\{f(x) \in Y : x \in X\}$ is called the **image of f** and denote by **Im f** .

Definition (2.17) [11]

Let μ and ν be a fuzzy sets on X . Define the fuzzy set $\mu \cap \nu$ as follows:
 $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in X$.

Definition (2.18) [11]

Let μ and ν be a fuzzy sets on X . Define the fuzzy set $\mu \cup \nu$ as follows:
 $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$ for all $x \in X$.

Definition (2.19) [5]

Let X be a non-empty set and let μ be the fuzzy subset of X for a fixed $0 \leq t \leq 1$, Then the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called an **upper level set of μ** .

Proposition (2.20)[12] Let A and B are fuzzy subKS-semigroup of X. Then $A \cap B$ is a fuzzy subKS-semigroup

Proposition(2.21)

Let μ and ν be fuzzy subKS-semigroups of X then $\mu \cup \nu$ is a fuzzy subKS-semigroup of X if $\mu \subseteq \nu$ or $\nu \subseteq \mu$.

Proof:

Let μ and ν are the fuzzy subKS-semigroup , and let $x, y \in \mu \cup \nu$ then

$$\begin{aligned}
 (\mu \cup \nu)(xy) &= \max\{\mu(xy), \nu(xy)\} \\
 &\geq \max\{\min\{\mu(x), \mu(y)\}, \min\{\nu(x), \nu(y)\}\} \quad [by \ hypothesis] \\
 &= \min\{\max\{\mu(x), \nu(x)\}, \max\{\mu(y), \nu(y)\}\} \quad [\mu \subseteq \nu \text{ or } \nu \subseteq \mu] \\
 &= \min\{(\mu \cup \nu)(x), (\mu \cup \nu)(y)\}.
 \end{aligned}$$

so ,

$$\begin{aligned}(\mu \cup \nu)(x * y) &= \max\{\mu(x * y), \nu((x * y))\} \\ &\geq \max\{\min\{\mu(x), \mu(y)\}, \min\{\nu(x), \nu(y)\}\} \quad [\text{by hypothesis}] \\ &= \min\{\max\{\mu(x), \nu(x)\}, \max\{\mu(y), \nu(y)\}\} \quad [\mu \subseteq \nu \text{ or } \nu \subseteq \mu] \\ &= \min\{(\mu \cup \nu)(x), (\mu \cup \nu)(y)\}.\end{aligned}$$

Hence $\mu \cup \nu$ is a fuzzy subKS-semigroup.

Definition (2. 22) [13]

Let $f : X \rightarrow Y$ be a mapping of KS-Semigroup and μ be a fuzzy subset of Y . The map μ^f is the **pre-image of μ under f** if $\mu^f = \mu(f(x)) \forall x \in X$

Definition (2.23) [13]

Let λ and μ be the fuzzy subsets in a set X the **cartesian product** $\lambda \times \mu : X \times X \rightarrow [0, 1]$ is defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\}$ for all $x, y \in X$.

Proposition(2.24)[12]

Let X be a KS-semigroup and let μ, ν be a fuzzy subKS-semigroup then $\mu \times \nu$ is a fuzzy subKS-semigroup .

Definition (2.25) [13]

Let ν be a fuzzy subset in X the strong fuzzy relation on X that is a fuzzy relation on X is ρ_ν given by $\rho_\nu(x, y) = \min\{\nu(x), \nu(y)\}$

Proposition (2.26)[12]

Let μ be a fuzzy subKS-semigroup then $G_\mu = \{x \in X : \mu(x) = \mu(0)\}$ is a subKS-semigroup

Proposition (2.27)[12]

Let X be a KS-semigroup, ν be fuzzy set Then ρ_ν is fuzzy subKS-semigroup if and only if ν is fuzzy subKS-semigroup.

Proposition (2.28)[12]

Let X be a KS-semigroup and μ, λ be two fuzzy sets in X such that $\mu \times \lambda$ is a fuzzy subKS-semigroup of $X \times X$. Then :

- 1) either $\mu(x) \leq \mu(0)$ or $\lambda(x) \leq \lambda(0)$ for all $x \in X$.
- 2) If $\mu(x) \leq \mu(0)$ for all $x \in X$ then either $\mu(x) \leq \lambda(0)$ or $\lambda(x) \leq \lambda(0)$.
- 3) If $\lambda(x) \leq \lambda(0)$ for all $x \in X$ then either $\mu(x) \leq \mu(0)$ or $\lambda(x) \leq \mu(0)$.
- 4) either μ or λ is a fuzzy subKS-semigroup of X .

3 Normal Fuzzy SubKS- Semigroups

In this section , we give the definition of normal fuzzy subKS-semigroup and give some properties ,like union , intersection , Cartesian product , image , G_μ and other properties of normal fuzzy subKS-semigroup, also we define maximal element of the normal fuzzy subKS-semigroup with some properties .

Definition (3.1)

A fuzzy subKS-semigroup μ of X is said to be **normal fuzzy subKS-semigroup** if there exists $x \in X$ such that $\mu(x) = 1$.

Remark (3.2)

A fuzzy subKS-semigroup μ of X is said to be normal fuzzy subKS-semigroup if and only if $\mu(0) = 1$.

Proof:

Let μ be a normal fuzzy subKS-semigroup of X . Then there exists $x \in X$ such that $\mu(x) = 1$ since by [12] we have $\mu(0) \geq \mu(x) \forall x \in X$

so $\mu(0) \geq 1$ then $\mu(0) = 1$.
conversely , it is clear .

Example (3.3)

Let $X = \{0, 1, 2\}$ be a KS-semigroup with binary operation "*" and "." defined by the following tables:

*	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

.	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

Define a fuzzy subset $\mu : X \rightarrow [0,1]$ by $\mu(0) = 1$ and $\mu(x) = 0.3 \quad \forall (x \neq 0) \in X$.Then μ is a normal fuzzy subKS- semigroup of X .

Proposition (3.4)

Let μ and ν be normal fuzzy subKS-semigroups of X .Then $\mu \cap \nu$ is a normal fuzzy subKS-semigroup of X .

Proof:

let μ and ν be normal fuzzy subKS-semigroups of X .Then by [proposition 2.20] $\mu \cap \nu$ is a fuzzy subKS-semigroup of X also $\mu(0) = 1$ and $\nu(0) = 1$ so

$(\mu \cap \nu)(0) = \min\{\mu(0), \nu(0)\} = 1$,therefore $\mu \cap \nu$ is a normal fuzzy subKS-semigroup .

Proposition (3.5)

Let μ and ν be normal fuzzy subKS-semigroups of X .Then $\mu \cup \nu$ be a normal fuzzy subKS-semigroup of X if $\mu \subseteq \nu$ or $\nu \subseteq \mu$.

Proof: It is clear

Proposition (3.6)

Let μ and ν be a normal fuzzy subKS-semigroup .Then $\mu \times \nu$ is a normal fuzzy subKS-semigroup .

