A New Complex Double Integral Transform and It's Applications in Partial Differential Equations

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Abstract: In this paper, we present a new complex double integral transform namely "Complex Double Sadik Transform", to solve general linear partial differential equations. Several functions are used (applied) to show the usefulness of this new double transformation.

***Keywords*:** Complex Double Sadik Transform, Double Integral, Partial differential Equation, Sadik Integral Transform.

1. Introduction

Partial differential equations are critical in mathematical physics, The heat, wave, Laplace, Poisson, telegraphy and Kelin-Gordan equations are examples of fundamental equations in mathematical physics, that occur in a variety of branches of physcis, including applied mathematics and engineering, [1] .

Lots of researchers have presented many papers in double integral transform and their applications in partial differential equations and their applications in engineering, physics and other fields of life,[1-4],[7].

The researchers Emad A. and Eman A. Mansour presented in 2022 double SEE integral transform,[3]. In addition to, the researchers Saed M. Turq and Emad A. Kuffi presented in 2022 double of the Emad-Falih transformation and its properties with some important applications,[7].

The aim of this paper is to find the solution of linear partial differential equations by using (applying) a new complex double integral transformation (Complex Double Sadik transform).

Definition 1.1. [5] Sadik transform:

The Sadik integral transform of $g(t)$ is defined as:

$$\mathbf{S}_a[g(t)] = \mathbf{F}(v^{\alpha},\beta) = \frac{1}{v^{\beta}}\int_0^{\infty} g(t)e^{-v^{\alpha}t}dt$$

Where v is a complex variable, α is any non zero real number, and β is any real number.

Definition 1.2. [6] The Complex Sadik Transform (CST) denoted by the operator $\mathbf{S}_a^c\{.\}$, the transform form is as follows:

$$\mathbf{S}_a^c[g(t)] = \mathbf{F}^c(s^{\alpha},\beta) = \frac{1}{s^{\beta}}\int_0^{\infty} g(t)e^{-is^{\alpha}t}dt$$

Where s is a complex variable, α is a any nonzero real number, and β is any real number.

Definition 1.3. A New Complex Double Integral Transform denoted by the operator $\mathbf{D}^c\{.\}$, the transform form is as follows:

$$\mathbf{D}^c[f(x,t)] = \mathbf{F}^c(u,v)$$
$$= \frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty} f(x,t)e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx.$$

Where u,v are a complex variables, α is any non zero real number, and β is any real number

2. A New Complex Double Integral Transform Properties

In this Section, we introduce the new Complex Double Integral Transform of some famous functions:

1 Let $f(x,t)=1$, then

$$\begin{aligned}\mathbf{D}^c[1] &= \frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty} 1e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx\\ &= \frac{1}{u^{\beta}}\int_0^{\infty} 1e^{-iu^{\alpha}x}dx\cdot\frac{1}{v^{\beta}}\int_0^{\infty} 1e^{-iv^{\alpha}t}dt\\ &= \mathbf{S}_a^c[1]\cdot\mathbf{S}_a^c[1]\\ &= \frac{-i}{u^{\alpha+\beta}}\cdot\frac{-i}{v^{\alpha+\beta}}\\ &= \frac{-1}{(uv)^{\alpha+\beta}}\end{aligned}$$

2 Let $f(x,t)=x^n t^m$, where n,m are positive integer numbers, then

$$\begin{aligned}\mathbf{D}^c[x^n t^m] &= \frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty} x^n t^m e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx\\ &= \frac{1}{u^{\beta}}\int_0^{\infty} x^n e^{-iu^{\alpha}x}dx\cdot\frac{1}{v^{\beta}}\int_0^{\infty} t^m e^{-iv^{\alpha}t}dt\\ &= \mathbf{S}_a^c[x^n]\cdot\mathbf{S}_a^c[t^n]\\ &= (-i)^{n+1}\frac{n!}{u^{n\alpha+(\alpha+\beta)}}\cdot(-i)^{m+1}\frac{m!}{v^{m\alpha+(\alpha+\beta)}}\\ &= \frac{(-i)^{n+m+2}n!\,m!}{u^{n\alpha+(\alpha+\beta)}\,.\,v^{m\alpha+(\alpha+\beta)}}\end{aligned}$$

3 Let $f(x,t)=e^{ax+bt}$, where a,b are real numbers,then

$$\begin{aligned}\mathbf{D}^c\left[e^{ax+bt}\right] &= \frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty} e^{ax+bt}e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx,\\ &= \frac{1}{u^{\beta}}\int_0^{\infty} e^{ax}e^{-iu^{\alpha}x}dx\cdot\frac{1}{v^{\beta}}\int_0^{\infty} e^{bt}e^{-iv^{\alpha}t}dt\\ &= \mathbf{S}_a^c[e^{ax}]\cdot\mathbf{S}_a^c[e^{bt}],\\ &= \frac{-1}{u^{\beta}}\left[\frac{a}{(u^{2\alpha}+a^2)}+i\frac{u^{\alpha}}{(u^{2\alpha}+a^2)}\right]\cdot\frac{-1}{v^{\beta}}\left[\frac{b}{(v^{2\alpha}+b^2)}+i\frac{v^{\alpha}}{(v^{2\alpha}+b^2)}\right],\\ &= \frac{1}{(uv)^{\beta}}\left[\frac{(a+iu^{\alpha})(b+iv^{\alpha})}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right].\end{aligned}$$

