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Analytical, Theoretical and Experimental Investigation of Linear Viscoelastic Thin Plate Deflection with Two Edges Fixed and the Other Two Edges Simply Supported

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Abstract

A polyester and composite polyester thin plate have been tested under a certain constant distributed load ($q=8N/mm^2$) so as to give a linear behavior over the plane of area at constant temperature (T=15ºC). The plate has been tested for thickness values (t=2,3,4mm). The type of boundary condition used here is two sides are fixed and the other two sides are simply supported. For the linear behavior of the polymer, a linear finite element program is achieved with the aid of FORTRAN power station program. The results of the FEM, theoretical and the experimental were compared. A good accuracy has been observed between the FEM, theoretical and the experimental work. In general the deflection behavior depends on the creep compliance. Increasing thickness results in decreasing the deflection, this decreasing depend on the value of thickness increasing. The deflection rate decrease during time for polyester, which depend on the creep function. The increasing in the plate dimensions ratio results in increasing the plate surface deflection for both plates (polyester and composite polyester plate). A composite plate shows a small increasing in the deflection during increasing the plate dimension ratio as comparing with that of polyester.

الخلاصــــــة

تم اختبار الصفائح الرقيقة من (بوليستر) و المركب من (بوليستر) تحت تأثير حمل موزع ثابت مقداره ($8N/m^2$) و تحت تأثير درجة حرارة ثابتة (15C°) و تم اختبار الصفيحة لثلاث قيم من السمك (2, 3, & 4 mm). شروط النهايات المستخدمة هي جانبين من الصفيحة مثبتة و الجانبين الآخرين مسنده بصورة مبسطة. تم استخدام ثلاث أنواع من نسب أبعاد الصفيحة. تصرّف الصفيحة يكون خطي حيث تم بناء برنامج العناصر المحددة لحالة المادة اللزجة المرنة و النتائج النظرية و العملية تم مقارنتها. تم ملاحظة دقة جيدة كمقارنة البرنامج مع الجزء النظري و العملي. بصورة عامة تصرف الانحناء يعتمد على مطاوعة الزحف. زيادة السمك يؤدي إلى نقصان في الانحناء للصفيحة و هذا النقصان يعتمد على مقدار زيادة السمك. معدل الانحناء خلال الزمن (للبولستر) يعتمد على مطاوعة الزحف. الزيادة في معدل نسب أبعاد الصفيحة يؤدي إلى زيادة في الانحناء للصفيحة لكلا (بوليستر) والمركب. الصفائح المركبة تظهر زيادة قليلة في الانحناء خلال زيادة نسبة الأبعاد كمقارنة مع (صفائح البولستر).

1. Introduction

The increasing use of polymers in engineering design is largely due to their high strength to weight ratio and to their corrosion resistance [1]. Viscoelastic materials experience both viscous and elastic phenomena as the name viscoelastic implies. There are some phenomena which are common to many viscoelastic materials [2], as illustrated in **Fig.(1)**.

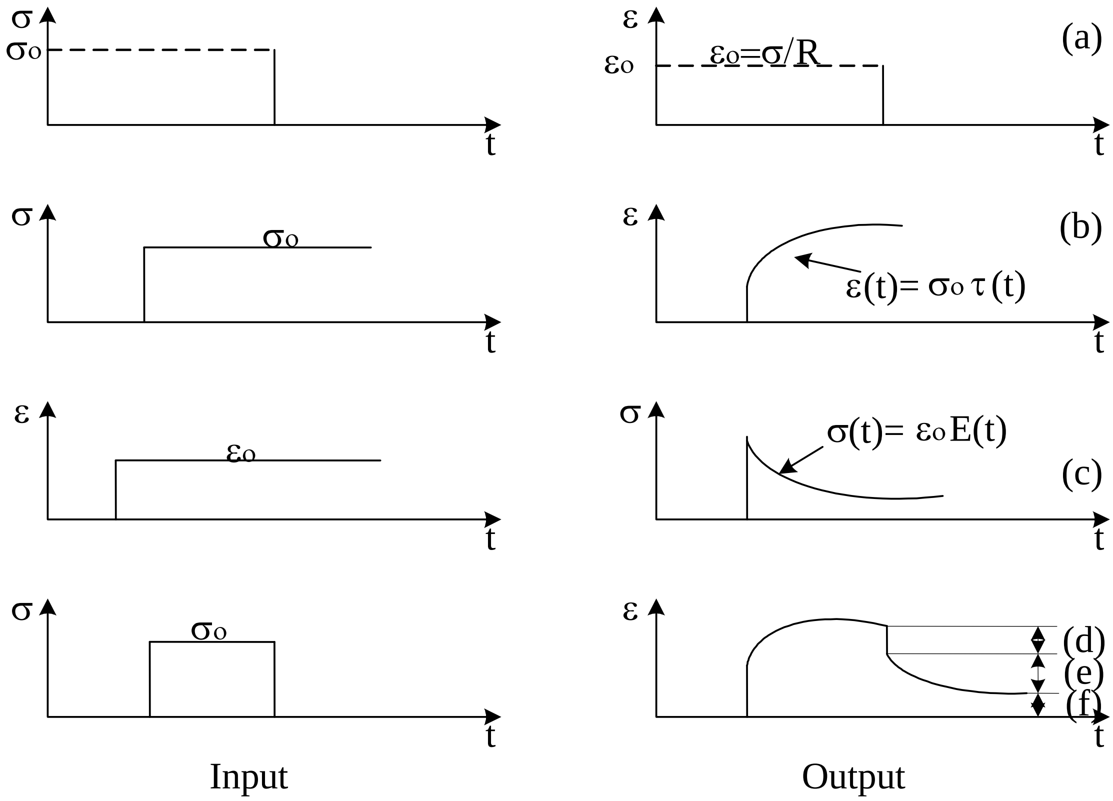

Figure (1) Phenomena common to many viscoelastic materials [2]

(a) Instantaneous elasticity.
(b) Creeps under constant stress.
(c) Stress relaxation under constant strain.
(d) Instantaneous recovery.
(e) Delayed recovery.
(f) Permanent set.

Most material exhibit linear or nearly linear behavior under small stress levels [3] .One of the most distinguishing features of viscoelastic materials is their response to so called a constant stresses (creep test). The creep test consists of measuring the time dependent strain resulting from the application of steady uni-axial stress as illustrated in **Fig.(2)**.