Proof:

Let μ and ν be normal fuzzy subKS-semigroup of X then ,since μ and ν are fuzzy subKS-semigroup so by [proposition 2.24] $\mu \times \nu$ is a fuzzy subKS-semigroup

Now , $(\mu \times \nu)(0,0) = \min\{\mu(0), \nu(0)\} = \min\{1,1\} = 1$ [since μ, ν are normal fuzzy subKS-semigroup].Hence $\mu \times \nu$ is normal fuzzy subKS-semigroup .

Proposition (3.7)

Let $f : X \rightarrow Y$ be a homomorphism if μ is a normal fuzzy subKS-semigroup of Y . Then μ^f is a normal fuzzy subKS-semigroup of X .

Proof:

Let μ is a normal fuzzy subKS-semigroup of Y and $x, y \in X$.Then

$$\begin{aligned} \mu^f(xy) &= \mu(f(xy)) = \mu(f(x) \cdot f(y)) \quad [\text{since } f \text{ is a homomorphism}] \\ &\geq \min\{\mu(f(x)), \mu(f(y))\} \quad [\text{since } \mu \text{ is a fuzzy subKS-semigroup}] .\text{So,} \\ &= \min\{\mu^f(x), \mu^f(y)\} \end{aligned}$$

$$\begin{aligned}\mu^f(x * y) &= \mu(f(x * y)) = \mu(f(x) * f(y)) \text{ [since } f \text{ is a homomorphism]} \\ &\geq \min\{\mu(f(x)), \mu(f(y))\} \text{ [since } \mu \text{ is a fuzzy subKS – semigroup]} \\ &= \min\{\mu^f(x), \mu^f(y)\}.\end{aligned}$$

Thus μ^f is a fuzzy subKS-semigroup . Now , $\mu^f(0) = \mu(f(0)) = \mu(0) = 1$ [since μ is a normal fuzzy subKS-semigroup]. Hence μ^f is a normal fuzzy subKS-semigroup .

Proposition (3.8)

Let $f : X \rightarrow Y$ be epimorphism if μ^f is a normal fuzzy subKS-semigroup of X then μ is a normal fuzzy subKS-semigroup of Y .

Proof:

Let μ^f be a normal fuzzy sub KS-semigroup of X .

let $x, y \in Y \quad \exists a, b \in X \quad \ni f(a) = x, f(b) = y$

$$\begin{aligned}\mu(xy) &= \mu(f(a)f(b)) = \mu(f(ab)) = \mu^f(ab) \geq \min\{\mu^f(a), \mu^f(b)\} = \min\{\mu(f(a)), \mu(f(b))\} \\ &= \min\{\mu(x), \mu(y)\}\end{aligned}$$

Also, in similar way we have $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$. Therefore, μ is a fuzzy subKS-semigroup .

Now , $\mu(0) = \mu(f(0)) = \mu^f(0) = 1$ [sine μ^f is a normal fuzzy subKS-semigroup] .

Hence μ is a normal fuzzy subKS-semigroup .

Proposition (3.9)

Let ν be a fuzzy subset of KS-semigroup X and ρ_ν be a strong fuzzy relation on X .Then ν is a normal fuzzy subKS-semigroup if and only if ρ_ν is a normal fuzzy subKS-semigroup .

Proof:

Let ν be a normal fuzzy subKS-semigroup of X , So by [proposition 2.27] ρ_ν is a fuzzy subKS-semigroup $\rho_\nu(0,0) = \min\{\nu(0), \nu(0)\} = \min\{1,1\} = 1$ [since ν is a normal fuzzy] .

Therefore ρ_ν is a normal fuzzy subKS-semigroup .Conversely, Let ρ_ν is a normal fuzzy subKS-semigroup of $X \times X$ then So by [proposition 2.27] ν is a fuzzy subKS-semigroup since $\rho_\nu(0,0) = 1 = \min\{\nu(0), \nu(0)\} = \nu(0)$ then $\nu(0) = 1$.Hence ν is a normal fuzzy subKS-semigroup .

Proposition (3.10)

Let μ be a fuzzy subKS-semigroup of X .Then a fuzzy set μ^+ defined by

$$\mu^+(x) = \mu(x) + 1 - \mu(0) \text{ is a normal fuzzy subKS-semigroup .}$$

Proof:

Let μ be a fuzzy subKS-semigroup of X , to prove μ^+ is a fuzzy subKS-semigroup let $x_1, x_2 \in X$ then

$$\begin{aligned}\mu^+(x_1 x_2) &= \mu(x_1 x_2) + 1 - \mu(0) \geq \min\{\mu(x_1), \mu(x_2)\} + 1 - \mu(0) \\ &= \min\{\mu(x_1) + 1 - \mu(0), \mu(x_2) + 1 - \mu(0)\} = \min\{\mu^+(x_1), \mu^+(x_2)\}\end{aligned}$$

and in similar way we have $\mu^+(x_1 * x_2) = \min\{\mu^+(x_1), \mu^+(x_2)\}$ hence μ^+ is fuzzy subKS-semigroup .Moreover $\mu^+(0) = \mu(0) + 1 - \mu(0) = 1 \quad \forall x \in X$ therefore μ^+ is a normal fuzzy subKS-semigroup .

Easily we can prove the corollaries 3.11 -3.17

Corollary (3.11)

Let μ and μ^+ be as in the above Proposition . Then

$\mu^+(x_0) = 0 \quad (\text{for some } x_0 \in X) \text{ implies } \mu(x_0) = 0$.Moreover μ is a normal if and only if μ^+ is a normal .

Corollary (3.12)

For every fuzzy subKS-semigroup μ defined on X, we have $(\mu^+)^+ = \mu^+$.Moreover if μ is a normal then $(\mu^+)^+ = \mu$.

Corollary (3.13)

If μ and ν are two fuzzy subKS-semigroup such that $\mu \subseteq \nu$ and $\mu(0) = \nu(0)$ then $G_\mu \subseteq G_\nu$.

Corollary (3.14)

If μ and ν are normal fuzzy subKS-semigroup of X such that $\mu \subseteq \nu$ then $G_\mu \subseteq G_\nu$.

Remark(3.15)

- 1) Denoted $NK(X)$ for the set of all normal fuzzy subKS-semigroup of X .
- 2) It is clear that $(NK(X), \subseteq)$, a poset .

Definition (3.16)

A fuzzy subset μ defined on KS-semigroup X is called maximal if it is non-constant and μ^+ is a maximal element of the poset $(NK(X), \subseteq)$.

Proposition (3.17)

Let μ be a non constant normal fuzzy subKS-semigroup of X if μ is a maximal element of $(NK(X), \subseteq)$.Then μ take only values 0 and 1 .