4 Let $f(x,t)=e^{-(ax+bt)}$, then

$$\begin{aligned}\mathbf{D}^c[e^{-(ax+bt)}] &= \frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty} e^{-(ax+bt)}e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx\\ &= \frac{1}{u^{\beta}}\int_0^{\infty} e^{-ax}e^{-iu^{\alpha}x}dx\cdot\frac{1}{v^{\beta}}\int_0^{\infty} e^{-bt}e^{-iv^{\alpha}t}dt\\ &= \mathbf{S}_a^c[e^{-ax}]\cdot\mathbf{S}_a^c[e^{-bt}]\\ &= \frac{-1}{u^{\beta}}\left[\frac{-a}{(u^{2\alpha}+(-a)^2)}+i\frac{u^{\alpha}}{(u^{2\alpha}+(-a)^2)}\right]\cdot\frac{-1}{v^{\beta}}\left[\frac{-b}{(v^{2\alpha}+(-b)^2)}+i\frac{v^{\alpha}}{(v^{2\alpha}+(-b)^2)}\right]\\ &= \frac{1}{(uv)^{\beta}}\left[\frac{(a-iu^{\alpha})(b-iv^{\alpha})}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]\end{aligned}$$

5 Let $f(x,t)=e^{i(ax+bt)}$, then

$$\begin{aligned}
D^c\left[e^{i(ax+bt)}\right] \quad & \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty e^{i(ax+bt)}e^{-i(u^\alpha x+v^\alpha t)}dtdx \\
&= \frac{1}{u^\beta}\int_0^\infty e^{-ix(u^\alpha-a)}dx\cdot\frac{1}{v^\beta}\int_0^\infty e^{-it(v^\alpha-b)}dt \\
&= \frac{1}{u^\beta}\frac{1}{-i(u^\alpha-a)}e^{-ix(u^\alpha-a)}\Big|_0^\infty\frac{1}{v^\beta}\frac{1}{-i(v^\alpha-b)}e^{-it(v^\alpha-b)}\Big|_0^\infty \\
&= \frac{1}{u^\beta}\frac{1}{-i(u^\alpha-a)}[0-1]\frac{1}{v^\beta}\frac{1}{-i(v^\alpha-b)}[0-1] \\
&= \frac{-i}{u^\beta(u^\alpha-a)}\frac{-i}{v^\beta(v^\alpha-b)} \\
&= \frac{-1}{(uv)^\beta(u^\alpha-a)(v^\alpha-b)}
\end{aligned}$$

6 Let $f(x,t)=e^{-i(ax+bt)}$, then

$$\begin{aligned}
D^c\left[e^{-i(ax+bt)}\right] &= \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty e^{-i(ax+bt)}e^{-i(u^\alpha x+v^\alpha t)}dtdx, \\
&= \frac{1}{u^\beta}\int_0^\infty e^{-ix(u^\alpha+a)}dx\cdot\frac{1}{v^\beta}\int_0^\infty e^{-it(v^\alpha+b)}dt, \\
&= \frac{1}{u^\beta}\frac{1}{-i(u^\alpha+a)}e^{-ix(u^\alpha+a)}\Big|_0^\infty\frac{1}{v^\beta}\frac{1}{-i(v^\alpha+b)}e^{-it(v^\alpha+b)}\Big|_0^\infty, \\
&= \frac{1}{u^\beta}\frac{1}{-i(u^\alpha+a)}[0-1]\frac{1}{v^\beta}\frac{1}{-i(v^\alpha+b)}[0-1], \\
&= \frac{-i}{u^\beta(u^\alpha+a)}\frac{-i}{v^\beta(v^\alpha+b)}, \\
&= \frac{-1}{(uv)^\beta(u^\alpha+a)(v^\alpha+b)}.
\end{aligned}$$

7 Let $f(x,t)=\sin(ax+bt)$, then

$$\begin{aligned}
D^c[\sin(ax+bt)] &= \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \sin(ax+bt)e^{-i(u^\alpha x+v^\alpha t)}dtdx \\
&, \text{we know } \sin(ax+bt)=\frac{e^{i(ax+bt)}-e^{-i(ax+bt)}}{2i} \\
&= \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\left[\frac{e^{i(ax+bt)}-e^{-i(ax+bt)}}{2i}\right]e^{-i(u^\alpha x+v^\alpha t)}dtdx \\
&= \frac{1}{2i}\left[D^c\left[e^{i(ax+bt)}\right]-D^c\left[e^{-i(ax+bt)}\right]\right], \\
&= \frac{1}{2i}\left[\frac{-1}{(uv)^\beta(u^\alpha-a)(v^\alpha-b)}+\frac{-1}{(uv)^\beta(u^\alpha+a)(v^\alpha+b)}\right] \\
&= \frac{1}{2i}\left[\frac{-1(u^\alpha+a)(v^\alpha+b)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}+\frac{-1(u^\alpha-a)(v^\alpha-b)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}\right] \\
&= i\left[\frac{av^\alpha+bu^\alpha}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}\right]
\end{aligned}$$

8 Let $f(x,t)=\cos(ax+bt)$, then

$$\begin{aligned}
D^c[\cos(ax+bt)] &= \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \cos(ax+bt)e^{-i(u^\alpha x+v^\alpha t)}dtdx \\
&, \text{we know } \cos(ax+bt)=\frac{e^{i(ax+bt)}+e^{-i(ax+bt)}}{2} \\
&= \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\left[\frac{e^{i(ax+bt)}+e^{-i(ax+bt)}}{2}\right]e^{-i(u^\alpha x+v^\alpha t)}dtdx \\
&= \frac{1}{2}\left[D^c\left[e^{i(ax+bt)}\right]+D^c\left[e^{-i(ax+bt)}\right]\right] \\
&= \frac{1}{2}\left[\frac{-1}{(uv)^\beta(u^\alpha-a)(v^\alpha-b)}+\frac{-1}{(uv)^\beta(u^\alpha+a)(v^\alpha+b)}\right] \\
&= \frac{1}{2}\left[\frac{-1(u^\alpha+a)(v^\alpha+b)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}+\frac{-1(u^\alpha-a)(v^\alpha-b)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}\right] \\
&= \frac{-(u^\alpha v^\alpha+ab)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}
\end{aligned}$$