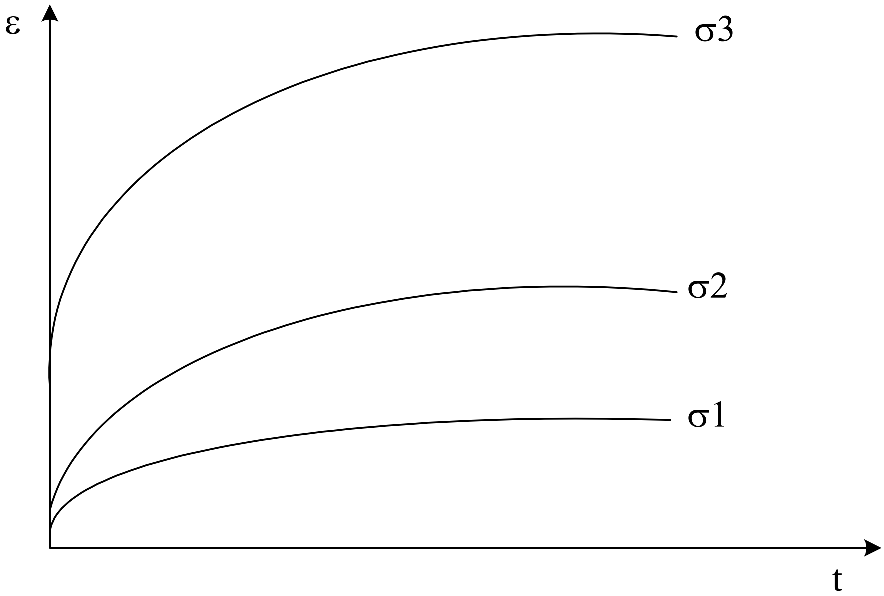

Figure (2) Creep strain at various stresses

These three curves are the strain measured at three different stress levels, each one twice the magnitude of the previous one (such that: $\sigma 3=2\sigma 2=4\sigma 1$).

Note that in **Fig.(2)** that when the stress is doubled, the resulting strain is doubled over its full range of time.

2. Theory of Linear Viscoelasticity

The response of viscoelastic material is called as a linear if the following condition is occurred [4]:

$$\varepsilon(a\sigma) = a\varepsilon(\sigma) \quad \text{.......... (1)}$$

The ratio of strain to stress is called the compliance and in the case of time-varying strain, arising from a constant stress the ratio is the creep compliance, D(t):

$$D(t)=\varepsilon(t)/\sigma \quad \text{.......... (2)}$$

The stress-strain time relation of viscoelastic material has been analyzed with the aid of mechanical models where the stress and strain are used instead of force and deformation of model [2]. All linear viscoelastic models are made up of linear springs and linear viscoelastic dashpot. In the linear springs shown in **Fig.(3a)**, the following relations can be written:

$$\sigma = R\varepsilon \quad \text{.......... (3)}$$

where:

R: linear spring constant or young modulus.

The spring element exhibit instantaneous elasticity and instantaneous recovery as shown in Fig.(3b). A linear dashpot element is shown in **Fig.(3c)** where:

$$\sigma = \eta \frac{d\varepsilon}{dt} = \eta \dot{\varepsilon} \quad \text{.......... (4)}$$

where:

$\dot{\varepsilon}$: strain rate during time

η: coefficient of viscoelasticity.

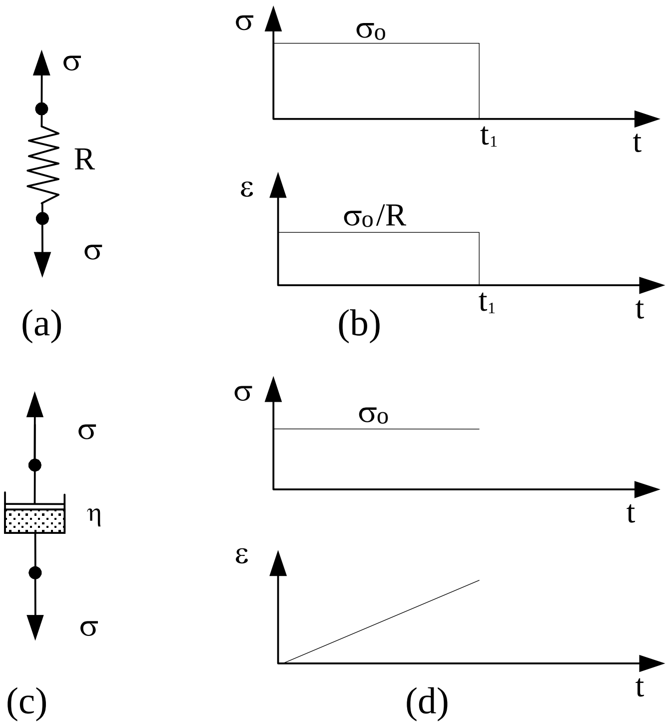

Figure (3) Behavior of linear spring and linear dashpot

Dashpot will be deformed continuously at constant rate when it subjected to a step of constant stress as shown in **Fig.(3d)**.

The Burgers model is shown in **Fig.(4a)** where the Maxwell and Kelvin model are connected in series [2].The total strain can be written as follows:

$$\varepsilon = \varepsilon 1 + \varepsilon 2 + \varepsilon 3 \quad \text{.......... (5)}$$

in which:

ε: total strain in burgers four-element model.

$\varepsilon 1$: the strain in spring for maxwell model.

$\varepsilon 2$: the strain in dashpot for maxwell model.

$\varepsilon 3$: the strain in kelvin model.

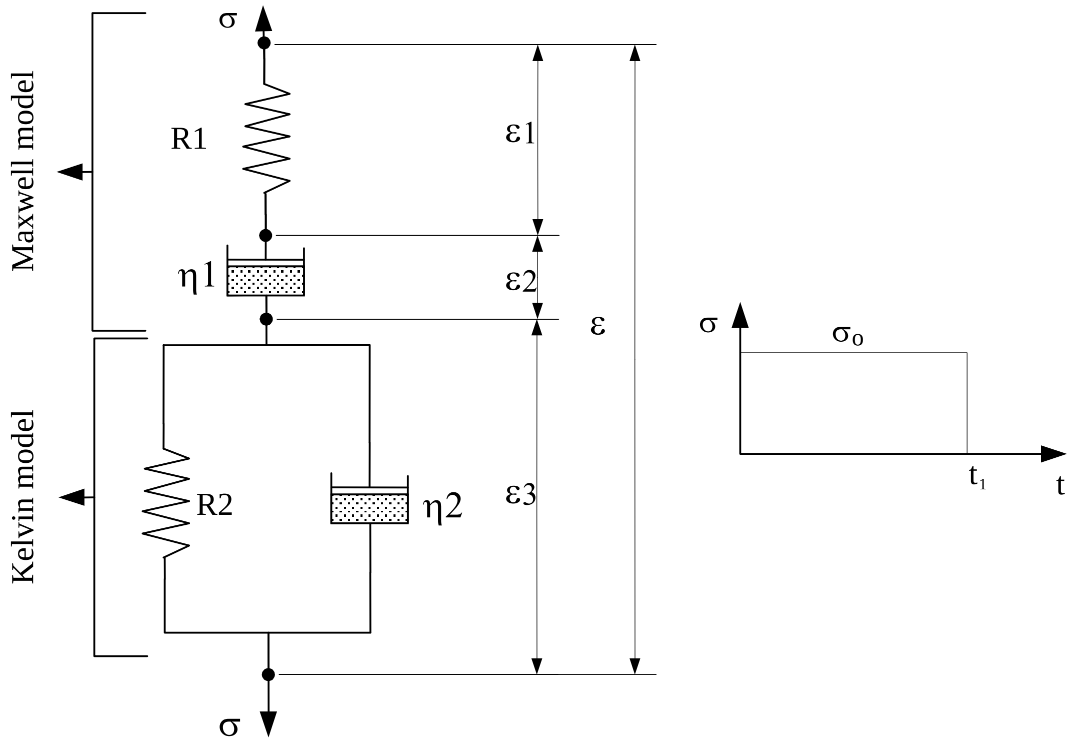

Figure (4) Behavior of a burger model

That is:

$$\varepsilon 1 = \frac{\sigma}{R1} \quad \text{......(6)}$$

$$\dot{\varepsilon} 2 = \frac{\dot{\sigma}}{\eta 1} \quad \text{......(7)}$$

$$\dot{\varepsilon} 3 + \frac{R2}{\eta 2}\varepsilon 3 = \frac{\sigma}{\eta 2} \quad \text{......(8)}$$

