

Analytical Model for Estimating Long-Term Deflections of Two-Way Reinforced Concrete Slabs^{*}

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Abstract

An analytical model for the calculation of long-term deflections of reinforced concrete two-way slab, flat slab and flat plate floor systems is proposed in this paper.

The proposed analytical model takes into account the significant factors influencing long-term slab deflections. It follows the short-term deflection calculation methods with realization of the time-dependent effects on the flexural rigidity of slabs.

The calculated deflections using the proposed model are compared with several field measured deflections and show good agreement with them.

الخلاصة

تم اقتراح نموذج تحليلي في هذا البحث لحساب الهطول طويل الأمد لأنظمة البلاطات الخرسانية المسلحة بأنواعها ثنائية الاتجاه والبلاطات المسطحة ذات المنهبطات والبلاطات المسطحة. إن النموذج التحليلي المقترح يأخذ بنظر الاعتبار العوامل المهمة المؤثرة على هطول البلاطات طويل الأمد انه يتبع طرق حساب الهطول قصير الأمد مع اعتبار التأثيرات المعتمدة على الزمن المؤثرة على الصلابة الانحنائية للبلاطات. إن الهطولات المحسوبة بهذا النموذج المقترح قد تمت مقارنتها مع عدة هطولات مسجلة موقعياً وقد أظهرت توافقاً جيداً معها.

^{*} Republished Paper (Accepted for Publishing in 1999)

1. Introduction

Common reinforced concrete two-way slabs used as floor systems in buildings are flat plates, flat slabs and two-way slabs with beams. An important design criterion is the control of deflections and cracking at service loads (serviceability requirements).

For crack control, limiting the spacing of bars and a minimum area of reinforcement can avoid excessive cracking of slabs.

For deflection control, the selected slab thickness (usually based on practicality) should be checked so that the calculated deflections are within allowable limits ^[1, 2].

The calculation of deflections (short-term and long-term) of reinforced concrete slabs is unpopular with designers because of the complexities involved in performing such calculations. Most of the methods available in the literature ^[2, 3] involve various factors some of which took considerable attention while others ignored.

Accordingly, the requirement should be a compromise between rationality and convenience. This statement should be the guide especially in view of the variability in measured deflections ^[2, 4].

This paper proposes a model for the calculation of long-term deflections of two-way slabs that takes care of the major influencing factors. The new proposal has the analytical approach in which the long-term deflections are calculated essentially similar to the short-term deflections. The proposal is compared with measured deflections found in the literature, and discussed.

2. Calculation of Slab Deflections

The initial deflection for uniformly loaded flat plates and two-way slabs can be approximated by Eqs. (1) and (2) ^[5, 6, 7]:

$$\text{Flat plates} \quad d_i = \varepsilon_{fp} q L^4 / E_{ci} I \dots\dots\dots (1)$$

$$\text{Two-way slabs} \quad d_i = \varepsilon_{tws} q L^4 / E_{ci} I \dots\dots\dots (2)$$

where:

I: the moment of inertia and;

q: the load refer to a unit width of the slab.

Deflection coefficients, ε_{fp} and ε_{tws} are given in **Table (1a)** and **(1b)** for interior panels. Note that these coefficients are dimensionless, so that q must be in load/length (kN/m).

Table (1-a) Deflection coefficients, ε_{fp} and ε_{tws} , for interior panels ($\times 10^{-5}$)

Type	Interior Panel Support			l/s					
				1.0	1.1	1.2	1.3	1.4	1.5
ε_{fp} (Flat Plates)	Zero edge beam stiffness	c/ℓ	0.0	581	487	428	387	358	337
			0.1*	441	372	320	283	260	243
ε_{tws} (Two-Way Slabs)	Elastically supported edges. The appropriate coefficient is in between the case of zero edge beams	Relatively flexible		308 to 250	330 to 230	290 to 210	260 to 190	240 to 170	230 to 160
				290 to 170	260 to 140	230 to 120	210 to 105	190 to 90	180 to 80

* Approximate

 c/ℓ = column / span ratio ℓ/s = longer span / shorter ratio

Table (1-b) Deflection coefficients* of flat-plate interior panels, from (5)

Center of Panel			
c/l, s/l ratios	0.0	0.1	0.2
1.0	0.00581	0.00441	0.00289
0.8	0.00420	0.00301	0.00189
0.6	0.00327	0.00234	0.00143
0.4	0.00284	0.00205	---

* All deflections are given as coefficient of qL^4/D .**Notation:**

c = width and depth of square column section, L = long span, S = short span, q = load per unit area, $D = Et^3/12(1-\nu^2)$, t = slab thickness, ν = Poisson's ratio, E = modulus of elasticity.