9 Let $f(x,t)=\sinh(ax+bt)$, then

$$\begin{aligned}
D^c[\sinh(ax+bt)] &= \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \sinh(ax+bt)e^{-i(u^\alpha x+v^\alpha t)}dtdx \\
&= \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\left[\frac{e^{(ax+bt)}-e^{-(ax+bt)}}{2}\right]e^{-i(u^\alpha x+v^\alpha t)}dtdx \\
&= \frac{1}{2}\left[D^c\left[e^{(ax+bt)}\right]-D^c\left[e^{-(ax+bt)}\right]\right] \\
&= \frac{1}{2}\left[\begin{array}{l}\frac{1}{(uv)^\beta}\left[\frac{(a+iu^\alpha)(b+iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]- \\ \frac{1}{(uv)^\beta}\left[\frac{(a-iu^\alpha)(b-iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]\end{array}\right] \\
&= i\left[\frac{av^\alpha+bu^\alpha}{(uv)^\beta(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]
\end{aligned}$$

10 Let $f(x,t)=\cosh(ax+bt)$, then

$$\begin{aligned}
D^c[\cosh(ax+bt)] &= \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \cosh(ax+bt)e^{-i(u^\alpha x+v^\alpha t)}dtdx \\
&, \text{we know } \cosh(ax+bt)=\frac{e^{(ax+bt)}+e^{-(ax+bt)}}{2} \\
&= \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty\left[\frac{e^{(ax+bt)}+e^{-(ax+bt)}}{2}\right]e^{-i(u^\alpha x+v^\alpha t)}dtdx \\
&= \frac{1}{2}\left[D^c\left[e^{(ax+bt)}\right]+D^c\left[e^{-(ax+bt)}\right]\right] \\
&= \frac{1}{2}\left[\begin{array}{l}\frac{1}{(uv)^\beta}\left[\frac{(a+iu^\alpha)(b+iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]+ \\ \frac{1}{(uv)^\beta}\left[\frac{(a-iu^\alpha)(b-iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]\end{array}\right] \\
&= \frac{ab-u^\alpha v^\alpha}{(uv)^\beta(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}
\end{aligned}$$

3. Summarization

The new complex double integral transform for some basic functions in the following table:

Table 1: The new complex double integral transform for some basic functions

Function	Complex Double Sadik transform
$f(x,t)$	$\mathbf{D}^c[f(x,t)]$
1	$\frac{-1}{(uv)^{\alpha+\beta}}$
$x^n t^m, m, n \in \mathbb{Z}^+$	$\frac{(-i)^{n+m+2} n!\, m!}{u^{n\alpha+(\alpha+\beta)} \cdot v^{m\alpha+(\alpha+\beta)}}$
e^{ax+bt}	$\frac{1}{(uv)^\beta}\left[\frac{(a+iu^\alpha)(b+iv^\alpha)}{(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]$
$\sin(ax+bt)$	$i\left[\frac{av^\alpha+bu^\alpha}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}\right]$
$\cos(ax+bt)$	$\frac{-(u^\alpha v^\alpha+ab)}{(uv)^\beta(u^{2\alpha}-a^2)(v^{2\alpha}-b^2)}$
$\sinh(ax+bt)$	$i\left[\frac{av^\alpha+bu^\alpha}{(uv)^\beta(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}\right]$
$\cosh(ax+bt)$	$\frac{ab-u^\alpha v^\alpha}{(uv)^\beta(u^{2\alpha}+a^2)(v^{2\alpha}+b^2)}$

4. Theorem and proof

$$\mathrm{D}^c\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \frac{\partial f(x,t)}{\partial x} e^{-i(u^\alpha x+v^\alpha t)}dtdx,$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}[\frac{1}{(u)^\beta}\underbrace{\int_0^\infty \frac{\partial f(x,t)}{\partial x} e^{-iu^\alpha x}dx}_{\text{Integration by parts}}]dt,$$

Let $\zeta = e^{-iu^\alpha x}$ $d\eta = \frac{\partial f(x,t)}{\partial x}dx$

$d\zeta = -iu^\alpha e^{-iu^\alpha x}dx$ $\eta = f(x,t)$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\left[\frac{1}{(u)^\beta}\left(\begin{array}{c} e^{-iu^\alpha x}f(x,t)\Big|_0^\infty + \\ iu^\alpha \int_0^\infty f(x,t)e^{-iu^\alpha x}dx \end{array}\right)\right]dt$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\left[-\frac{f(0,t)}{u^\beta} + \frac{iu^\alpha}{u^\beta}\int_0^\infty f(x,t)e^{-iu^\alpha x}dx\right]dt$$

$$= -\frac{1}{u^\beta}\left[\frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}f(0,t)dt\right] + iu^\alpha\left[\frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty f(x,t)e^{-i(u^\alpha x+v^\alpha t)}dtdx\right],$$

$$= -\frac{1}{u^\beta}\mathrm{F}^c(0,v) + iu^\alpha \mathrm{F}^c(u,v).$$

$$\mathrm{D}^c\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{1}{(uv)^\beta}\int_0^\infty\int_0^\infty \frac{\partial^2 f(x,t)}{\partial x^2} e^{-i(u^\alpha x+v^\alpha t)}dtdx,$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}[\frac{1}{(u)^\beta}\underbrace{\int_0^\infty \frac{\partial^2 f(x,t)}{\partial x^2} e^{-iu^\alpha x}dx}_{\text{Integration by parts}}]dt,$$

Let $\zeta = e^{-iu^\alpha x}$ $d\eta = \frac{\partial^2 f(x,t)}{\partial x^2}dx$