From Eqs.(6-8), may obtained the following second order differential equation between stress and strain:

$$\sigma + \left(\frac{\eta 1}{R1} + \frac{\eta 1}{R2} + \frac{\eta 2}{R2}\right)\dot{\sigma} + \frac{\eta 1 \eta 2}{R1R2}\ddot{\sigma} = \eta 1 \dot{\varepsilon} + \frac{\eta 1 \eta 2}{R2}\ddot{\varepsilon} \quad \text{......(9)}$$

The Laplace transformation method is used in solving the differential equations (9) to illustrate the creep behavior as follows:

$$\varepsilon(t)=\frac{\sigma o}{R1}+\frac{\sigma o}{\eta 1}t+\frac{\sigma o}{R2}(1-e^{-R2t/\eta 2}) \quad \text{.. (10)}$$

The material constant R1, R2, η1, η2 can be determined from the experimental data for creep test in linear viscoelasticity behavior. The stress relaxation behavior of the Burger model for a step of strain can be obtained from Eq.(9) as follows [2]:

$$\sigma(t)=\frac{\varepsilon o}{A}[(q1-q2r1)e^{-r1t}-(q1-q2r2)e^{-r2t}] \quad \text{................................ (11)}$$

where:

$$r1=\frac{p1-A}{2p2} \quad ,P1=(\frac{\eta 1}{R1}+\frac{\eta 1}{R2}+\frac{\eta 2}{R2}) \quad ,P2=\frac{\eta 1\eta 2}{R1R2} \quad \text{........................ (12)}$$

$$r2=\frac{p1+A}{2p2} \quad ,q1=\eta 1 \quad ,q2=\frac{\eta 1\eta 2}{R2} \quad \text{...................................... (13)}$$

$$A=\sqrt{p1^2-4p2} \quad \text{... (14)}$$

Creep compliance can be obtained from Eq.(10) as:

$$D(t)=\frac{\varepsilon(t)}{\sigma_o}=\frac{1}{R1}+\frac{1}{\eta 1}t+\frac{1}{R2}(1-e^{-R2t/\eta 2}) \quad \text{.................................... (15)}$$

If the material behavior is linear, the stress can be represented by:

$$\sigma(t)=\varepsilon_o E(t) \quad \text{.. (16)}$$

$$E(t)=\sigma(t)/\varepsilon_o \quad \text{.. (17)}$$

The function E(t) thus obtained and called the relaxation modulus. the equation of relaxation modulus can be obtained using Eqs.(11,17) as follows:

$$E(t)=\sigma(t)/\varepsilon_o=\frac{1}{A}[(q1-q2r1)e^{-r1t}-(q1-q2r2)e^{-r2t}] \quad \text{..................... (18)}$$

Plate can be considered as thin when its thickness is about fifteen times smaller than the shortest span length.

3. Theoretical Plate Deflection

The solution of the equilibrium differential equation for bending two opposite side simply supported and other clamped **Fig.(5)**, can be written as follows [5]:

$$w = \left(\frac{4qa^4}{\pi^5 D} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m^5} \left(1 - \frac{\alpha m \tanh\alpha m + 2}{2\cosh\alpha m} \cosh\frac{2y\alpha m}{b} + \frac{1}{2\cosh\alpha m} \frac{2y}{b} \sin\frac{2y\alpha m}{b}\right) \sin\frac{m\pi x}{a} \right)$$

$$- \left\{ - \frac{2qa^4}{\pi^5 D} \sum_{m=1,3,5} \left(\frac{\sin\frac{m\pi x}{a}}{m^5 \cosh\alpha m} \, \frac{\alpha m - \tanh\alpha m(1 + \alpha m \tanh\alpha m)}{\alpha m - \tanh\alpha m(\alpha m \tanh\alpha m - 1)} \right) * \left(\frac{m\pi y}{a} \sinh\frac{m\pi y}{a} - \alpha m \tanh\alpha m \cosh\frac{m\pi y}{a} \right) \right\} \quad \ldots\ldots (19)$$

where:

- w: the plate deflection(mm) .
- q: the distributed load (N/mm^2).
- a, b: plate dimensions.
- D: the flexural rigidity of plate, given by:

$$D = \frac{Et^3}{12(1 - v^2)} \quad \text{and} \quad \alpha m = \frac{m\pi b}{2a}$$

Hence, the modulus of elasticity (E), can be represented here in the form of function of time as in Eq.(18).

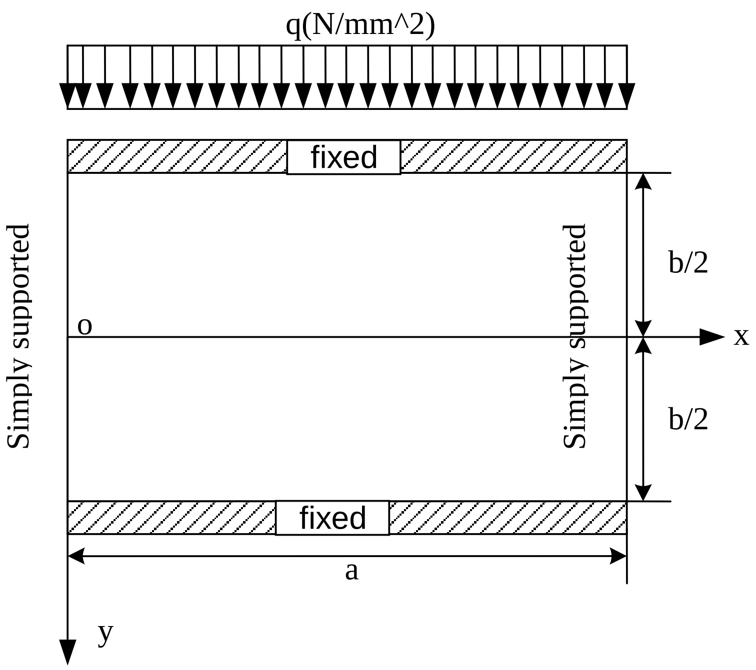

Figure (5) Rectangular plate with two opposite edges simply supported and the other two edges fixed

4. Finite Element Approach for Linear Thin Viscoelastic Plate

Linear viscoelastic plate equations analysis has been derived to formulate an approach model in (FEM) depending on standard element stiffness derived from [6]. In general, the program gives the following data for each node:

1. The deflection (w) and the slopes (θx, θy).
2. The strains (εx, εy, γxy).
3. The stresses (σx, σy, τxy).

The following system of equilibrium equation for FEM has been used:

$$\left(\int [B]^T [D][B] dA\right)\{u\} - \int q(x,y)[N] dA = 0 \quad \text{.......... (20)}$$

where the element stiffness [K]e is given by:

$$[K]_e = \iint [B]^T [D][B] dxdy \quad \text{.......... (21)}$$

The matrix [K] for each element has been formulated from [7,8] and the vector of the equivalent nodal force {f}e is:

$$\{f\}_e = \iint q(x,y)[N] dA \quad \text{.......... (22)}$$

Thus, the equilibrium for plate element can be expressed in the concise form:

$$\{f\}_e = [K]_e \{u\}_e \quad \text{.......... (23)}$$

$$[D] = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} \quad \text{.......... (24)}$$

where:

[B]: strain matrix.