An approximate method based on the equivalent frame method can be used [8]. The deflection at the center of a rectangular panel as shown in **Fig.(1)** is the sum of the mid-span deflection of the column strip in one direction and that of the middle strip in the other direction, thus

$$d = d_{ex} + d_{my} \dots \dots \dots (3)$$

or:

$$d = d_{cy} + d_{mx} \dots \dots \dots (4)$$

At first, a "reference" deflection is computed:

$$d_{f,ref} = \frac{WL^4}{384E_c I_{frame}} \dots\dots\dots (5)$$

where:

W: is the load per meter along the span of length L and;

I_{frame} : is the moment of inertia of full-width panel including the contribution of the column-line beam or drop panels and column capitals if present.

E_c : is the concrete short-term modulus of elasticity.

The deflection of strips will be:

$$d_{f,strip} = d_{f,ref} \frac{M_{strip} E_c I_{frame}}{M_{frame} E_c I_{strip}} \dots\dots\dots (6)$$

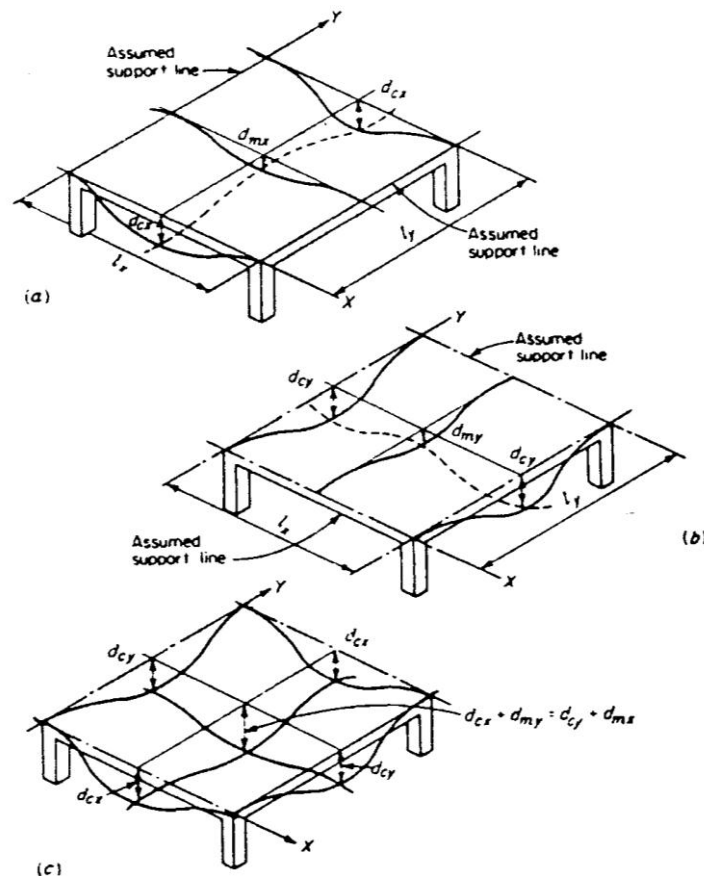


Figure (1) Basis of equivalent frame method for deflection analysis:
 (a) X-direction bending. (b) Y-direction bending. (c) Combined bending.

It may be noted that the ratios M_{strip}/M_{frame} identical to the lateral moment distribution factors given by the ACI Code ^[1] as shown in **Table (2)**.