Proposition (3.18)

Let I be a non-empty subset of X then I is a subKS-semigroup if and only if χ_I is a normal fuzzy subKS-semigroup where

$$\chi_I = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Proof:

Let I be a subKS-semigroup of X and let $x, y \in I$ such that

$x * y \in I$ and $xy \in I$ [I is a subKS – semigroup] .Then

$\chi_I(x * y) = 1 \geq \min\{\chi_I(x), \chi_I(y)\}$. So $\chi_I(xy) = 1 \geq \min\{\chi_I(x), \chi_I(y)\}$.

Then χ_I is a fuzzy subKS-semigroup .Now , since I is a subKS-semigroup and I is a non-empty subset of X thus $\exists x \in I$ such that $0 = x * x \in I$. Therefore $\chi_I(0) = 1$. Hence χ_I is a normal fuzzy subKS-semigroup .Conversely ,suppose that χ_I is a normal fuzzy subKS-semigroup ,so χ_I is a fuzzy subKS-semigroup let $x, y \in I$.To prove $x * y \in I$ since

$$\chi_I(x * y) \geq \min\{\chi_I(x), \chi_I(y)\}$$

$$\chi_I(x) = 1, \chi_I(y) = 1. \text{ So } \chi_I(x * y) \geq 1 \Rightarrow \chi_I(x * y) = 1.$$

So $x * y \in I$. Now, let $x, y \in I$ so $\chi_I(x) = 1, \chi_I(y) = 1$. Then $\chi_I(xy) \geq \min\{\chi_I(x), \chi_I(y)\} = 1$, we get $xy \in I$.

4. Fuzzy Normal SubKS- Semigroup of KS- Semigroup

In this section we give the definition of fuzzy normal subKS-semigroup of X with related some properties, like union, intersection, Cartesian product, strong, image and other properties of fuzzy subKS-semigroup.

Definition (4.1)

Let X be a KS-semigroups and μ a fuzzy set on X. Then μ is called a **fuzzy normal subKS-semigroup** of X if it satisfies the following conditions:

- 1) μ is a fuzzy subKS-semigroup of X.
- 2) $\mu(x * y) = \mu(y * x) \quad \forall x, y \in X \setminus \{0\}$.
- 3) $\mu(xy) = \mu(yx) \quad \forall x, y \in X$.

Remark(4.2)

The notions of fuzzy normal subKS-semigroup appear in 2011 such that $\mu(x * y) = \mu(y * x), \quad \forall x, y \in X$ since we find this condition imply to be μ constant since

$$\mu(0 * y) = \mu(y * 0) \Rightarrow \mu(0) = \mu(y), \quad \forall y \in X$$

then the results will be simple therefore we omitted $\{0\}$ in our definition for the second condition.

Example (4.3)

Let $X = \{0, 1, 2\}$ be a KS-semigroup with binary operation "*" and "." defined by the following tables:

*	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

.	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

Define a fuzzy subset $\mu: X \rightarrow [0,1]$ by $\mu(0) = 0.6$ and $\mu(x) = 0.2 \quad \forall (x \neq 0) \in X$ then μ is a fuzzy normal subKS-semigroup of X.

Proposition (4.4)

Let μ and ν be fuzzy normal subKS-semigroups of X then $\mu \cap \nu$ be a fuzzy normal sub KS-semigroup.

Proof:

Let μ and ν be fuzzy normal subKS-semigroups of X.

Then $\mu \cap \nu$ is a fuzzy subKS-semigroup of X by [proposition 2.20]. Now,

$$\begin{aligned} (\mu \cap \nu)(xy) &= \min\{\mu(xy), \nu(xy)\} \\ &= \min\{\mu(yx), \nu(yx)\} \quad [\mu, \nu \text{ are fuzzy normal subKS - semigroups}] \text{ so,} \\ &= (\mu \cap \nu)(yx), \quad \forall x, y \in X. \\ (\mu \cap \nu)(x * y) &= \min\{\mu(x * y), \nu(x * y)\} \\ &= \min\{\mu(y * x), \nu(y * x)\} \quad [\mu, \nu \text{ are fuzzy normal subKS - semigroups}] \\ &= (\mu \cap \nu)(y * x) \quad \forall x, y \in X \setminus \{0\}. \end{aligned}$$

Therefore $\mu \cap \nu$ is a fuzzy normal subKS-semigroup.

Proposition (4.5)

Let μ and ν be fuzzy normal subKS-semigroup of X such that $\mu \subseteq \nu$ or $\nu \subseteq \mu$. Then $\mu \cup \nu$ be a fuzzy normal subKS-semigroup.

Proof:

Suppose that μ and ν be fuzzy normal subKS-semigroups,

then μ and ν are fuzzy sub KS-semigroups then

$\mu \cup \nu$ be a fuzzy subKS-semigroups [proposition 2.21]. Now,

$$\begin{aligned} (\mu \cup \nu)(xy) &= \max\{\mu(xy), \nu(xy)\} = \max\{\mu(yx), \nu(yx)\} & [\text{by hypothesis}] \\ &= (\mu \cup \nu)(yx) \quad \forall x, y \in X. \end{aligned}$$

so,

$$\begin{aligned} (\mu \cup \nu)(x * y) &= \max\{\mu(x * y), \nu(x * y)\} = \max\{\mu(y * x), \nu(y * x)\} & [\text{by hypothesis}] \\ &= (\mu \cup \nu)(y * x) \quad \forall x, y \in X \setminus \{0\}. \end{aligned}$$

Hence $\mu \cup \nu$ is a fuzzy normal subKS-semigroup .

Proposition (4.6)

Let λ and μ be fuzzy normal subKS-semigroups of X then $\lambda \times \mu$ is a fuzzy normal subKS-semigroup of $X \times X$.

Proof:

Let λ and μ be a fuzzy normal subKS-semigroups of X and let $(x_1, x_2), (y_1, y_2) \in X \times X$ where $x_1, x_2, y_1, y_2 \in X \ni x = (x_1, x_2), y = (y_1, y_2)$ then λ and μ be a fuzzy subKS-semigroups of X so $\lambda \times \mu$ is a fuzzy subKS-semigroup [proposition 2.24].

$$\begin{aligned} (\lambda \times \mu)(xy) &= (\lambda \times \mu)((x_1, x_2) \cdot (y_1, y_2)) = (\lambda \times \mu)(x_1 y_1, x_2 y_2) = \min\{\lambda(x_1 y_1), \mu(x_2 y_2)\} \\ &= \min\{\lambda(y_1 x_1), \mu(y_2 x_2)\} \quad [\lambda, \mu \text{ are fuzzy normal subKS – semigroups}] \\ &= (\lambda \times \mu)((y_1, y_2) \cdot (x_1, x_2)) = (\lambda \times \mu)(yx) \end{aligned}$$

and so ,

let $(x_1, x_2), (y_1, y_2) \in X \times X$ where $x_1, x_2, y_1, y_2 \in X \setminus \{0\}$ such that

$$x = (x_1, x_2), y = (y_1, y_2) \in X \times X$$

$$\begin{aligned} (\lambda \times \mu)(x * y) &= (\lambda \times \mu)((x_1, x_2) * (y_1, y_2)) = (\lambda \times \mu)(x_1 * y_1, x_2 * y_2) = \min\{\lambda(x_1 * y_1), \mu(x_2 * y_2)\} \\ &= \min\{\lambda(y_1 * x_1), \mu(y_2 * x_2)\} \quad [\lambda, \mu \text{ are fuzzy normal subKS – semigroups}] \\ &= (\lambda \times \mu)((y_1, y_2) * (x_1, x_2)) = (\lambda \times \mu)(y * x). \end{aligned}$$

Therefore $\lambda \times \mu$ is a fuzzy normal subKS-semigroup .