$d\zeta = -iu^\alpha e^{-iu^\alpha x}dx$ $\eta = \frac{\partial f(x,t)}{\partial x}$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\left[\frac{1}{(u)^\beta}\left(e^{-iu^\alpha x}\frac{\partial f(x,t)}{\partial x}\Big|_0^\infty + iu^\alpha\int_0^\infty \frac{\partial f(x,t)}{\partial x}e^{-iu^\alpha x}dx\right)\right]dt,$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\left[\frac{1}{(u)^\beta}e^{-iu^\alpha x}\frac{\partial f(x,t)}{\partial x}\Big|_0^\infty + \frac{iu^\alpha}{u^\beta}\int_0^\infty \frac{\partial f(x,t)}{\partial x}e^{-iu^\alpha x}dx\right]dt,$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}[-\frac{1}{(u)^\beta}\frac{\partial f(0,t)}{\partial x} + \frac{iu^\alpha}{u^\beta}\underbrace{\int_0^\infty \frac{\partial f(x,t)}{\partial x}e^{-iu^\alpha x}dx}_{\text{Integration by parts}}]dt,$$

Let $\zeta = e^{-iu^{\alpha}x}\ d\eta = \dfrac{\partial f(x,t)}{\partial x}dx$

$d\zeta = -iu^{\alpha}e^{-iu^{\alpha}x}dx\ \eta = f(x,t)$

$$= \frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}\left[-\frac{1}{(u)^{\beta}}\frac{\partial f(0,t)}{\partial x}\right.$$

$$\left.+\frac{iu^{\alpha}}{u^{\beta}}\left(e^{-iu^{\alpha}x}f(x,t)\Big|_0^{\infty} + iu^{\alpha}\int_0^{\infty} f(x,t)e^{-iu^{\alpha}x}dx\right)\right]dt$$

$$= \frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}\left[-\frac{1}{(u)^{\beta}}\frac{\partial f(0,t)}{\partial x}\right.$$

$$\left.+\frac{iu^{\alpha}}{u^{\beta}}\left(-f(0,t) + iu^{\alpha}\int_0^{\infty} f(x,t)e^{-iu^{\alpha}x}dx\right)\right]dt,$$

$$= \frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}\left[-\frac{1}{(u)^{\beta}}\frac{\partial f(0,t)}{\partial x}\right.$$

$$\left.-\frac{iu^{\alpha}}{u^{\beta}}f(0,t) + \frac{(iu^{\alpha})^2}{u^{\beta}}\int_0^{\infty} f(x,t)e^{-iu^{\alpha}x}dx\right]dt,$$

$$= -\frac{1}{(u)^{\beta}}\left[\frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}\frac{\partial f(0,t)}{\partial x}\right] - \frac{iu^{\alpha}}{u^{\beta}}\left[\frac{1}{(v)^{\beta}}\int_0^{\infty} e^{-iv^{\alpha}t}f(0,t)\right]$$

$$+(iu^{\alpha})^2\left[\frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty} f(x,t)e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx\right],$$

$$= -\frac{1}{u^{\beta}}\frac{\partial \mathrm{F}^c(0,v)}{\partial x} - \frac{iu^{\alpha}}{u^{\beta}}\mathrm{F}^c(0,v) + (iu^{\alpha})^2\mathrm{F}^c(u,v).$$

in general

$$let\ ab = \frac{\partial^{n-1}\mathrm{F}^c(0,v)}{\partial x^{n-1}} + iu^{\alpha}\frac{\partial^{n-2}\mathrm{F}^c(0,v)}{\partial x^{n-2}} + (iu^{\alpha})^2\frac{\partial^{n-3}\mathrm{F}^c(0,v)}{\partial x^{n-3}}$$

$$\mathrm{D}^c\left[\frac{\partial^n f(x,t)}{\partial x^n}\right] = (iu^{\alpha})^n\mathrm{F}^c(u,v) - \frac{1}{u^{\beta}}\left[\mathrm{ab} + \cdots + (iu^{\alpha})^{n-2}\frac{\partial \mathrm{F}^c(0,v)}{\partial x} + (iu^{\alpha})^{n-1}\mathrm{F}^c(0,v)\right].$$

or $$\mathrm{D}^c\left[\frac{\partial^n f(x,t)}{\partial x^n}\right] = (iu^{\alpha})^n\mathrm{F}^c(u,v) - \frac{1}{u^{\beta}}\left[\sum_{k=1}^{n}(iu^{\alpha})^{k-1}\frac{\partial^{n-k}\mathrm{F}^c(0,v)}{\partial x^{n-k}}\right]$$

$$\mathrm{D}^c\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty}\frac{\partial f(x,t)}{\partial t}e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx,$$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[\frac{1}{(v)^{\beta}}\int_0^{\infty}\frac{\partial f(x,t)}{\partial t}e^{-iv^{\alpha}t}dt\right]dx$$

Let $\zeta = e^{-iv^{\alpha}t}d\eta = \dfrac{df(x,t)}{dt}dt$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[\frac{1}{(v)^{\beta}}\left(e^{-iv^{\alpha}t}f(x,t)\Big|_0^{\infty} + iv^{\alpha}\int_0^{\infty} f(x,t)e^{-iv^{\alpha}t}dt\right)\right]dx$$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[-\frac{f(x,0)}{v^{\beta}} + \frac{iv^{\alpha}}{v^{\beta}}\int_0^{\infty} f(x,t)e^{-iv^{\alpha}t}dt\right]dx$$

$$= -\frac{1}{v^{\beta}}\left[\frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}f(x,0)dx\right] + iv^{\alpha}\left[\frac{1}{(uv)^{\beta}}\int_0^{\infty}\int e_0^{\infty}f(x,t)e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx\right]$$

$$= -\frac{1}{v^{\beta}}\mathrm{F}^c(u,0) + iv^{\alpha}\mathrm{F}^c(u,v)$$

$$\mathrm{D}^c\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty}\frac{\partial^2 f(x,t)}{\partial t^2}e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx$$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[\frac{1}{(v)^{\beta}}\underbrace{\int_0^{\infty}\frac{\partial^2 f(x,t)}{\partial t^2}e^{-iv^{\alpha}t}dt}_{\text{Integration by parts}}\right]dx,$$