The element deformed shape can be approximated with a suitable set of shape functions Ni (x,y):

$$w(x,y) = \sum_i Ni(x,y) ui = [N]\{u\} \quad \text{.......... (25)}$$

A rectangular four node element of plate (ijkl) coinciding with (xy) planes is used with three degree of freedom for each node (w, θx, θy) ,where the total degree of freedom for each element is (12 degree of freedom), **Fig.(6)**. At each node displacements (an) are introduced and defined by (for node i):

$$\{a\}^i = \begin{Bmatrix} wi \\ \theta xi \\ \theta yi \end{Bmatrix} = \begin{Bmatrix} wi \\ -\left(\frac{\partial w}{\partial y}\right)_i \\ -\left(\frac{\partial w}{\partial x}\right)_i \end{Bmatrix} \quad \text{.......... (26)}$$

The strain can be found according to the following formula:

$$\varepsilon = [B]\{a\}^e \quad \text{.......... (27)}$$

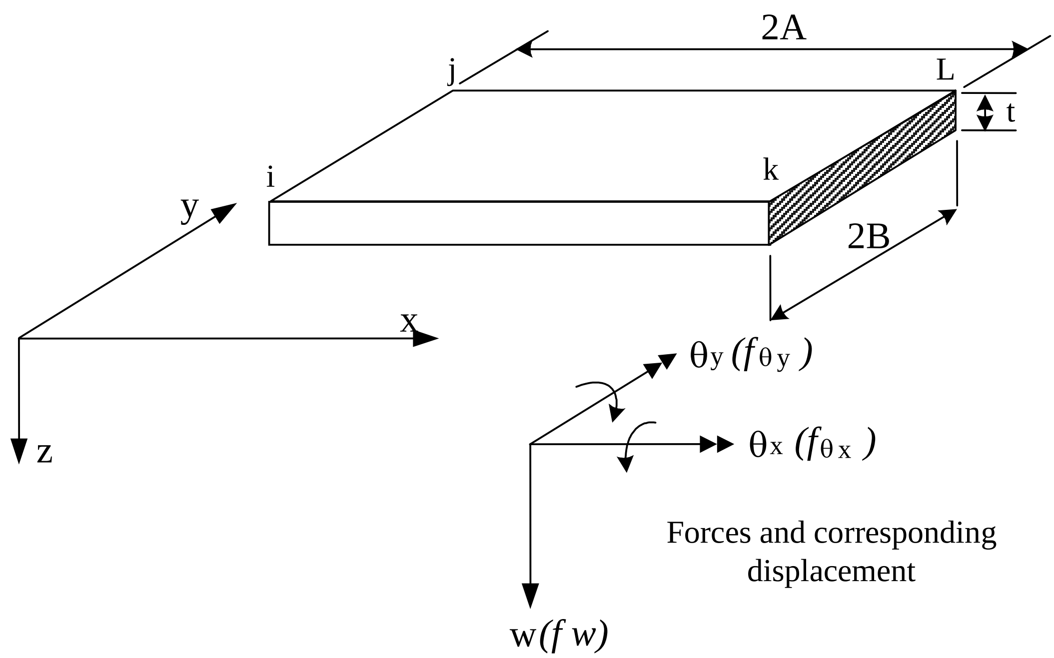

Figure (6) Rectangular plate element

5. Linear Viscoelasticity Computer Program

In this software the stress, strain, deflection and the rotations about the x-axis and y-axis have been determined for each time step according to the stress relation function, E(t). The program has been done for thin plate analysis for linear viscoelasticity. The program contains the following subroutine written in FORTRAN power station language:

1. Data subroutine: the required input data can be classified as:

- Stress relaxation function constants Eq.(18)
- Plate element dimensions: A, B, thickness (see **Fig.(6)**).
- Poisson's ratio ν.
- Mesh data: number of nodes, elements and the DOF.

- Element number and its nodes sequence.
- Boundary conditions.
- Node numbers and its Cartesian (x, y).
- Time to be read and write result.

2. Loading subroutine: the job of this subroutine is to assemble the element nodal force evaluated from equation (23).

3. Stiffness matrix assemble: in this subroutine, the element stiffness (which has been evaluated from Eq.(21) will be assembled to give the global stiffness matrix.

4. Reducer subroutine: to reduce the global stiffness matrix by applying the boundary conditions.

5. Solver subroutine: to solve the system equations to give the nodal deflections (w, θx, θy).

6. Displacement subroutine: in this subroutine the output data (w, θx, θy) has been written in the output data file.

7. Stress subroutine: the strain (εx, εy, γxy) which has been evaluated from Eq.(27) has been written in the data file as well as for each node at each time the stress result written here. The Fortran power station has been done with the aid of [9,10].

Figure (7) shows the block diagram of this program subroutine.