Proposition (4.7)

Let $\mu \times \lambda$ be a fuzzy normal subKS-semigroup of X . Then either λ or μ is a fuzzy normal subKS-semigroup of X .

Proof:

Let $\mu \times \lambda$ be a fuzzy normal subKS-semigroup of X . Then $\mu \times \lambda$ be a fuzzy subKS-semigroup of X then by [theorem 2.28] , either λ or μ is a fuzzy subKS-semigroup of X if λ be a fuzzy subKS-semigroup of X so by [theorem 2.28] we have $\lambda(x) \leq \mu(0)$ to prove λ is a normal let $x_1, x_2 \in X$. Then

$$\begin{aligned} \lambda(x_1 x_2) &= \min\{\mu(0), \lambda(x_1 x_2)\} = (\mu \times \lambda)(0, x_1 x_2) = (\mu \times \lambda)((0, x_1) \cdot (0, x_2)) \\ &= (\mu \times \lambda)((0, x_2) \cdot (0, x_1)) = (\mu \times \lambda)(0, x_2 x_1) = \min\{\mu(0), \lambda(x_2 x_1)\} = \lambda(x_2 x_1) \end{aligned}$$

Now , let $x_1, x_2 \in X \setminus \{0\}$

$$\begin{aligned} \lambda(x_1 * x_2) &= \min\{\mu(0), \lambda(x_1 * x_2)\} = (\mu \times \lambda)(0, x_1 * x_2) = (\mu \times \lambda)((0, x_1) * (0, x_2)) \\ &= (\mu \times \lambda)((0, x_2) * (0, x_1)) = (\mu \times \lambda)(0, x_2 * x_1) = \min\{\mu(0), \lambda(x_2 * x_1)\} = \lambda(x_2 * x_1) \end{aligned}$$

Hence λ is a fuzzy normal subKS-semigroup .In a similar way if $\mu \times \lambda$ is a fuzzy normal subKS-semigroup and μ is a fuzzy subKS-semigroup .We can prove that μ is a fuzzy normal subKS-semigroup .

Proposition (4.8)

Let A be a subKS-semigroup and let μ defined by

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in A \\ t_2 & \text{if } x \notin A \end{cases}, \text{ where } t_1 < t_2. \text{ Then } \mu \text{ is a fuzzy normal sub KS-semigroup .}$$

Proof:

Let A be a subKS-semigroup and let μ defined by

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in A \\ t_2 & \text{if } x \notin A \end{cases}, \text{ where } t_1 < t_2$$

In first we prove μ is a fuzzy subKS-semigroup of X ,let $x, y \in X$. Then

- 1) If $x, y \in A$ then $xy \in A \rightarrow \mu(xy) = t_1 \geq \min\{\mu(x), \mu(y)\}$
- 2) If $x, y \notin A$ then $xy \notin A \rightarrow \mu(xy) = t_2 \geq \min\{\mu(x), \mu(y)\}$
- 3) If $x \in A, y \notin A$ then $\mu(x) = t_1, \mu(y) = t_2$ and $(xy) \notin A$ so $\mu(xy) = t_2 \geq \min\{\mu(x), \mu(y)\}$
 $= \min\{t_1, t_2\} = t_1$
- 4) If $x \notin A, y \in A$ then $\mu(x) = t_2, \mu(y) = t_1$ and $(xy) \notin A$ so $\mu(xy) = t_2 \geq \min\{\mu(x), \mu(y)\}$
 $= \min\{t_2, t_1\} = t_1$

so $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ by a similar way $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \quad \forall x, y \in X \setminus \{0\}$.

Therefore , μ is a fuzzy subKS-semigroup .Now,

- 1) if $x, y \in A$ since A is subKS – semigrroupp $\Rightarrow xy \in A, yx \in A$,then $\mu(xy) = t_1 = \mu(yx)$.
- 2) if $x, y \notin A$ since A is subKS – semigrroupp $\Rightarrow xy \notin A, yx \notin A$,then $\mu(xy) = t_2 = \mu(yx)$.
- 3) if $x \in A, y \notin A$ then $xy \notin A, yx \notin A$. Thus $\mu(xy) = t_2 = \mu(yx)$.
- 4) if $x \notin A, y \in A$ then $xy \notin A, yx \notin A$.Thus $\mu(xy) = t_2 = \mu(yx)$.

so $\mu(xy) = \mu(yx)$ for all $x, y \in X$. Also,

- 1) If $x, y \in A$ [since A is subKS – semigrroupp $\Rightarrow x * y \in A, y * x \in A$]
then $\mu(x * y) = t_1 = \mu(y * x)$.
- 2) If $x, y \notin A$ [since A is subKS – semigrroupp $\Rightarrow x * y \notin A, y * x \notin A$]
then $\mu(x * y) = t_2 = \mu(y * x)$.

3 If $x \in A, y \notin A$ then $x * y \notin A, y * x \notin A$. Thus $\mu(x * y) = t_2 = \mu(y * x)$.

4) If $x \notin A, y \in A$ then $x * y \notin A, y * x \notin A$. Thus $\mu(x * y) = t_2 = \mu(y * x)$.

so $\mu(x * y) = \mu(y * x)$ for all $x, y \in X \setminus \{0\}$. Hence μ is a fuzzy normal subKS-semigroup .

Proposition (4.9)

Let ν be a fuzzy subKS-semigroup of X and ρ_ν strongest fuzzy relation on X. Then ν is a fuzzy normal subKS-semigroup if and only if ρ_ν is a fuzzy normal subKS-semigroup of $X \times X$.