Let $\zeta = e^{-iv^{\alpha}t}\ d\eta = \dfrac{\partial^2 f(x,t)}{\partial t^2}dt$

$d\zeta = -iv^{\alpha}e^{-iv^{\alpha}t}dt\ \eta = \dfrac{\partial f(x,t)}{\partial t}$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[\frac{1}{(v)^{\beta}}\left(e^{-iv^{\alpha}t}\frac{\partial f(x,t)}{\partial t}\Big|_0^{\infty} + iv^{\alpha}\int_0^{\infty}\frac{\partial f(x,t)}{\partial t}e^{-iv^{\alpha}t}dt\right)\right]dx,$$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[\frac{1}{(v)^{\beta}}e^{-iv^{\alpha}t}\frac{\partial f(x,t)}{\partial t}\Big|_0^{\infty} + \frac{iv^{\alpha}}{v^{\beta}}\int_0^{\infty}\frac{\partial f(x,t)}{\partial t}e^{-iv^{\alpha}t}dt\right]dx,$$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[-\frac{1}{(v)^{\beta}}\frac{\partial f(x,0)}{\partial t} + \frac{iv^{\alpha}}{v^{\beta}}\underbrace{\int_0^{\infty}\frac{\partial f(x,t)}{\partial t}e^{-iv^{\alpha}t}dt}_{\text{Integration by parts}}\right]dx,$$

Let $\zeta = e^{-iv^{\alpha}t}\ d\eta = \dfrac{\partial f(x,t)}{\partial t}dt$

$d\zeta = -iv^{\alpha}e^{-iv^{\alpha}t}dt\ \eta = f(x,t)$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[-\frac{1}{(v)^{\beta}}\frac{\partial f(x,0)}{\partial t}\right.$$

$$\left.+\frac{iv^{\alpha}}{v^{\beta}}\left(e^{-iv^{\alpha}t}f(x,t)\Big|_0^{\infty} + iv^{\alpha}\int_0^{\infty} f(x,t)e^{-iv^{\alpha}t}dt\right)\right]dx$$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[-\frac{1}{(v)^{\beta}}\frac{\partial f(x,0)}{\partial t}\right.$$

$$\left.+\frac{iv^{\alpha}}{v^{\beta}}\left(-f(x,0) + iv^{\alpha}\int_0^{\infty} f(x,t)e^{-iv^{\alpha}t}dt\right)\right]dx,$$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[-\frac{1}{(v)^{\beta}}\frac{\partial f(x,0)}{\partial t}\right.$$

$$\left.-\frac{iv^{\alpha}}{v^{\beta}}f(x,0) + \frac{(iv^{\alpha})^2}{v^{\beta}}\int_0^{\infty} f(x,t)e^{-iv^{\alpha}t}dt\right]dx,$$

$$= -\frac{1}{(v)^{\beta}}\left[\frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\frac{\partial f(x,0)}{\partial t}\right] - \frac{iv^{\alpha}}{v^{\beta}}\left[\frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}f(x,0)\right]$$

$$+(iv^{\alpha})^2\left[\frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty} f(x,t)e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx\right],$$

$$= -\frac{1}{v^{\beta}}\frac{\partial \mathrm{F}^c(u,0)}{\partial t} - \frac{iv^{\alpha}}{v^{\beta}}\mathrm{F}^c(u,0) + (iv^{\alpha})^2\mathrm{F}^c(u,v).$$

$$\mathrm{D}^c\left[\frac{\partial^2 f(x,t)}{\partial t\partial x}\right] = \frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty}\frac{\partial^2 f(x,t)}{\partial t\partial x}e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx$$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[\frac{1}{(v)^{\beta}}\underbrace{\int_0^{\infty}\frac{\partial^2 f(x,t)}{\partial t\partial x}e^{-iv^{\alpha}t}dt}_{\text{Integration by parts}}\right]dx,$$

Let $\zeta = e^{-iv^{\alpha}t}\ d\eta = \dfrac{\partial^2 f(x,t)}{\partial t\partial x}dt$

$d\zeta = -iv^{\alpha}e^{-iv^{\alpha}t}dt\ \eta = \dfrac{\partial f(x,t)}{\partial x}$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[\frac{1}{(v)^{\beta}}\left(e^{-iv^{\alpha}t}\frac{\partial f(x,t)}{\partial x}\Big|_0^{\infty} + iv^{\alpha}\int_0^{\infty}\frac{\partial f(x,t)}{\partial x}e^{-iv^{\alpha}t}dt\right)\right]dx,$$

$$= \frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\left[\frac{1}{(v)^{\beta}}e^{-iv^{\alpha}t}\frac{\partial f(x,t)}{\partial x}\Big|_0^{\infty} + \frac{iv^{\alpha}}{v^{\beta}}\int_0^{\infty}\frac{\partial f(x,t)}{\partial x}e^{-iv^{\alpha}t}dt\right]dx,$$

$$= -\frac{1}{v^{\beta}}\left[\frac{1}{(u)^{\beta}}\int_0^{\infty} e^{-iu^{\alpha}x}\frac{\partial f(x,0)}{\partial x}\right] + iv^{\alpha}\left[\frac{1}{(uv)^{\beta}}\int_0^{\infty}\int_0^{\infty}\frac{\partial f(x,t)}{\partial x}e^{-i(u^{\alpha}x+v^{\alpha}t)}dtdx\right],$$

$$= -\frac{1}{v^{\beta}}\frac{\partial \mathrm{F}^c(u,0)}{\partial x} + iv^{\alpha}\left[-\frac{1}{u^{\beta}}\mathrm{F}^c(0,v) + iu^{\alpha}\mathrm{F}^c(u,v)\right],$$

$$= -\frac{1}{v^{\beta}}\frac{\partial \mathrm{F}^c(u,0)}{\partial x} - \frac{iv^{\alpha}}{u^{\beta}}\mathrm{F}^c(0,v) + i^2u^{\alpha}v^{\alpha}\mathrm{F}^c(u,v)$$