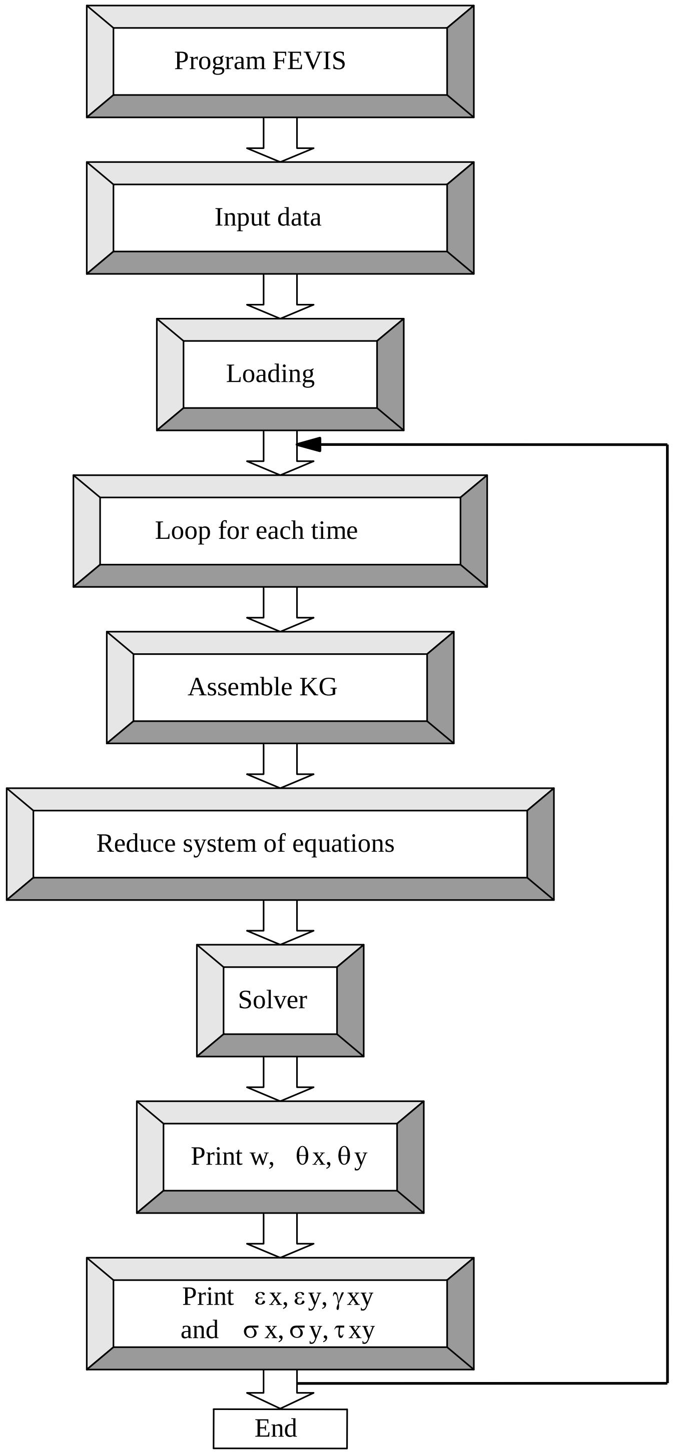

Figure (7) Block diagram for linear viscoelatcity thin plate

6. Experimental Work

Polyester specimen has been prepared by mixing two liquid substances. The procedure used to achieve the creep test is by reading the instantaneous deformation of specimen from the movement of dial gage pointer step by step with (mm) unit during the subsequence time. The compliance equation has been evaluated according to Burger four element equation (15). The final creep result for compliance equation can be written as follows:

$$D(t)=3.566e\text{-}3+(1.0767e\text{-}6).t+9.535e\text{-}4.(1-EXP(-0.02123.t)) \quad \ldots\ldots\ldots\ldots (28)$$

According to Eq.(18) the relaxation formula can be written as:

$$E(t)=220.EXP(-r1.t) + 60.3.EXP(-r2.t) \quad \ldots\ldots\ldots\ldots (29)$$

The same above procedure has been done for composite polyester specimen with volume fraction (Vf=0.3). **Figure (8)** shows a schematic graph of standard creep test specimen [11].

The creep compliance has the following function:

$$D(t)=1.09e\text{-}3+(1.3687e\text{-}5).t+1.835e\text{-}3.(1-EXP(-0.127.t)) \quad \ldots\ldots\ldots\ldots (30)$$

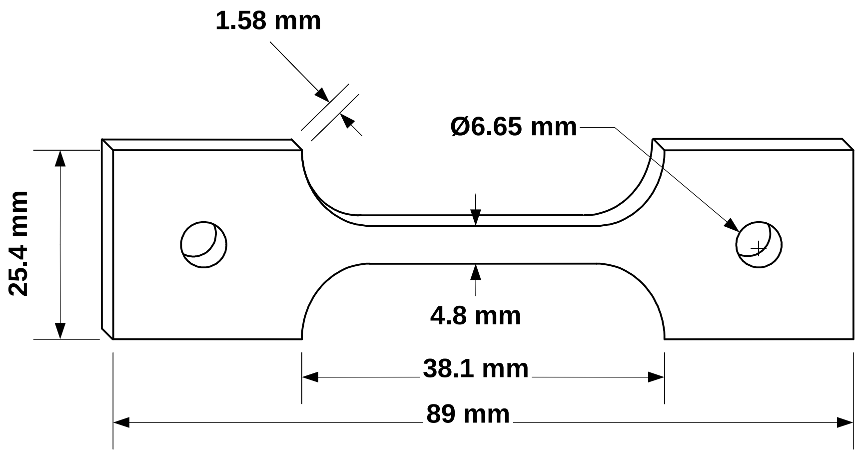

Figure (8) Standard creep test specimen

Figure (9) and **Fig.(10)** show the graph of function Eq.(28) and Eq.(30) respectively.