Proof:

Let ν be a fuzzy normal subKS-semigroup of X , $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X \times X$. Now

clearly ρ_V is a fuzzy subKS-semigroup by [proposition 2.27] . Now,

$$\begin{aligned}\rho_V(xy) &= \rho_V(x_1 y_1, x_2 y_2) = \min\{\nu(x_1 y_1), \nu(x_2 y_2)\} \\ &= \min\{\nu(y_1 x_1), \nu(y_2 x_2)\} \quad [\nu \text{ is a fuzzy normal subKS - semigroup}] \\ &= \rho_V(y_1 x_1, y_2 x_2) \\ &= \rho_V((y_1, y_2) \cdot (x_1, x_2)) = \rho_V(yx)\end{aligned}$$

and , let $x, y \in X \setminus \{0\}$

$$\begin{aligned}\rho_V(x * y) &= \rho_V(x_1 * y_1, x_2 * y_2) = \min\{\nu(x_1 * y_1), \nu(x_2 * y_2)\} \\ &= \min\{\nu(y_1 * x_1), \nu(y_2 * x_2)\} \quad [\nu \text{ is a fuzzy normal subK - semigroup}] \\ &= \rho_V(y_1 * x_1, y_2 * x_2) = \rho_V((y_1, y_2) * (x_1, x_2)) = \rho_V(y * x)\end{aligned}$$

Hence ν is a fuzzy normal subKS-semigroup .Conversely , assume that ρ_V is a fuzzy normal subKS semigroup of $X \times X$ then for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X \times X$. We know that ν is a fuzzy subKS-semigroup. To prove that ν is a fuzzy normal subKS-semigroup there are two cases :

1) If $\nu(x_1 y_1) \leq \nu(x_2 y_2)$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in X \times X$. Then

$$\begin{aligned}\nu(x_1 y_1) &= \min\{\nu(x_1 y_1), \nu(x_2 y_2)\} = \rho_V(xy) = \rho_V(yx) = \rho_V[(y_1, y_2) \cdot (x_1, x_2)] = \rho_V(y_1 x_1, y_2 x_2) \\ &= \min[\nu(y_1 x_1), \nu(y_2 x_2)] = \nu(y_1 x_1)\end{aligned}$$

since if $\nu(x_1 y_1) = \nu(y_2 x_2)$, we have

$$\nu(x_1 y_1) < \nu(x_2 y_2) \quad \text{and} \quad \nu(y_2 x_2) < \nu(y_1 x_1)$$

$$\text{so } \nu(x_1 y_1) < \min\{\nu(x_2 y_2), \nu(y_1 x_1)\} = \rho_V((x_2 y_2), (y_1 x_1))$$

$$\begin{aligned}\text{then } \nu(x_1 y_1) &< \rho_V((x_2 y_1), (y_2 x_1)) = \rho_V((y_2 x_1), (x_2 y_1)) = \rho_V(y_2 x_2, x_1 y_1) \\ &= \min\{\nu(y_2 x_2), \nu(x_1 y_1)\} = \nu(x_1 y_1)\end{aligned}$$

where $(x_2 y_1), (y_2 x_1) \in X \times X$ since $x_1, x_2, y_1, y_2 \in X$

so $\nu(x_1 y_1) < \nu(x_1 y_1)$. This contradiction

$$\text{we get } \nu(x_1 y_1) = \nu(y_1 x_1) \quad \forall (x_1, x_2), (y_1, y_2) \in X \times X.$$

2) If $\nu(x_2 y_2) \leq \nu(x_1 y_1)$

In a similar way we get $\nu(x_2 y_2) = \nu(y_2 x_2) \quad \forall (x_1, x_2), (y_1, y_2) \in X \times X$

Now , let $x = (x_1, x_2)$, $y = (y_1, y_2) \in X \times X$ such that $x_1, x_2, y_1, y_2 \in X \setminus \{0\}$ there are two cases :

1) If $\nu(x_1 * y_1) \leq \nu(x_2 * y_2)$ then

$$\begin{aligned}\nu(x_1 * y_1) &= \min\{\nu(x_1 * y_1), \nu(x_2 * y_2)\} = \rho_V(x * y) = \rho_V(y * x) \\ &= \rho_V[(y_1, y_2) * (x_1, x_2)] = \min[\nu(y_1 * x_1), \nu(y_2 * x_2)] = \nu(y_1 * x_1).\end{aligned}$$

$$\text{so } \nu(x_1 * y_1) = \nu(y_1 * x_1) \quad \forall x_1, x_2, y_1, y_2 \in X \setminus \{0\}$$

2) If $\nu(x_2 * y_2) \leq \nu(x_1 * y_1)$.In similar way we get $\nu(x_2 * y_2) = \nu(y_2 * x_2)$.

Hence ν is a fuzzy normal subKS-semigroup .

Remark(4.10)

If μ is a fuzzy set such that for all $x \in X$, $\mu(x) = 1$ then μ is a normal fuzzy subKS-semigroup and μ is a fuzzy normal subKS-semigroup .

proof: it is clear

Proposition (4.11)

Let $f : X \rightarrow Y$ be a homomorphism if μ is a fuzzy normal subKS-semigroup of Y . Then μ^f is a fuzzy normal subKS-semigroup of X .

Proof:

Let μ be a fuzzy normal subKS-semigroup of Y .Then

μ^f is a fuzzy subKS-semigroup [by Proposition (3.7)].Now , let $x, y \in X$,so

$$\begin{aligned}\mu^f(xy) &= \mu(f(xy)) = \mu(f(x) \cdot f(y)) \quad [f \text{ is a homomorphism}] \\ &= \mu(f(y) \cdot f(x)) \quad [\text{since } \mu \text{ is fuzzy normal subKS – semigroup}] \\ &= \mu(f(yx)) = \mu^f(yx)\end{aligned}$$

and let $x, y \in X/\{0\}$, so

$$\begin{aligned}\mu^f(x * y) &= \mu(f(x * y)) = \mu(f(x) * f(y)) \quad [f \text{ is a homomorphism}] \\ &= \mu(f(y) * f(x)) \quad [\text{since } \mu \text{ is fuzzy normal subKS – semigroup}] \\ &= \mu(f(y * x)) = \mu^f(y * x).\end{aligned}$$

Hence μ^f is a fuzzy normal subKS-semigroup .

Proposition (4.12)

Let $f : X \rightarrow Y$ be an epimorphism if μ^f is a fuzzy normal subKS-semigroup of X . Then μ is a fuzzy normal subKS-semigroup of Y .

Proof:

Let μ^f be a fuzzy normal subKS-semigroup of X so μ is a fuzzy subKS-semigroup [by Proposition (3.8)]

let $x, y \in Y \quad \exists a, b \in X$ such that $f(a) = x$, $f(b) = y$ [f is an epimorphism]

$$\begin{aligned}\mu(xy) &= \mu(f(a) \cdot f(b)) = \mu(f(ab)) = \mu^f(ab) = \mu^f(ba) \quad [\text{since } \mu^f \text{ is a fuzzy normal subKS – semigroup}] \\ &= \mu(f(ba)) = \mu(f(b) \cdot f(a)) = \mu(yx).\end{aligned}$$

and let $x, y \in Y/\{0\}$, so $\exists a, b \in X$ such that $f(a) = x$, $f(b) = y$, so $a, b \neq 0$ since $f(0) = 0$ in a similar way we can prove $\mu(x * y) = \mu(y * x)$.