In general

$$\mathbf{D}^c\left[\frac{\partial^n f(x,t)}{\partial t^n}\right] = (iv^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{v^\beta}\left[\frac{\partial^{n-1}\mathbf{F}^c(u,0)}{\partial t^{n-1}} + iv^\alpha \frac{\partial^{n-2}\mathbf{F}^c(u,0)}{\partial t^{n-2}} + (iv^\alpha)^2 \frac{\partial^{n-3}\mathbf{F}^c(u,0)}{\partial t^{n-3}} + \cdots + (iv^\alpha)^{n-2}\frac{\partial \mathbf{F}^c(u,0)}{\partial t} + (iv^\alpha)^{n-1}\mathbf{F}^c(u,0)\right]$$

$$\text{or } = (iv^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{v^\beta}\left[\sum_{k=1}^{n} (iv^\alpha)^{k-1}\frac{\partial^{n-k}\mathbf{F}^c(u,0)}{\partial t^{n-k}}\right]$$

$$\mathbf{D}^c\left[\frac{\partial^2 f(x,t)}{\partial x\partial t}\right] = \frac{1}{(uv)^\beta}\int_0^\infty \int_0^\infty \frac{\partial^2 f(x,t)}{\partial x\partial t} e^{-i(u^\alpha x+v^\alpha t)} dt dx,$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}[\frac{1}{(u)^\beta}\underbrace{\int_0^\infty \frac{\partial^2 f(x,t)}{\partial x\partial t} e^{-iu^\alpha x} dx}_{\text{Integration by parts}}]dt,$$

Let $\zeta = e^{-iu^\alpha x}\ d\eta = \frac{\partial^2 f(x,t)}{\partial x\partial t}dx$

$d\zeta = -iu^\alpha e^{-iu^\alpha x}dx\ \eta = \frac{\partial f(x,t)}{\partial t}$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\left[\frac{1}{(u)^\beta}\left(e^{-iu^\alpha x}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty + iu^\alpha \int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-iu^\alpha x} dx\right)\right] dt,$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\left[\frac{1}{(u)^\beta} e^{-iu^\alpha x}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty + \frac{iu^\alpha}{u^\beta}\int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-iu^\alpha x} dx\right] dt,$$

$$= \frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\left[-\frac{1}{(u)^\beta}\frac{\partial f(0,t)}{\partial t} + \frac{iu^\alpha}{u^\beta}\int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-iu^\alpha x} dx\right] dt$$

$$= -\frac{1}{u^\beta}\left[\frac{1}{(v)^\beta}\int_0^\infty e^{-iv^\alpha t}\frac{\partial f(0,t)}{\partial t}\right] + iu^\alpha\left[\frac{1}{(uv)^\beta}\int_0^\infty \int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-i(u^\alpha x+v^\alpha t)} dt dx\right],$$

$$= -\frac{1}{u^\beta}\frac{\partial \mathbf{F}^c(0,v)}{\partial t} + iu^\alpha\left[-\frac{1}{v^\beta}\mathbf{F}^c(u,0) + iv^\alpha \mathbf{F}^c(u,v)\right],$$

$$= -\frac{1}{u^\beta}\frac{\partial \mathbf{F}^c(0,v)}{\partial t} - \frac{iu^\alpha}{v^\beta}\mathbf{F}^c(u,0) + i^2 u^\alpha v^\alpha \mathbf{F}^c(u,v).$$

Tabel(2): Summarization

Function	Complex Double Sadik transform
$f(x,t)$	$\mathbf{D}^c[f(x,t)] = \mathbf{F}^c(u,v)$
$\frac{\partial f(x,t)}{\partial x}$	$-\frac{1}{u^\beta}\mathbf{F}^c(0,v) + iu^\alpha \mathbf{F}^c(u,v)$
$\frac{\partial^2 f(x,t)}{\partial x^2}$	$-\frac{1}{u^\beta}\frac{\partial \mathbf{F}^c(0,v)}{\partial x} - \frac{iu^\alpha}{u^\beta}\mathbf{F}^c(0,v) + (iu^\alpha)^2 \mathbf{F}^c(u,v)$
$\frac{\partial^n f(x,t)}{\partial x^n}$	$(iu^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{u^\beta}\left[\sum_{k=1}^n (iu^\alpha)^{k-1}\frac{\partial^{n-k}\mathbf{F}^c(0,v)}{\partial x^{n-k}}\right]$
$\frac{\partial f(x,t)}{\partial t}$	$-\frac{1}{v^\beta}\mathbf{F}^c(u,0) + iv^\alpha \mathbf{F}^c(u,v)$
$\frac{\partial^2 f(x,t)}{\partial t^2}$	$-\frac{1}{v^\beta}\frac{\partial \mathbf{F}^c(u,0)}{\partial t} - \frac{iv^\alpha}{v^\beta}\mathbf{F}^c(u,0) + (iv^\alpha)^2 \mathbf{F}^c(u,v)$
$\frac{\partial^n f(x,t)}{\partial t^n}$	$(iv^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{v^\beta}\left[\sum_{k=1}^n (iv^\alpha)^{k-1}\frac{\partial^{n-k}\mathbf{F}^c(u,0)}{\partial t^{n-k}}\right]$
$\frac{\partial^2 f(x,t)}{\partial t\,\partial x}$	$-\frac{1}{v^\beta}\frac{\partial \mathbf{F}^c(u,0)}{\partial x} - \frac{iv^\alpha}{u^\beta}\mathbf{F}^c(0,v) + i^2 u^\alpha v^\alpha \mathbf{F}^c(u,v)$
$\frac{\partial^2 f(x,t)}{\partial x\,\partial t}$	$-\frac{1}{u^\beta}\frac{\partial \mathbf{F}^c(0,v)}{\partial t} - \frac{iu^\alpha}{v^\beta}\mathbf{F}^c(u,0) + i^2 u^\alpha v^\alpha \mathbf{F}^c(u,v)$