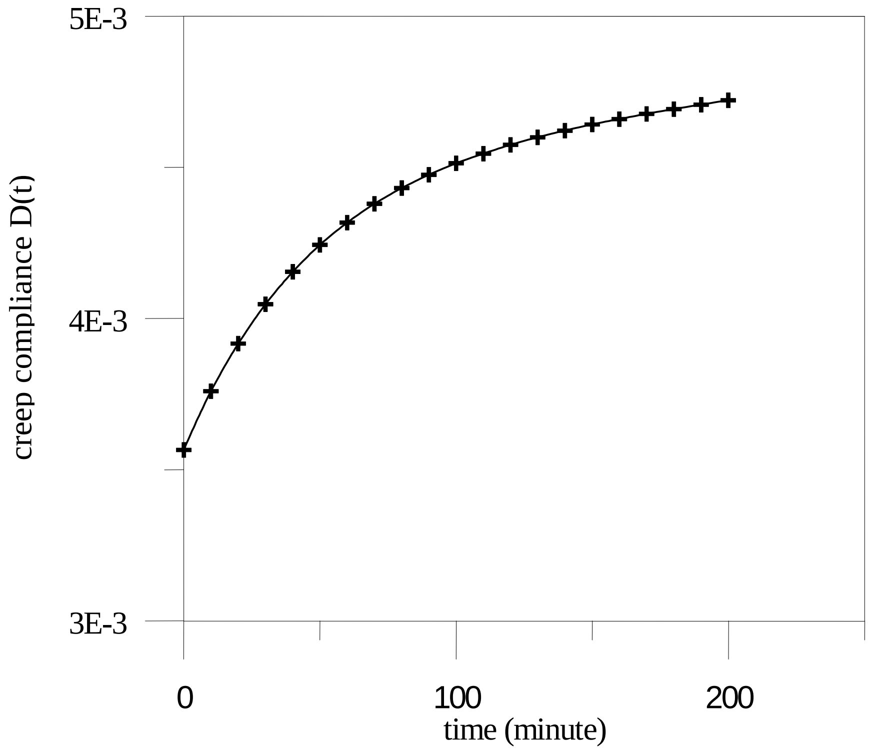

Figure (9) Creep compliance for linear viscoelastic polyester

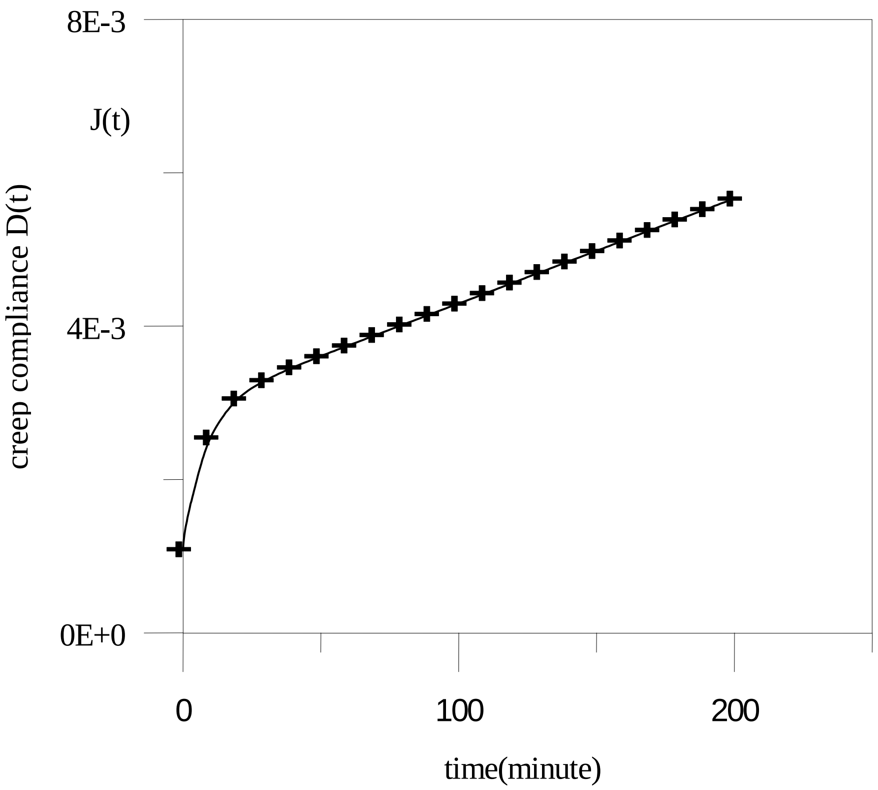

Figure (10) Creep compliance for linear composite viscoelastic polyester

The secondary experimental tests, which include the deflection of thin plate for linear tests, will be compared with both theoretical, finite element proposed models. The equipment shown in **Fig.(11)** has been used to carry the experimental tests for the plates. The plate has been mounted on the stand of the equipment and the distributed load is applied by steel plate from which the value of the distributed load is calculated. For linear behavior the value of (q) must be choose so as to give the condition that the behavior of plate will be as linear (q=8N/mm^2). This value is tested at temperature (T=15°C.). The linear behavior of polyester plate has been done for three different thickness (t=2, 3, 4mm) such that the plate is thin. The first two sides of plate were fixed and the other two sides were simply supported **Fig.(5)**.

The deflection of plate has been record by dial gauge equipment and can be recorded by digital equipment with a strain gauge assembly but this method is cost.

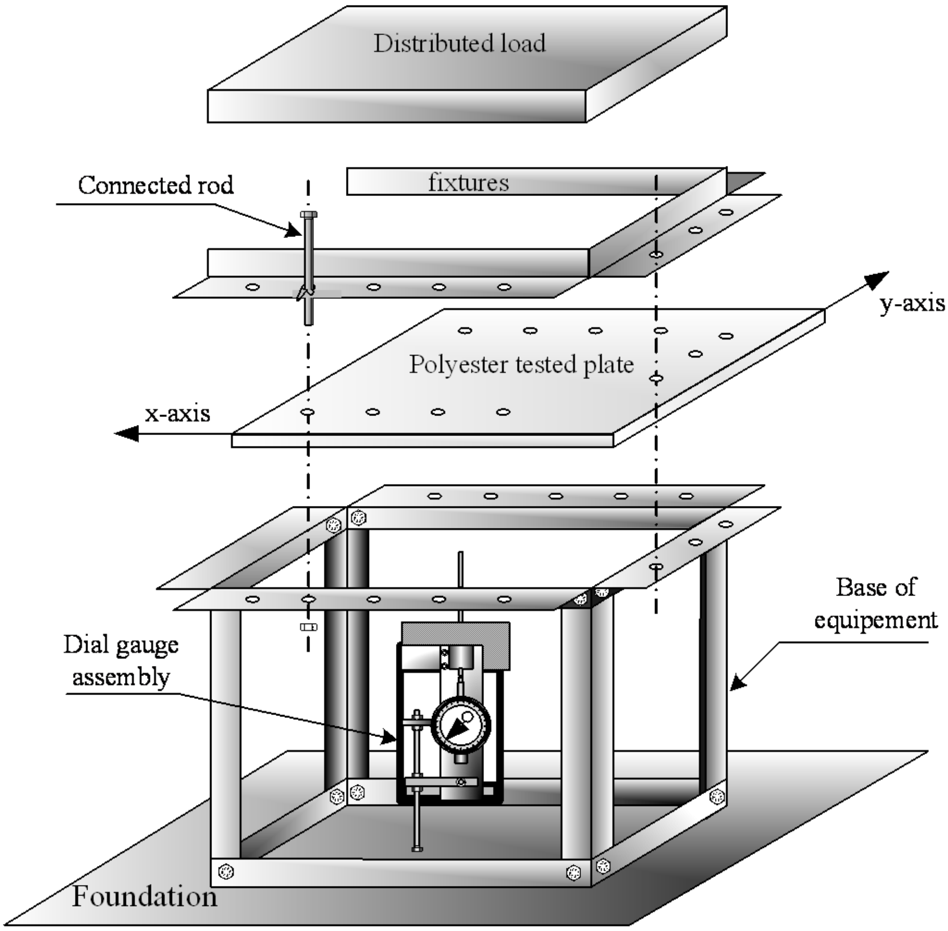

Figure (11) Plate deflection test equipment assembly

7. Result and Discussion

The result of the experimental work will be explained and discuss. The comparison included the thickness variation, variation the plate dimensions. The plate dimension ratio (a/b) has been used here for three values (a/b= 0.5, 0.75, 1).

The central deflection of plates has been analyzed for both polyester and composite polyester thin plates. The results have been shown in **Figs.(12-17)**. The comparison explained three types of thickness so as to ensure the thin condition of plate, say, (t=2,3,4mm). The values of central point of thin plates deflection has been tested for nearly three hours.

The deflection (w) is chosen for a maximum value of variation at the center point of plate (the mid distance of length and width). This is due to the symmetry of the tested plate (two opposite side fixed and the other two side simply supported); hence, the slope of the deflection will be zero at the middle point which give the maximum deflection.

The constant applied distributed load (q) give a constant stresses at each node which results in increasing the strain in the plate (as in creep test for this material). Consequently, the deflection of plate increase with time for each thickness as it has been observed in **Figs.(12-17)**.

A small difference has been observed for the deflection behavior between the FEM and the theoretical analysis with error percentage reaches a range (3-5%). This difference between the calculated data represents the error between the two materials. This error may reduce by increasing the number of elements used for thin plate in the FEM or increasing the number of node for each element which required another element type with a large number of total degree of freedom (more than 12 degree).

Increasing the thickness of plate results in increasing the central deflection for the same time period .This result is due to the fact that the deflection equation (19) depends on the flexural rigidity of plate which depends on the thickness of plate.

The central deflection of plate decrease higher as increasing the thickness from (t=2mm) to (3, 4 mm) and this is due to the fact that the flexural rigidity depend on the cubic value of thickness.

Also increasing thickness will reduce the applied stress and strain in the plane of the plate and consequently the deflection of plate will be reduced.

Higher stress with small thickness (t=2mm) gives a higher deflection rate as comparing with small stress (3, 4 mm).

From **Figs.(12-17)**, comparing the results of experimental central deflection with that of the theoretical and FEM gives good agreements with small error. Normally, this error may presented for this type of equipment of plate deflection measurements and this error may reduce by replace the mechanical dial gauge equipment by a digital equipment to increase the instantaneous response for reading the deflection of plate.

Due to the stiffer creeps compliance for composite material as comparing with that of polyester this results that the composite plate shows a small increasing in the deflection during increasing the plate dimension ratio as comparing with that of polyester.