Hence μ is a fuzzy normal subKS-semigroup .

Proposition (4.13)

Let μ be a non constant fuzzy normal subKS-semigroup of X such that μ is a fuzzy KS-ideal of X then

$$\mu = \begin{cases} \mu(0) & \text{if } x = 0 \\ t & \text{if } x \neq 0 \end{cases} , \text{ where } t \in [0, \mu(0))$$

Proof:

Let μ be a non constant fuzzy normal subKS-semigroup of X ,

and let $x, y \in X/\{0\}$, since μ is a fuzzy KS-ideal ,so

$\mu(x) \geq \min\{\mu(x * y), \mu(y)\} = \min\{\mu(y * x), \mu(y)\}$ [since μ is a fuzzy normal subKS – semigroup]
 but $\mu(y * x) \geq \mu(y)$.Hence $\mu(x) \geq \mu(y)$,
 but $\mu(x) \geq \mu(y) \geq \min\{\mu(y * x), \mu(x)\}$ [by definition of KS – ideal]
 $= \min\{\mu(x * y), \mu(x)\} = \mu(x)$

since $\mu(x * y) \geq \mu(x)$ implies $\mu(y) = \mu(x)$. for all $x, y \in X$ such that $x, y \neq 0$
 that is mean μ is a constant for each $x, y \neq 0$

since $\mu(0) \geq \mu(x)$ for all $x \in X$, and μ is a non constant, so

$$\mu = \begin{cases} \mu(0) & \text{if } x = 0 \\ t & \text{if } x \neq 0 \end{cases} \text{ where } t \in [0, \mu(0)) .$$

Proposition (4.14)

Let μ be a fuzzy normal subKS-semigroup of X .Then the fuzzy set μ^+ defined by
 $\mu^+(x) = \mu(x) + 1 - \mu(0)$ is a fuzzy normal subKS-semigroup .

Proof:it is clear

Proposition (4.15)

Let I be a non-empty subset of X . Then I is a subKS-semigroup if only if χ_I is a fuzzy normal subKS-semigroup, where

$$\chi_I = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Proof:

Let I is a subKS-semigroup of X .Then

χ_I is a fuzzy subKS-semigroup [by Proposition (3.18)]

Now, $x, y \in X$ then there are several cases :

- 1) If $x, y \in I$ then $xy \in I$ and $yx \in I$ then $\chi_I(xy) = \chi_I(yx) = 1$.
- 2) If $x, y \notin I$ then $xy \notin I$ and $yx \notin I$ then $\chi_I(xy) = \chi_I(yx) = 0$.
- 3) If $x \in I, y \notin I$ then $xy, yx \notin I$ then $\chi_I(xy) = \chi_I(yx) = 0$.
- 4) If $x \notin I, y \in I$ then $xy, yx \notin I$ then $\chi_I(xy) = \chi_I(yx) = 0$.

thus $\chi_I(xy) = \chi_I(yx)$.

In a similar way we can prove $\chi_I(x * y) = \chi_I(y * x) \quad \forall x, y \in X \setminus \{0\}$.

Hence χ_I is a fuzzy normal subKS-semigroup .

Conversely ,

let χ_I is a fuzzy normal subKS-semigroup we can prove I is a subKS-semigroup of X in a similar way of Proposition (3.18) .

Proposition (4.16)

Let X be a KS-semigroup and $a, b \in X \setminus \{0\}$ such that $a^2 = a, b^2 = b$ and $ax = a, bx = b \quad \forall x \in X \setminus \{0\}$. Then $\mu(a) = \mu(b)$ where μ is a fuzzy normal subKS-semigroup .

Proof:

Let μ be a fuzzy normal subKS-semigroup of X and

Let $a, b \in X \setminus \{0\}$ such that $a^2 = a, b^2 = b$.Then

$\mu(a) = \mu(ax)$ [$a = ax \quad \forall x \in X \setminus \{0\}$] let $x = b$

$\mu(a) = \mu(ab) = \mu(ba)$ [μ is a normal fuzzy subKS – semigroup]

so $\mu(a) = \mu(b)$ [since $bx = b \quad \forall x \in X \setminus \{0\}$] .

5 . Fuzzy Normal SubKS- Semigroup of Fuzzy SubKS- Semigroup

In this section , we give the definition of fuzzy normal subKS-semigroup of fuzzy subKS-semigroup of X and give some of properties ,like union , intersection , Cartesian product ,image and other properties of fuzzy subKS-semigroup of fuzzy subKS-semigroup .

Definition (5.1)

Let X be a KS-semigroups , μ and ν are fuzzy subKS-semigroups of X such that $\mu \subseteq \nu$ then μ is called **fuzzy normal subKS-semigroup of fuzzy subKS-semigroup** ν if :

- (1) $\mu(y * x) \geq \min\{\mu(x * y), \nu(y)\}$
- (2) $\mu(yx) \geq \min\{\mu(xy), \nu(y)\}$, $\forall x, y \in X$.

Example (5.2)

Let $X = \{0, 1, 2, 3\}$ be a KS-semigroup with binary operation "*" and "." defined by the following tables:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	0
3	3	2	1	0

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Define two fuzzy sets $\mu, \nu : X \rightarrow [0,1]$, by

$\nu(0) = 0.8$, $\nu(1) = 0.4$, $\nu(2) = 0.2$, $\nu(3) = 0.1$ and

$\mu(0) = 0.5$, $\mu(1) = 0.4$, $\mu(2) = 0.2$, $\mu(3) = 0.1$ then by usual calculations we can prove that μ is a fuzzy normal subKS- semigroup of ν .

Proposition (5.3)

Let X be a KS-semigroup and let μ and λ be fuzzy normal subKS-semigroup of fuzzy subKS-semigroups ν . Then $\mu \cap \lambda$ is a fuzzy normal sub KS-semigroup of ν .

Proof:

Let μ and λ are fuzzy normal subKS-semigroup of fuzzy subKS-semigroup ν .

then $\mu \cap \lambda$ is a fuzzy subKS-semigroup [proposition 2.20].

Now , let $x, y \in X$, since $\mu(y * x) \geq \min\{\mu(x * y), \nu(y)\}$, $\lambda(y * x) \geq \min\{\lambda(x * y), \nu(y)\}$ and $\mu(yx) \geq \min\{\mu(xy), \nu(y)\}$, $\lambda(yx) \geq \min\{\lambda(xy), \nu(y)\}$.Therefore

$$1) (\mu \cap \lambda)(yx) = \min\{\mu(yx), \lambda(yx)\} \geq \min\{\min\{\mu(xy), \nu(y)\}, \min\{\lambda(xy), \nu(y)\}\} \\ = \min\{\min\{\mu(xy), \lambda(xy)\}, \min\{\nu(y), \nu(y)\}\} = \min\{(\mu \cap \lambda)(xy), \nu(y)\}$$

and,

$$2) (\mu \cap \lambda)(y * x) = \min\{\mu(y * x), \lambda(y * x)\} \geq \min\{\min\{\mu(x * y), \nu(y)\}, \min\{\lambda(x * y), \nu(y)\}\} \\ = \min\{\min\{\mu(x * y), \lambda(x * y)\}, \min\{\nu(y), \nu(y)\}\} = \min\{(\mu \cap \lambda)(x * y), \nu(y)\}$$

Hence $\mu \cap \lambda$ is a fuzzy normal subKS-semigroup of ν .