Table (2) Ratios M strip/M frame (ACI 318-95)

		I_2 / I_1		
		0.5	1.0	2.0
Interior negative moment:				
$\alpha_1 I_2 / I_1 = 0.0$		75	75	75
$\alpha_1 I_2 / I_1 \geq 1.0$		90	75	45
Exterior negative moment:				
$\beta_t = 0$		100	100	100
$\beta_t \geq 2.5$		75	75	75
$\alpha_1 I_2 / I_1 \geq 0.0$	$\beta_t = 0$	100	100	100
	$\beta_t \geq 0$	90	75	45
Positive moment:				
$\alpha_1 I_2 / I_1 = 0.0$		60	60	60
$\alpha_1 I_2 / I_1 \geq 1.0$		90	75	45

$$\alpha = \frac{E_{cb} I_b}{E_{cs} I_s}$$

$$\beta_t = \frac{E_{cb} C}{2E_{cs} I_s}$$

The rotation of the column at the floor is given by:

$$\theta = \frac{M_{net}}{K_{ec}} \dots \dots \dots (7)$$

where:

θ : is the angle change in radians,

M_{net} : is the difference in floor moments to left and right of column and;

K_{ec} : is the stiffness of equivalent column (ACI code ^[1] section 13.7).

Knowing that for a member with an end rotation of θ radians, the far end fixed, the mid-span deflection is:

$$d_{\theta} = \frac{\theta L}{8} \dots \dots \dots (8)$$

Thus the total deflection at midspan of column strip or middle strip is:

$$d_{strip} = d_{f,strip} + d_{\theta left} + d_{\theta right} \dots \dots \dots (9)$$

The calculations described (Eq.3 to Eq.9) are repeated in the second direction of the panel, and the center deflection is obtained as indicated in Eqs. (3) or (4).

3. Flexural Rigidity of Slabs

The flexural rigidity of slabs varies considerably and relies mainly on empirical equations to account for cracking, creep and shrinkage.

The most important aspect of deflection calculation is the extent of cracking of the slab. Any attempt at predicting deflections should address the problem of cracking.

The ACI code suggests the use of an effective moment of inertia, I_e , based on the work of Branson^[9,10], given by:

$$I_e = \left(\frac{M_{cr}}{M_a} \right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a} \right)^3 \right] I_{cr} \leq I_g \quad \dots\dots\dots (10)$$

where:

- I_g = gross moment of inertia of concrete section;
- I_{cr} = moment of inertia of the cracked transformed section;
- M_{cr} = cracking moment and;
- M_a = applied moment.

Prior to calculating I_{cr} , the position of the neutral axis has to be determined. For a singly reinforced rectangular section, i.e. a section with tension reinforcement only, as shown in **Fig.(2)**, the following expressions are established:

$$k = \sqrt{(\rho n)^2 + 2\rho n} - \rho n \quad \dots\dots\dots (11)$$

$$I_{cr} = \frac{1}{3} b(kd)^3 + nA_s(d - kd)^2 \quad \dots\dots\dots (12)$$

where:

- n : modular ratio = E_s/E_{ci} ,
- E_s = modulus of elasticity of steel = 200000 MPa and;
- E_{ci} = modulus of elasticity of concrete (short-term).

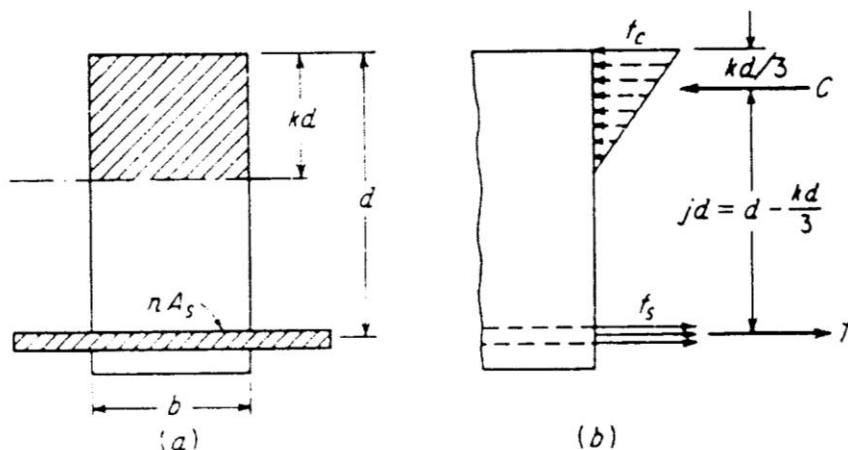


Figure (2) Cracked transformed section

4. Slab Deflections Due to Long-Term Loads

If loads are sustained over a long period of time, deflections are increased significantly due to the effects of creep and shrinkage.

