Other Properties of the Class D(T)

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Abstract— *The class of D(T)- operators are equivalent to the class of quasi-normal operators. This paper discusses additional properties of this class of operators. Assuming that if the operator T is not far from normality and U serves as an interrupter, it follows that the operator U will be both unique and positive. Moreover, we explore other properties that merge when the operator T commutes with T^* T. In one of our main theorems, we demonstrate that the operator T in the class D(T) is also normal when it is invertible.*

Keywords: D(T)-operators, Hilbert space.

1. Introduction

Hilbert space is an inner product space which is complete with respect to the norm induced by its inner. Many authors have

$$T_{jk=}\begin{cases}0 & if \quad j\neq k+1\\ 1 & if \quad j=k+1.\end{cases}$$

Example:
we consider T^* , the adjoint of the unilateral shift $T: \ell_2 \to \ell_2$; recall that T has matrix entries
And U is the diagonal operator with diagonal entries $\left\{\frac{n+1}{n+2}: n = 0,1,2,\cdots\right\}$,
Then

$$T^*TU = \begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0&0\\0&\frac{2}{3}&0\\0&0&\frac{3}{4}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2}&0&0\\0&\frac{2}{3}&0\\0&0&0\end{bmatrix} = UT^*T.$$

Thus, the operator $U \in D(T)$.

2. Properties of the class D(T)

In this section, we will demonstrate several properties of the class $D(T)$.
If $T\epsilon B(H)$ and in the class of $D(T)-$ operators, we assume that T is relatively close to being a normal operator by considering the operator U in $D(T)$ as an interrupter of T, such that $T^*T = TUT^*$. Consequently, we will impose additional requirements on the operator U, demanding that it to be self-adjoint, positive, and unique.
We refer to the following lemma due to it is significance in proving Theorem (2-2).

Lemma (2-1): ([3])
deduce that $U_1 - U_2 = 0$, confirming the uniqueness of U.

Theorem (2-2):
If $U \in D(T)$, then U is a self adjoint operator.
<u>Proof:</u>
Given $U \in D(T)$, *we have* $TUT^* \in D(T^*)$ due to Theorem (2-1).
Furthermore, since $T^*T = TUT^*$ and it is self – adjoint, each T must satisfy $TUT^* = T^*U^*T$.
Implying that U is a self-adjoint operator.

Theorem (2-3):
$If\ U \in D(T), then\ the\ operator\ U\ is$
$positive\ on\ Ran(T^*).$

Proof:

We start by observing that
$< UT^*x, T^*x > = < TUT^*x, x >$
$= < T^*Tx, x >$
$= < Tx, Tx > = \|Tx\|^2 \quad for\ all\ x \in H.$
Consequently, we establish that
$< UT^*x, T^*x > \geq 0$, indicating the positivity of $U\ on\ Ran(T^*)$.

Theorem (2-4):
If $T^* has\ a\ dense\ range\ on\ Ran(T^*), then\ U$ is unique

Proof:

Let $U_1\ and\ U_2$ be two operators in $D(T)$. Considering this, we have $TU_1T^* = T^*T = TU_2T^*$.
Therefore, $T(U_1 - U_2)T^* = 0$.
Given that T^* has a dense range, and $\overline{Ran(T^*)} = N(T)^{\perp}$.
We conclude that
$N(T) = \{0\}$, establishing that $T\ is$ injective.
As a result, $(U_1 - U_2)T^* = 0$. Using the fact that T^* has a dense range once more, we deduce that $U_1 - U_2 = 0$, confirming the uniqueness of U.

3. Other properties of D(T) when T

commutes with T^* T.

We begin this section by exploring the relationship between this class and normality. An operator T in D(T) does not necessarily have to be normal, but the converse is true.

Theorem (3-1):
If $T\ is\ a\ normal\ operator, then\ T \in D(T)$.

Proof:

When T is normal, we have $(T^*T)T = (TT^*)T = T(T^*T)$. Therefor, $T \epsilon D(T)$.