Proposition (5.4)

Let X be a KS-semigroups and let μ and λ are fuzzy normal subKS-semigroup of fuzzy subKS-semigroup ν then $\mu \cup \lambda$ is a fuzzy normal subKS-semigroup of ν if $\mu \subseteq \lambda$ or $\lambda \subseteq \mu$.

Proof:

Let μ and λ are fuzzy normal subKS-semigroup of fuzzy subKS-semigroup ν . $\mu \cup \lambda$ is a fuzzy subKS-semigroup so by [proposition 2.21] . Now , let $x, y \in X$.Then

$$\begin{aligned}
 1) (\mu \cup \lambda)(yx) &= \max\{\mu(yx), \lambda(yx)\} \geq \max\{\min\{\mu(xy), \nu(y)\}, \min\{\lambda(xy), \nu(y)\}\} \\
 &= \min\{\max\{\mu(xy), \lambda(xy)\}, \max\{\nu(y), \nu(y)\}\} [\text{since } \mu \subseteq \lambda \text{ or } \lambda \subseteq \mu] \\
 &= \min\{(\mu \cup \lambda)(xy), \nu(y)\}
 \end{aligned}$$

and so ,

$$\begin{aligned}
 2) (\mu \cup \lambda)(y * x) &= \max\{\mu(y * x), \lambda(y * x)\} \\
 &\geq \max\{\min\{\mu(x * y), \nu(y)\}, \min\{\lambda(x * y), \nu(y)\}\} \\
 &= \min\{\max\{\mu(x * y), \lambda(x * y)\}, \max\{\nu(y), \nu(y)\}\} [\mu \subseteq \lambda \text{ or } \lambda \subseteq \mu] \\
 &= \min\{(\mu \cup \lambda)(x * y), \nu(y)\}
 \end{aligned}$$

Hence $\mu \cup \lambda$ is a fuzzy normal subKS-semigroup of ν .

Proposition (5.5)

Let A, ν be fuzzy subKS-semigroup in X and ρ_ν strongest fuzzy relation on X if A is a fuzzy normal subKS-semigroup of ν . Then ρ_A is a fuzzy normal subKS-semigroup of ρ_ν .

Proof:

Let A be a fuzzy normal subKS-semigroup of ν so A is a fuzzy subKS-semigroup of X so by [proposition 2.27] ρ_A is a fuzzy subKS-semigroup of X and ρ_ν is a fuzzy subKS-semigroup of X Now, to prove ρ_A be a fuzzy normal subKS-semigroup of ρ_ν .

$$\begin{aligned}
 1) \rho_A(yx) &= \rho_A(y_1 x_1, y_2 x_2) = \min\{A(y_1 x_1), A(y_2 x_2)\} \\
 &\geq \min\{\min\{A(x_1 y_1), \nu(y_1)\}, \min\{A(x_2 y_2), \nu(y_2)\}\} [A \text{ is a fuzzy normal of } \nu] \\
 &= \min\{\min\{A(x_1 y_1), A(x_2 y_2)\}, \min\{\nu(y_1), \nu(y_2)\}\} = \min\{\rho_A(x_1 y_1, x_2 y_2), \rho_\nu(y)\} \\
 &= \min\{\rho_A(xy), \rho_\nu(y)\}
 \end{aligned}$$

$$\begin{aligned}
 2) \rho_A(y * x) &= \rho_A(y_1 * x_1, y_2 * x_2) = \min\{A(y_1 * x_1), A(y_2 * x_2)\} \\
 &\geq \min\{\min\{A(x_1 * y_1), \nu(y_1)\}, \min\{A(x_2 * y_2), \nu(y_2)\}\} [A \text{ is a fuzzy normal of } \nu] \\
 &= \min\{\min\{A(x_1 * y_1), A(x_2 * y_2)\}, \min\{\nu(y_1), \nu(y_2)\}\} \\
 &= \min\{\rho_A(x_1 * y_1, x_2 * y_2), \rho_\nu(y)\} = \min\{\rho_A(x * y), \rho_\nu(y)\}.
 \end{aligned}$$

Hence ρ_A is a fuzzy normal subKS-semigroup of ρ_ν .

Proposition (5.6)

If μ and λ are fuzzy normal subKS-semigroup of fuzzy subKS-semigroup ν then $\mu \times \lambda$ is a fuzzy normal subKS-semigroup of $\nu \times \nu$.

Proof:

Let μ and λ are fuzzy normal subKS-semigroup of ν .

let $(x_1, x_2), (y_1, y_2) \in X \times X$ such that $x = (x_1, x_2), y = (y_1, y_2)$

so ν, μ, λ are fuzzy subKS-semigroup of X , then by [proposition 2.24] $\nu \times \nu$ is a fuzzy subKS-semigroup , so by [proposition 2.24] then $\mu \times \lambda$ is a fuzzy subKS-semigroup of $X \times X$. Now, to prove $\mu \times \lambda$ is a fuzzy normal subKS-semigroup of $\nu \times \nu$

$$\begin{aligned}
 (\mu \times \lambda)(yx) &= (\mu \times \lambda)((y_1, y_2)(x_1, x_2)) = (\mu \times \lambda)(y_1 x_1, y_2 x_2) = \min\{\mu(y_1 x_1), \lambda(y_2 x_2)\} \\
 &\geq \min\{\min\{\mu(x_1 y_1), \nu(y_1)\}, \min\{\lambda(x_2 y_2), \nu(y_2)\}\} \\
 &= \min\{\min\{\mu(x_1 y_1), \lambda(x_2 y_2)\}, \min\{\nu(y_1), \nu(y_2)\}\} \\
 &= \min\{(\mu \times \lambda)((x_1, x_2)(y_1, y_2)), \nu \times \nu(y_1, y_2)\} = \min\{(\mu \times \lambda)(xy), \nu \times \nu(y)\}
 \end{aligned}$$

and so ,

$$\begin{aligned}
 (\mu \times \lambda)(y * x) &= (\mu \times \lambda)((y_1, y_2) * (x_1, x_2)) = (\mu \times \lambda)(y_1 * x_1, y_2 * x_2) \\
 &= \min\{\mu(y_1 * x_1), \lambda(y_2 * x_2)\} \geq \min\{\min\{\mu(x_1 * y_1), \nu(y_1)\}, \min\{\lambda(x_2 * y_2), \nu(y_2)\}\} \\
 &= \min\{\min\{\mu(x_1 * y_1), \lambda(x_2 * y_2)\}, \min\{\nu(y_1), \nu(y_2)\}\} \\
 &= \min\{(\mu \times \lambda)((x_1, x_2) * (y_1, y_2)), \nu \times \nu(y_1, y_2)\} = \min\{(\mu \times \lambda)(x * y), \nu \times \nu(y)\}
 \end{aligned}$$

Hence $\mu \times \lambda$ is a fuzzy normal subKS-semigroup of $\nu \times \nu$.