Theorem 4.1. Let $\mathbf{F}^c(u,v)$ is the complex new integral transform of $f(x,t)(\mathbf{F}^c(u,v) = \mathbf{D}^c[f(x,t)])$, then

$$\mathbf{D}^c\left[\frac{\partial^n f(x,t)}{\partial x^n}\right] = (iu^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{u^\beta}\left[\sum_{k=1}^n (iu^\alpha)^{k-1}\frac{\partial^{n-k}\mathbf{F}^c(0,v)}{\partial x^{n-k}}\right] \quad (1)$$

$$\mathbf{D}^c\left[\frac{\partial^n f(x,t)}{\partial t^n}\right] = (iv^\alpha)^n \mathbf{F}^c(u,v) - \frac{1}{v^\beta}\left[\sum_{k=1}^n (iv^\alpha)^{k-1}\frac{\partial^{n-k}\mathbf{F}^c(u,0)}{\partial t^{n-k}}\right] \quad (2)$$

Proof. Firstly, We want prove (1) by mathematical induction

1 for $n = 1$

$$\mathbf{D}^c\left[\frac{\partial f(x,t)}{\partial x}\right] = (iu^\alpha)^1 \mathbf{F}^c(u,v) - \frac{1}{u^\beta}\left[\sum_{k=1}^{1} (iu^\alpha)^{k-1}\frac{\partial^{1-k}\mathbf{F}^c(0,v)}{\partial x^{1-k}}\right]$$

$$= iu^\alpha \mathbf{F}^c(u,v) - \frac{1}{u^\beta}\mathbf{F}^c(0,v)$$

Thus, true for $n = 1$

2 Assume true for $n = m$ that mean

$$\mathbf{D}^c\left[\frac{\partial^m f(x,t)}{\partial x^m}\right] = (iu^\alpha)^m \mathbf{F}^c(u,v) - \frac{1}{u^\beta}\left[\sum_{k=1}^{m} (iu^\alpha)^{k-1}\frac{\partial^{m-k}\mathbf{F}^c(0,v)}{\partial x^{m-k}}\right]$$

3 we want to prove (1) for $n = m + 1$

$$\mathbf{D}^c\left[\frac{\partial^{m+1} f(x,t)}{\partial x^{m+1}}\right]=\mathbf{D}^c\left[\frac{\partial}{\partial x}\left[\frac{\partial^m f(x,t)}{\partial x^m}\right]\right]$$
$$=iu^\alpha\mathbf{D}^c\left[\frac{\partial^m f(x,t)}{\partial x^m}\right]-\frac{1}{u^\beta}\frac{\partial^m \mathbf{F}^c(0,v)}{\partial x^m},$$
$$=iu^\alpha\left[(iu^\alpha)^m\mathbf{F}^c(u,v)-\frac{1}{u^\beta}\left[\sum_{k=1}^{m}(iu^\alpha)^{k-1}\frac{\partial^{m-k}\mathbf{F}^c(0,v)}{\partial x^{m-k}}\right]\right]-\frac{1}{u^\beta}\frac{\partial^m \mathbf{F}^c(0,v)}{\partial x^m},$$
$$=(iu^\alpha)^{m+1}\mathbf{F}^c(u,v)-\frac{1}{u^\beta}\left[\sum_{k=1}^{m}(iu^\alpha)^{k}\frac{\partial^{m-k}\mathbf{F}^c(0,v)}{\partial x^{m-k}}\right]-\frac{1}{u^\beta}\frac{\partial^m \mathbf{F}^c(0,v)}{\partial x^m},$$
$$=(iu^\alpha)^{m+1}\mathbf{F}^c(u,v)-\frac{1}{u^\beta}\left[\sum_{k=1}^{m}(iu^\alpha)^{k}\frac{\partial^{m-k}\mathbf{F}^c(0,v)}{\partial x^{m-k}}+\frac{\partial^m \mathbf{F}^c(0,v)}{\partial x^m}\right],$$
$$=(iu^\alpha)^{m+1}\mathbf{F}^c(u,v)-\frac{1}{u^\beta}\sum_{k=0}^{m}(iu^\alpha)^{k}\frac{\partial^{m-k}\mathbf{F}^c(0,v)}{\partial x^{m-k}},$$
$$=(iu^\alpha)^{m+1}\mathbf{F}^c(u,v)-\frac{1}{u^\beta}\sum_{k=1}^{m+1}(iu^\alpha)^{k-1}\frac{\partial^{m-(k-1)}\mathbf{F}^c(0,v)}{\partial x^{m-(k-1)}},$$
$$=(iu^\alpha)^{m+1}\mathbf{F}^c(u,v)-\frac{1}{u^\beta}\sum_{k=1}^{m+1}(iu^\alpha)^{k-1}\frac{\partial^{m+1-k}\mathbf{F}^c(0,v)}{\partial x^{m+1-k}},$$
$$=\mathbf{D}^c\left[\frac{\partial^{m+1} f(x,t)}{\partial x^{m+1}}\right]$$

So Theorem is true for $n\in\mathbb{N}$

Finally, We want prove (2) by mathematical induction

1 for $n=1$

$$\mathbf{D}^c\left[\frac{\partial f(x,t)}{\partial t}\right]=(iv^\alpha)^1\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\left[\sum_{k=1}^{1}(iv^\alpha)^{k-1}\frac{\partial^{1-k}\mathbf{F}^c(u,0)}{\partial t^{1-k}}\right]$$
$$=iv^\alpha\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\mathbf{F}^c(u,0)$$