Theorem (3-2):
If T is an invertible operator and $T \in D(T), then$:

1. $T\ is\ a\ normal\ operator.$
2. $(TT^*)T^* = T^*(T^*T)$
3. $T(T^*T) = (T^*T)T$
4. $T^{-1}(T^*T) = (T^*T)T^{-1}$

Proof:

1- Given $T \in D(T)$:
$(T^*T)T = T(T^*T)$
Multiplying both sides by T^{-1}yields: $(T^*T)TT^{-1} = (TT^*)TT^{-1}$
Hence $T^*T = TT^*$, making T normal.
It is evident that (1) implies (2), (3) and (4).
Now let $T = A + Bi$ represent the Cartesian form of any operator T, where $A = Re\,T = \frac{T+T^*}{2}$, and $B = Im\,T = \frac{T-T^*}{2i}$ denote the real and imaginary components of T. Berberian [1] introduced a theorem that demonstrates T being normal if and only if $AB = BA$. This result is further extended and generalized in the subsequent Theorem (3-3). In fact, we will establish that $T \in D(T)$ if $AB = BA$, although the converse does not hold true.
The proof of Theorem (3-3) is straightforward; thus, we choose to omit it.

Theorem (3-3):
If T is an operator in $B(H)$, then

1. If $AB = BA, then\ T \in D(T)$.
2. If $T \in D(T), then\ ABT = TBA.$

Theorem (3-4):
If $T\ is\ an\ operator\ such\ that\ T^*T$ commutes with A and B, then $T \epsilon D(T)$.

Proof:

Given that T commutes with both A and B, we have:
$T^*T(A + Bi) = (A + Bi)T^*T$ and $T^*TT = TT^*T.$
Thus, it follows that $T \in D(T)$.

Theorem (3-5):
If $T\ is\ an\ operator\ such\ that\ TT^*$ commutes with $A\ and\ B, and\ TTT^* = T^*TT$, then $T\epsilon\ D(T)$.

Proof:

Since $TT^*A = ATT^*\ and\ TT^*B = BTT^*$, then
$$TT^*\left(\frac{T+T^*}{2}\right) = \left(\frac{T+T^*}{2}\right)TT^*$$
and
$$TT^*\left(\frac{T-T^*}{2i}\right) = \left(\frac{T-T^*}{2i}\right)TT^*$$
This gives: $TT^*T = TTT^*$.
Since $TTT^* = T^*TT$, we have:
$$TT^*T = TTT^* = T^*TT.$$
$$T^*TT = TT^*T.$$
Thus, $T\epsilon(T)$.

Furthermore, within the context of the $D(T)$ class, if an operator T exhibits invertibility, two results can be obtained.

Theorem (3-6):
If $T \in D(T)$ be an invertible operator. Then TT^* commutes with A and B.
Proof:
Since $T \in D(T)$, we have:

$$TT^*A = TT^*\left(\frac{T+T^*}{2}\right)$$
$$= \frac{TT^*T + TT^*T^*}{2}$$

Hence, $(TT^*)T^* = T^*(T^*T)$

$$\Rightarrow \frac{TT^*T + TT^*T^*}{2} = \frac{T^*TT + T^*TT^*}{2}$$
$$= \frac{T(TT^*) + (T^*T)T^*}{2}$$
$$= \left(\frac{T+T^*}{2}\right)TT^*$$
$$= ATT^* .$$

Similarly, $TT^*B = BTT^*$.

Theorem (3-7):
If T is an invertible operator and $T \in D(T)$, then T^*T commutes with A and B.
Proof:
1)Since
$T\ is\ an\ invertible\ operator\ in\ the\ class\ of\ operators\ D(T),$
$we\ have$:

$$T^*TA = T^*T\left(\frac{T+T^*}{2}\right) = \frac{T^*TT + T^*TT^*}{2}$$
$$= \frac{TT^*T + {T^*}^2T}{2}$$
$$= \left(\frac{T+T^*}{2}\right)T^*T.$$

= AT^*T.
2) Similarly, $T^*TB = BT^*T$.

IV Conclusion
This paper discusses other properties of operators are called $D(T)$ −operator. The main results have been proved in this paper, which are:

1. $If\ U \in D(T), then\ U\ is\ a\ self\ adjoint\ operator.$.
2. The operator $U\ is\ positive\ on\ Ran(T^*)$.
3. If $T^*has\ a\ dense\ range\ on\ Ran(T^*),$ $then\ U\ is\ unique.$
4. If $T\ is\ a\ normal\ operator, then\ T \in D(T)$.
5. If T is an invertible operator and $T \in D(T), then (TT^*)T^* = T^*(T^*T)\ and\ T(T^*T) = (T^*T)T$.

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