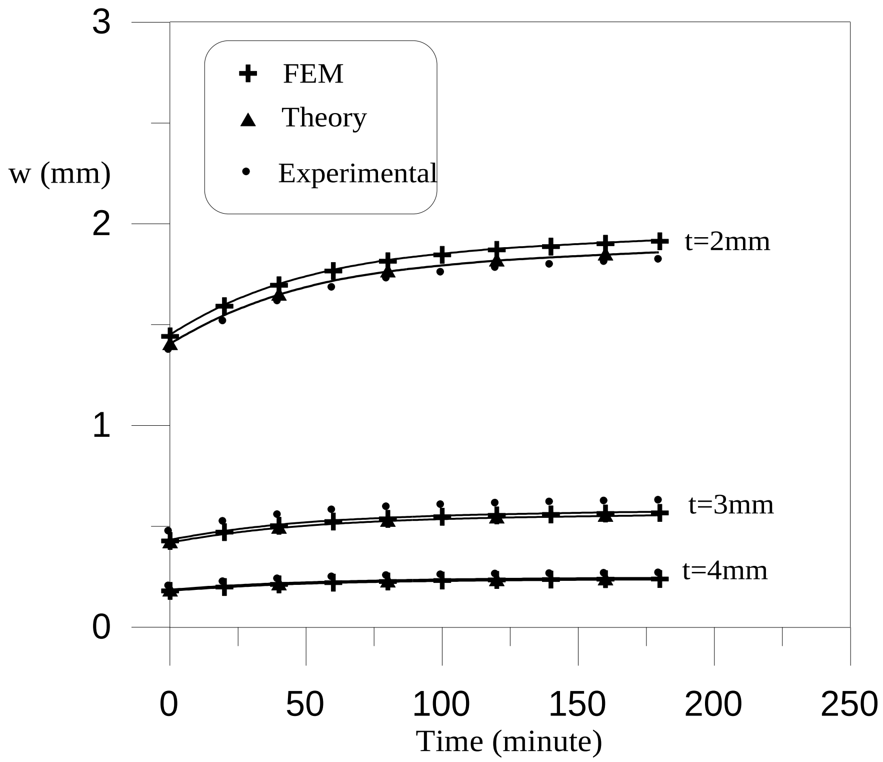

Figure (12) Variation of central point deflection with thickness for linear polyester (q= 8N/mm^2), (a/b)= 0.5

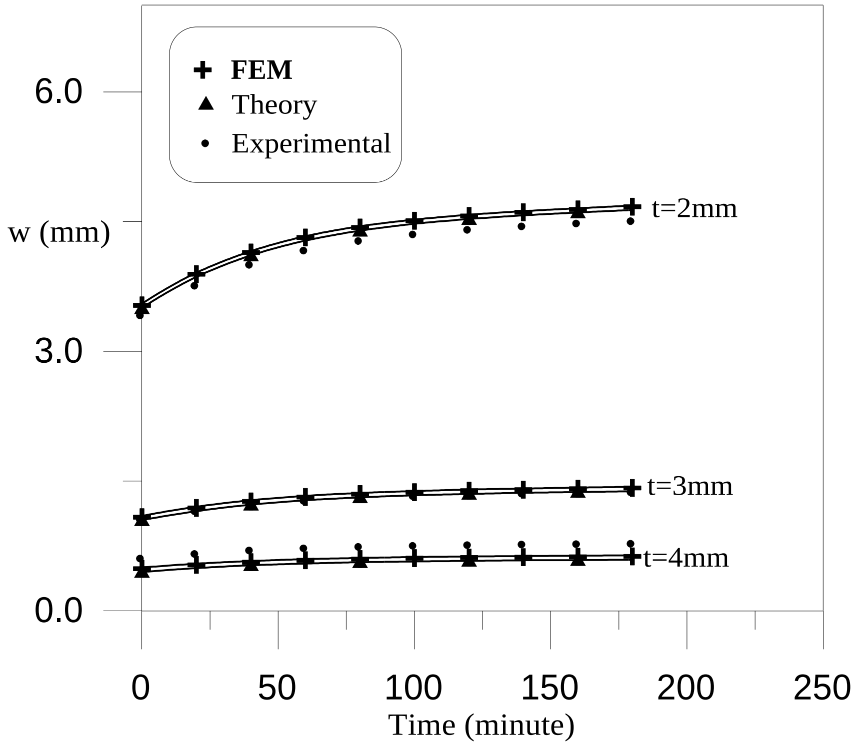

Figure (13) Variation of central point deflection with thickness for linear polyester (q= 8N/mm^2), (a/b)= 0.75

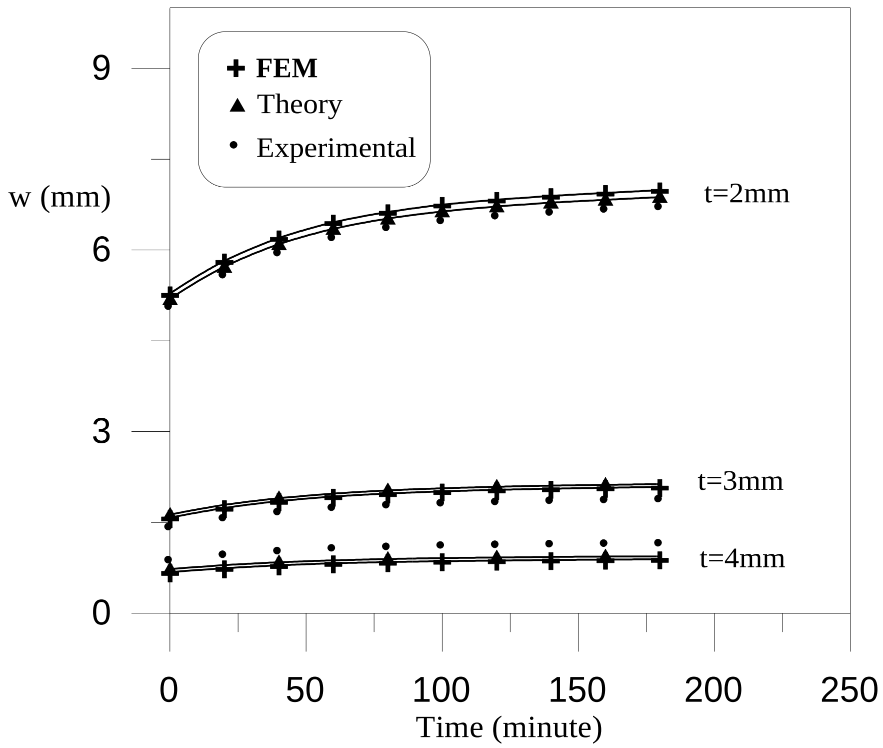

Figure (14) Variation of central point deflection with thickness for linear polyester (q= 8N/mm^2), (a/b)= 1