Proposition (5.7)

Let $f : X \rightarrow Y$ be a homomorphism if μ is a fuzzy normal subKS-semigroup of a fuzzy subKS-semigroup ν . Then μ^f is a fuzzy normal subKS-semigroup of ν^f .

Proof:

Let μ is a fuzzy normal subKS-semigroup of ν . Then

$$\mu^f(x) = \mu(f(x)) \leq \nu(f(x)) = \nu^f(x) \quad [\text{since } \mu \subseteq \nu] \text{ and } \mu^f \text{ is a fuzzy subKS-semigroup}$$

where by Proposition (3.8) ν^f is a fuzzy subKS-semigroup.

Now, to prove μ^f is a fuzzy normal subKS-semigroup of ν^f . Since

$$\begin{aligned}
 \mu^f(yx) &= \mu(f(yx)) = \mu(f(y)f(x)) \geq \min\{\mu(f(x)f(y)), \nu(f(y))\} \\
 &= \min\{\mu(f(xy)), \nu^f(y)\} = \min\{\mu^f(xy), \nu^f(y)\}
 \end{aligned}$$

so,

$$\begin{aligned}
 \mu^f(y * x) &= \mu(f(y * x)) = \mu(f(y) * f(x)) \geq \min\{\mu(f(x) * f(y)), \nu(f(y))\} \\
 &= \min\{\mu(f(x * y)), \nu^f(y)\} = \min\{\mu^f(x * y), \nu^f(y)\}
 \end{aligned}$$

Hence μ^f is a fuzzy normal subKS-semigroup of ν^f .

Proposition (5.8)

Let $f : X \rightarrow Y$ epimorphism if μ^f is a fuzzy normal subKS-semigroup of ν^f . Then μ is a fuzzy normal subKS-semigroup of ν .

Proof:

Let μ^f is a fuzzy normal subKS-semigroup of ν^f then since f is an epimorphism if $x \in Y \exists a \in X$ such that $f(a) = x$

$\mu(x) = \mu(f(a)) = \mu^f(a) \leq \nu^f(a) = \nu(f(a)) = \nu(x)$ so $\mu(x) \subseteq \nu(x) \forall x \in Y$ and μ is a fuzzy subKS-semigroup [by Proposition (3.8)]

Now, let $x, y \in Y \exists a, b \in X$ such that $f(a) = x, f(b) = y$. Then

$$\begin{aligned}
 \mu(yx) &= \mu(f(b)f(a)) = \mu(f(ba)) = \mu^f(ba) \geq \min\{\mu^f(ab), \nu^f(b)\} \\
 &= \min\{\mu(f(ab)), \nu(f(b))\} = \min\{\mu(f(a)f(b)), \nu(y)\} = \min\{\mu(xy), \nu(y)\}
 \end{aligned}$$

and so,

$$\begin{aligned}
 \mu(y * x) &= \mu(f(b) * f(a)) = \mu(f(b * a)) = \mu^f(b * a) \geq \min\{\mu^f(a * b), \nu^f(b)\} \\
 &= \min\{\mu(f(a * b)), \nu(f(b))\} = \min\{\mu(f(a) * f(b)), \nu(y)\} = \min\{\mu(x * y), \nu(y)\}.
 \end{aligned}$$

Hence μ is a fuzzy normal subKS-semigroup of ν .

Proposition (5.9)

Let μ be a fuzzy normal subKS-semigroup of ν and μ is a normal fuzzy subKS-semigroup of X . Then $\mu = \nu$.

Proof:

let μ be a fuzzy normal subKS-semigroup of ν , so $\mu \subseteq \nu$ since $\mu(y * x) \geq \min\{\mu(x * y), \nu(y)\}$ since μ is a normal fuzzy subKS-semigroup of X so $\mu(0) = 1$. Now, if $x = 0$ we have $\mu(y) \geq \min\{\mu(0), \nu(y)\} = \min\{1, \nu(y)\} = \nu(y)$. But $\mu(y) \leq \nu(y)$, $\forall y \in X$ so $\mu(y) = \nu(y)$, $\forall y \in X$

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References

- [1] Y.Imai and K.Iseki, "On Axiom Systems of Propositional Calculi XIV ", Proc.Japan Acad,42 ,19-22 , 1966
- [2] K.Iseki , " An Algebra Related with a Propositional Calculus" , Japan Acad . ,42 1966 .
- [3] K. H. Kim ,"On Structure of KS-Semigroups" ,Int . Math. Forum , 1, No. 2 , 67-76 , 2006 .
- [4] Joncelyn S.Paradero-Vilela and Mila Cawi " On KS-Semigroup Homomorphism " International Mathematical Forum ,4, no. 23, 1129- 1138 , 2009 .
- [5] Williams, D.R, Prince and Husain Shamshad , "On Fuzzy KS- Semigroup" . International Mathematical Forum ,2, No. 32, 1577-1586 , 2007 .
- [6] Celestin Lele and Congxin Wu Harbin "Generalization of Some Fuzzy Ideal in BCK- Algebras" , Institute of Technology , Department of Mathematics, Harbin 150001,China , 2001 .
- [7] Won Kyun Jeong , " On Anti Fuzzy Prime Ideal in BCK-Algebras" Journal of the Chungcheong Mathematical Society Volume 12,August 1999.
- [8] Young Bae Jun And Seok Zun Song ,"Fuzzy Set Theory Applied To Implicative Ideals in BCK-Algebras " ; Bull . Korean Math. Soc.43, No.3, pp.461- 470, 2006 .
- [9] Petrich, Mario."Introduction to Semigroups" Charles E.Merrill Publishing Company A Bell and Howell Company ,USA. 1973 .
- [10] Tero Harju," Lecture Notes on Semigroups " Department of Mathematics University of Turku, 2010 . Available at [http:// Users. utu. Fi / harju /](http://Users.utu.fi/harju/)
- [11] L.A Zadeh ,"Fuzzy Sets" ,Information Control ,8 ,338-353 ,1965.
- [12] sajda. kadhun mohammed and Sundus Najah Jaber " Some Results on Fuzzy subKs-Semigroups ", Journal of kerbala university ,Vol.11,No.3 ,Scientific ,2013.
- [13] Kuroki " Fuzzy Bi-ideals in Semigroups " Comment Math . Univ. St. Paul ,28 , 17-21 , 1979 .