ACI code ^[1] specifies that additional long-term deflections due to the combined effects of creep and shrinkage shall be calculated by multiplying the immediate deflection by the factor.

$$\lambda = \frac{\varepsilon}{1 + 50\rho'} \dots\dots\dots (13)$$

where:

$\rho' = A_s' / bd$ and;

ε : is a time-dependent coefficient. Because compression steel is seldom used in slabs,

$\lambda = 2.0$.

Critiques ^[10] indicate that this may seriously underestimate long-term slab deflections in view of various tests and field-measured slab deflections.

To examine the behaviour of a singly reinforced rectangular section under sustained load, see **Fig.(3)**. The concrete strain, at top fiber, increases. The strain diagram rotates and the neutral axis moves down as a result of creep.

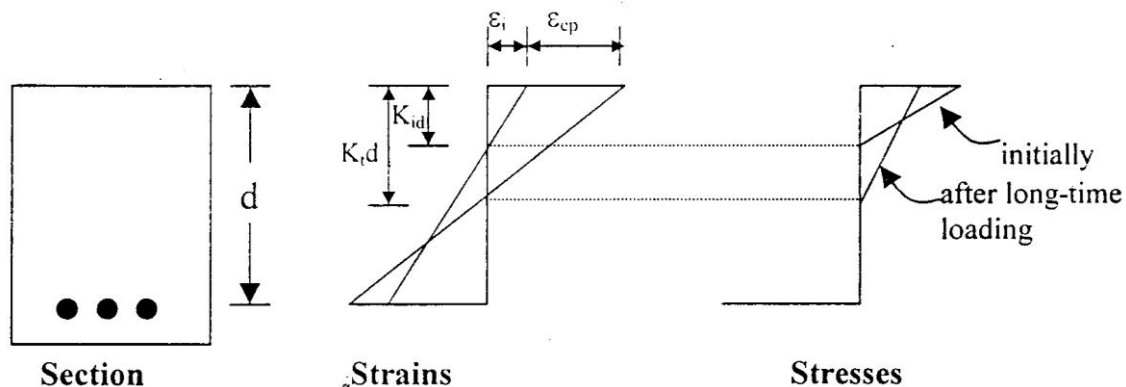


Figure (3) Strain and stress distributions initially and after long-time loading in a flexural member

5. Procedure Proposed

A procedure is suggested for calculating long-term deflection analogous to that of the short-term deflection (Eqs. 1 & 2 or Eqs. 3 to 9) as follows:

5.1 Long-Term Flexural Rigidity of Slabs

In this proposal, the analysis shall be made on "aged" reinforced concrete sections. The use will be made here of $E_{c(t)}$ to mean the age-adjusted effective modulus of elasticity and $I_{e(t)}$

to mean the moment of inertia of an age-adjusted section in which the transformed section is composed of the area of concrete plus $(E_s/E_{c(t)})$ multiplied by the area of steel.

The age-adjusted modulus of elasticity is defined by ^[7, 11]:

$$E_{e(t)} = E_{ci} / (1 + \chi v_t) \dots \dots \dots (14)$$

where:

$$v = \text{creep coefficient at time } t = \frac{(t - t_o)^{0.6}}{10 + (t - t_o)^{0.6}} v_u,$$

v = ultimate creep coefficient ^[7] and;

χ = aging coefficient which depends on age at the time (t_o) when the structure begins carrying the load and on the load duration as shown in **Table (3)**.

Table (3) Aging coefficients

$t - t_{\ell a}$ days	v_u	$t_{\ell a}$ in days			
		10^1	10^2	10^3	10^4
10^1	0.5	0.525	0.804	0.811	0.809
	1.5	0.720	0.826	0.825	0.820
	2.5	0.774	0.842	0.837	0.830
	3.5	0.806	0.856	0.848	0.839
10^2	0.5	0.505	0.888	0.916	0.915
	1.5	0.739	0.919	0.932	0.928
	2.5	0.804	0.935	0.943	0.938
	3.5	0.839	0.946	0.951	0.946
10^3	0.5	0.511	0.912	0.973	0.981
	1.5	0.732	0.943	0.981	0.985
	2.5	0.795	0.956	0.985	0.988
	3.5	0.830	0.964	0.987	0.990
10^4	0.5	0.501	0.899	0.976	0.994
	1.5	0.717	0.934	0.983	0.995
	2.5	0.781	0.949	0.986	0.996
	3.5	0.818	0.958	0.989	0.997