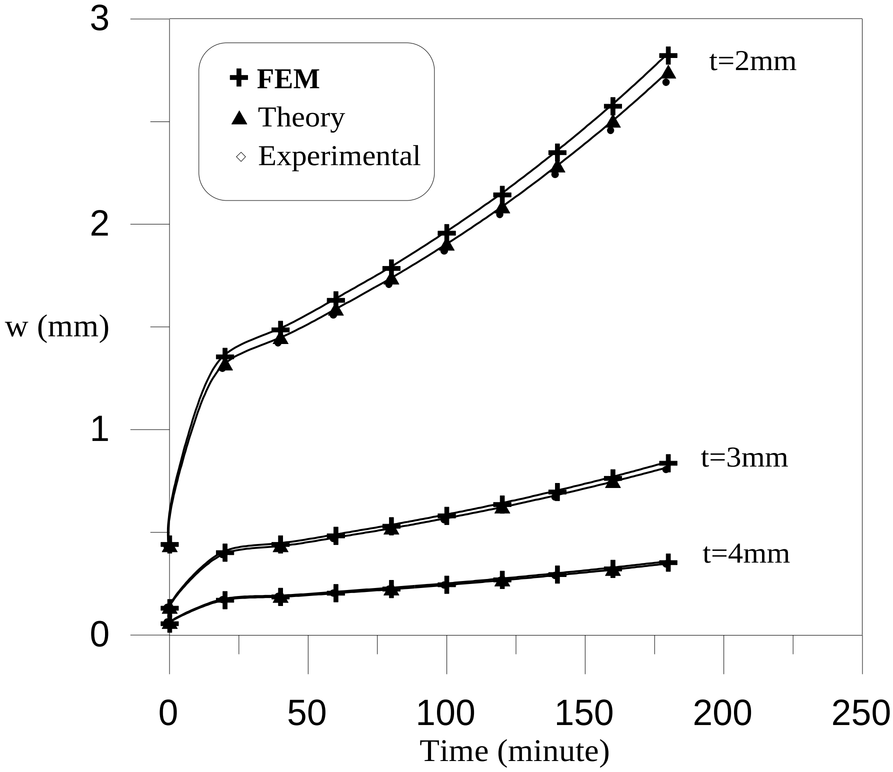

Figure (15) Variation of central point deflection with thickness for linear composite polyester (q= 8N/mm^2), (a/b)= 0.5

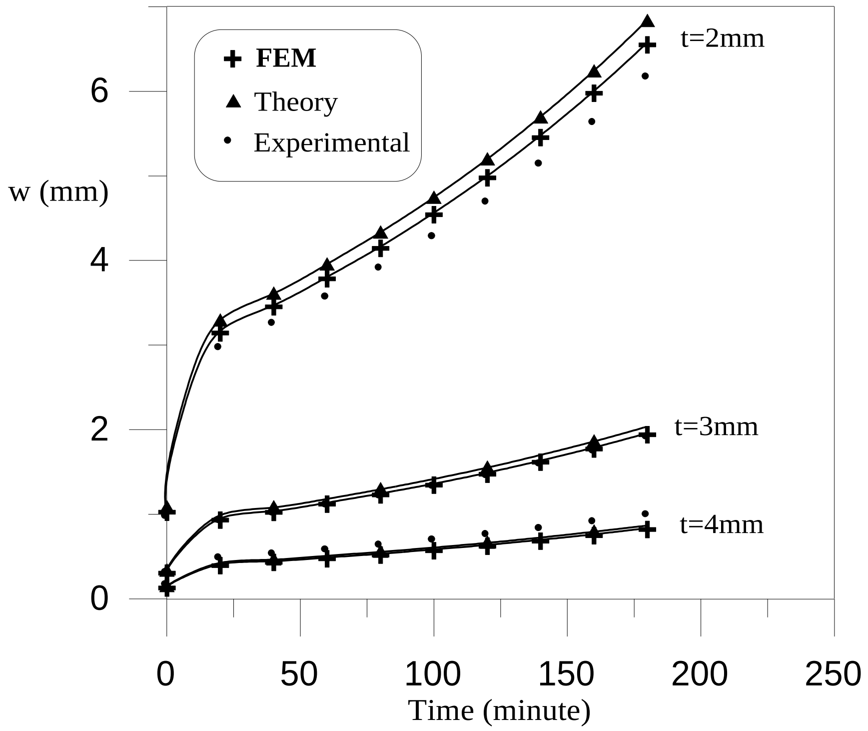

Figure (16) Variation of central point deflection with thickness for linear composite polyester (q= 8N/mm^2), (a/b)= 0.75

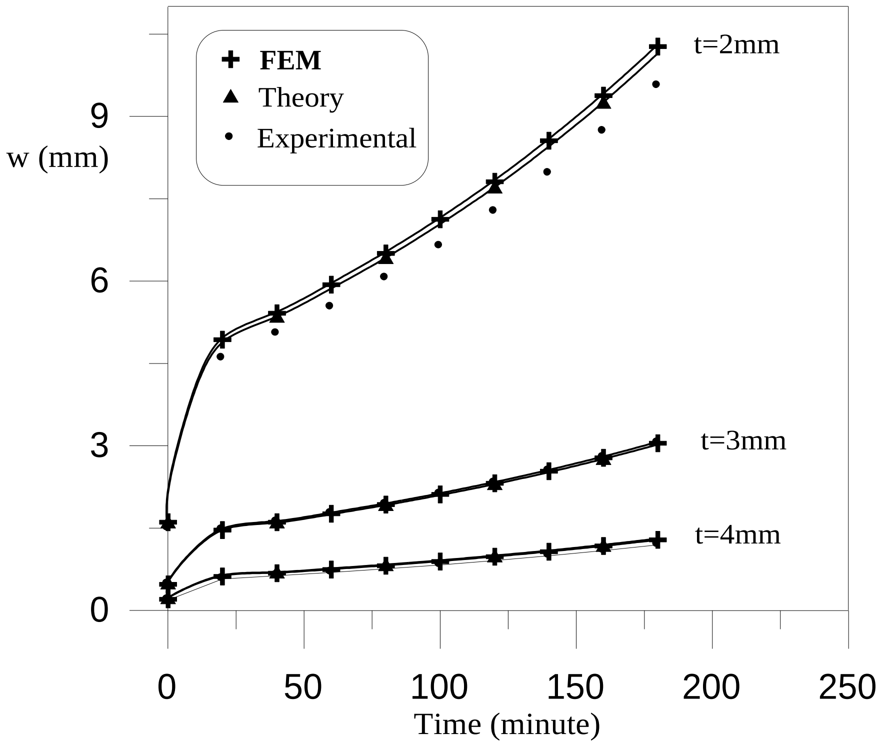

Figure (17) Variation of central point deflection with thickness for linear composite polyester (q= 8N/mm^2), (a/b)= 1

8. Conclusions

From the present work, the following conclusions may be listed:

1. The creep compliance behavior of polyester and composite polyester shows that the viscoelastic material affected by time and hence the strain increase during time.
2. The creep behavior differs as comparing between the polyester and composite polyester.
3. The creep rate for polyester decrease during time and the same behavior has been shown for composite polyester during the first short time.
4. Nearly a constant strain rate was observed for the wide time range.
5. A good accuracy has been shown for FEM, theoretical and experimental work for thin plate polyester and composite polyester. Hence the error percentage reaches a value (3-5%) as comparing between the FEM and the analytical.
6. In general, increasing the plate thickness gives a decreasing in the value of the central plate.
7. The deflection rate decrease during time.
8. A composite plate shows a small increasing in the deflection during increasing the plate dimension ratio as comparing with that of polyester.

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