The report of ACI committee 209 ^[7] recommended that the value of v_u is $(2.35 \gamma_c)$ where γ_c is a correction factor, the product of several multipliers depending upon ambient relative humidity, average thickness of the member or its volume-to-surface ratio and on the temperature. For relative humidity 40% average thickness 150 mm or volume-to-surface ratio of 37.5 mm and temperature 21°C, all the Multipliers are equal to 1.0. In this case, γ_c is calculated as: $\gamma_c = 1.25 t_o^{-0.118}$. (Moist-cured concrete).

The time-dependent effective moment of inertia $I_{e(t)}$, can be computed according to Eq. (10) observing the following, variation: In computing $I_{cr(t)}$ the position of the neutral axis is shifted: kd to $(kd)_t$, and the modular ratio n (E_s/E_c) is changed into $n(t)$: ($E_s/E_{c(t)}$) according, to Eq. (14), as follows:

Equilibrium requires $b(kd)_t^2 / 2 = n(t) A_s (d - (kd)_t)$, Therefore:

$$n_{(t)} = \left(\frac{bd}{2A_s} \right) \left(\frac{k_{(t)}^2}{1 - k_{(t)}} \right) = \frac{1}{2\rho} \frac{k_{(t)}^2}{1 - k_{(t)}} \dots\dots\dots (15)$$

Substituting (15) in (12) gives:

$$I_{cr} = \frac{bd^3}{6} (3k_{(t)}^2) \dots\dots\dots (16)$$

An illustration of the shifting of neutral axis downward with time in a singly-reinforced section is given in **Fig.(4)**. About 50% increase in the neutral axis depth occurred after a long time (1000 days) over the initial depth. The long-term flexural rigidity is now defined by Equations (10), (14), (15) and (16).

To show the significant decline in flexural rigidity with time. **Figure (5)** presents an illustration. For a particular slab section, a decrease of about 65% in the flexural rigidity occurred after 1000 days in the case of uncracked section $\left(\frac{M_a}{M_{cr}} \right) \leq 1.0$. When $\left(\frac{M_a}{M_{cr}} \right) = 2.0$, a decrease in flexural rigidity of about 50% occurred, and of about 40% when the section is fully cracked.

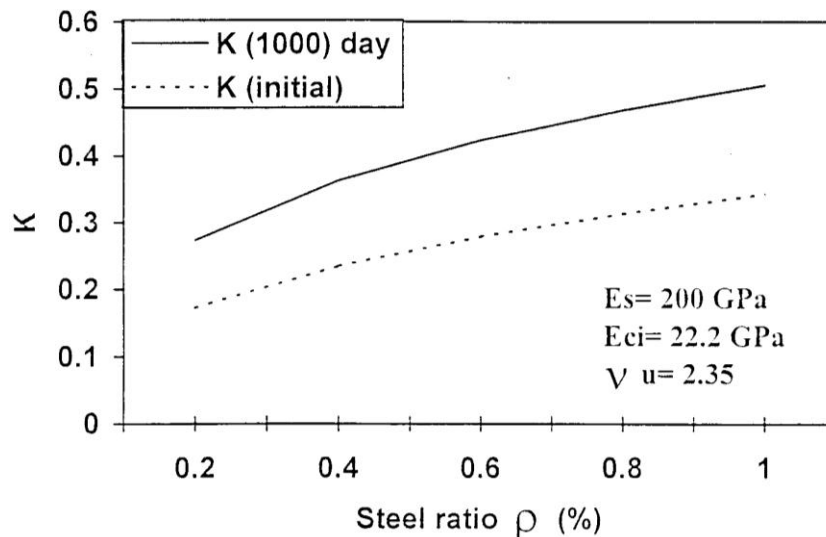


Figure (4) Initial and long-term neutral axis position of slab section

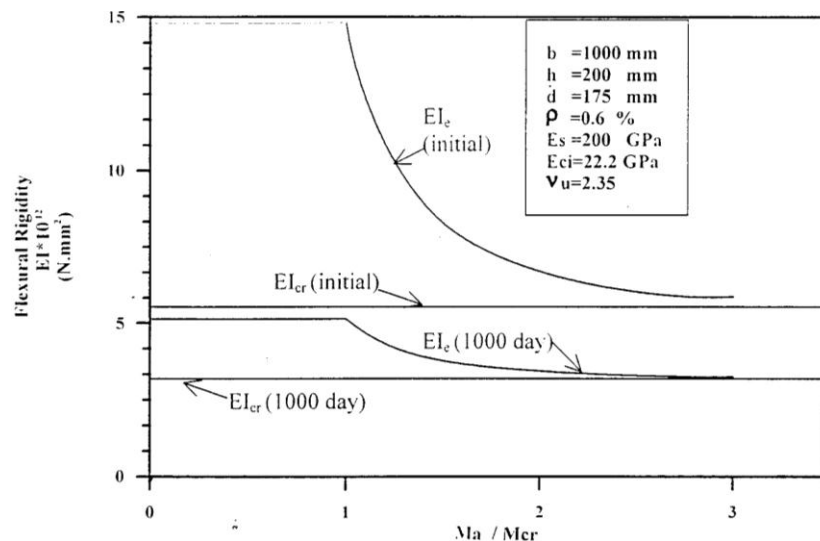


Figure (5) Initial and long-term flexural rigidity of slab section

5.2 Long-Term Deflection of Slab

Follow the same procedure performed for short-term deflection calculation (Eqs.1 & 2 or Eqs. 3 to 9) to calculate the long-term deflection of two-way slabs. The calculations should take care of the time-dependent flexural rigidity established in section (5-1). These effects shall take place in (Eqs.1 & 2) and (Eqs.5 & 6).

For continuous slabs (at one or both ends), satisfactory results are expected using an average of the positive and negative moment region values from Eq.10^[1]. However, it has been shown^[12] that Eq.10 can be used along with an average of the positive and negative moment region values as follows:

Flat plates: $I_{e(t)}$, positive and negative, for the long direction column strip.

Two-way slabs: $I_{e(t)}$, positive and negative, for the short direction middle strip.

The center of interior panels normally remains uncracked in common designs of the slabs.

6. Comparison with Field-Measured Long-Term Slab Deflections

Field data on long-term deflections of two-way slabs, flat slabs and flat plates are scarce. However, several investigations^[13, 14, 15, 16, 17] have been reported. **Table (4)** presents summary of data for long-term deflections calculations. The values of E_c are either those reported by the investigators or computed using the ACI Code expression:

$$E_c = 0.043 W_c^{1.5} \sqrt{f'_c} \dots\dots\dots (17)$$

where:

W_c = the density of concrete (kg/m^3).

Table (5) compares the predictions of this paper with the field-measured deflections.

Table (4) Summary of data for long-term deflections

References	Structure	Dimension (m)	Thickness (mm)	Sustained load (kN/m ²)	Modulus of Elasticity (MPa)
Taylor (13)	Flat plate	6.34×5.07	200	5.5	21400
Jokinen (14)	Flat slab	9.0×9.0	200	5.5	27800
Heiman (15)	Flat slab	7.54×7.24	240	5.5	28500
Sbarounis (16)*	Flat plate	6.7×6.7	185	4.2	18000
Jawad (17)	Flat slab	7.0×7.0	220	7.3	21700

* Light weight concrete (1760 kg/m³)

Table (5) Comparison of measured and calculated deflections

References	Measured Deflections (mm)	Calculated Deflections (mm)	Calculated/Measured
(13)	24.4 (9-year)	26.1	1.07
(14)	33 (1-year)	36.4	1.04
(15)	21.6 (9-year)	23.1	1.07
(16)	34.3 (1-year)	28.5	0.83
(17)	33.6 (15-year)	38.7	1.15

The correlation between calculated and measured deflections is shown to be good. The mean value of calculated/measured deflections is (1.03) with a coefficient of variation of (11) percent.

It is worth mentioning that the analytical model presented in this paper requires that the creep characteristics of concrete are needed to be estimated as accurately as possible. Needless to say that such information's are absent in the reported investigations and the ACI 209 recommendations ^[7] were, therefore, applied in the calculated values. Reference 17, however, estimated creep factor for Iraqi concrete to lie between (1.9) and (2.2) for its particular investigation. A value of (2.0) was used in the calculations.

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