Thus, true for $n=1$

2 Assume true for $n=m$ that mean

$$\mathbf{D}^c\left[\frac{\partial^m f(x,t)}{\partial t^m}\right]=(iv^\alpha)^m\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\left[\sum_{k=1}^{m}(iv^\alpha)^{k-1}\frac{\partial^{m-k}\mathbf{F}^c(u,0)}{\partial t^{m-k}}\right]$$

3 We want to prove (2) for $n=m+1$

$$\mathbf{D}^c\left[\frac{\partial^{m+1} f(x,t)}{\partial t^{m+1}}\right]=\mathbf{D}^c\left[\frac{\partial}{\partial t}\left[\frac{\partial^m f(x,t)}{\partial t^m}\right]\right]$$
$$=iv^\alpha\mathbf{D}^c\left[\frac{\partial^m f(x,t)}{\partial t^m}\right]-\frac{1}{v^\beta}\frac{\partial^m \mathbf{F}^c(u,0)}{\partial t^m},$$
$$=iv^\alpha\left[(iv^\alpha)^m\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\left[\sum_{k=1}^{m}(iv^\alpha)^{k-1}\frac{\partial^{m-k}\mathbf{F}^c(u,0)}{\partial t^{m-k}}\right]\right]-\frac{1}{v^\beta}\frac{\partial^m \mathbf{F}^c(u,0)}{\partial t^m},$$
$$=(iv^\alpha)^{m+1}\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\left[\sum_{k=1}^{m}(iv^\alpha)^{k}\frac{\partial^{m-k}\mathbf{F}^c(u,0)}{\partial t^{m-k}}\right]-\frac{1}{v^\beta}\frac{\partial^m \mathbf{F}^c(u,0)}{\partial t^m},$$
$$=(iv^\alpha)^{m+1}\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\left[\sum_{k=1}^{m}(iv^\alpha)^{k}\frac{\partial^{m-k}\mathbf{F}^c(u,0)}{\partial t^{m-k}}+\frac{\partial^m \mathbf{F}^c(u,0)}{\partial t^m}\right],$$
$$=(iv^\alpha)^{m+1}\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\sum_{k=0}^{m}(iv^\alpha)^{k}\frac{\partial^{m-k}\mathbf{F}^c(u,0)}{\partial t^{m-k}},$$
$$=(iv^\alpha)^{m+1}\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\sum_{k=1}^{m+1}(iv^\alpha)^{k-1}\frac{\partial^{m-(k-1)}\mathbf{F}^c(u,0)}{\partial t^{m-(k-1)}},$$
$$=(iv^\alpha)^{m+1}\mathbf{F}^c(u,v)-\frac{1}{v^\beta}\sum_{k=1}^{m+1}(iv^\alpha)^{k-1}\frac{\partial^{m+1-k}\mathbf{F}^c(u,0)}{\partial t^{m+1-k}},$$
$$=\mathbf{D}^c\left[\frac{\partial^{m+1} f(x,t)}{\partial t^{m+1}}\right]$$

So Theorem is true for $n\in\mathbb{N}$

5. Example

Example 5.1. Consider the following partial differential equation

$$u_{tt}=u_{xx} \qquad (3)$$

with conditions

$$u(x,0)=\sin x, u_t(x,0)=2$$
$$u(0,t)=2t, u_x(0,t)=\cos t$$

Solution:

we take Double complex Sadik transform in Equation (3)

$$\mathbf{D}^c[u_{tt}]=\mathbf{D}^c[u_{xx}]$$

we obtain

$$-\frac{1}{v^\beta}\frac{\partial \mathrm{F}^c(u,0)}{\partial t}-\frac{iv^\alpha}{v^\beta}\mathrm{F}^c(u,0)+(iv^\alpha)^2\mathrm{F}^c(u,v)=-\frac{1}{u^\beta}\frac{\partial \mathrm{F}^c(0,v)}{\partial x}-\frac{iu^\alpha}{u^\beta}\mathrm{F}^c(0,v)+(iu^\alpha)^2\mathrm{F}^c(u,v)$$

$$-(iu^\alpha)^2\mathbf{F}^c(u,v)+(iv^\alpha)^2\mathbf{F}^c(u,v)=\frac{1}{v^\beta}\frac{\partial \mathbf{F}^c(u,0)}{\partial t}+\frac{iv^\alpha}{v^\beta}\mathbf{F}^c(u,0)-\frac{1}{u^\beta}\frac{\partial \mathbf{F}^c(0,v)}{\partial x}-\frac{iu^\alpha}{u^\beta}\mathbf{F}^c(0,v)$$

$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^c(u,v)=\frac{1}{v^\beta}\frac{\partial \mathbf{F}^c(u,0)}{\partial t}+\frac{iv^\alpha}{v^\beta}\mathbf{F}^c(u,0)-\frac{1}{u^\beta}\frac{\partial \mathbf{F}^c(0,v)}{\partial x}-\frac{iu^\alpha}{u^\beta}\mathbf{F}^c(0,v)$$

substitute $\mathbf{F}^c(u,0), \mathbf{F}^c(0,v), \frac{\partial \mathbf{F}^c(0,v)}{\partial x}$ and $\frac{\partial \mathbf{F}^c(u,0)}{\partial t}$, we obtain

$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^c(u,v)=\frac{1}{v^\beta}\frac{-2i}{u^{\alpha+\beta}}+\frac{iv^\alpha}{v^\beta}\frac{-1}{u^\beta(u^{2\alpha}-1)}-\frac{1}{u^\beta}\frac{-iv^\alpha}{v^\beta(v^{2\alpha}-1)}-\frac{iu^\alpha}{u^\beta}\frac{-2}{v^{2\alpha+\beta}}$$

$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^c(u,v)=-\frac{2i}{(uv)^\beta u^\alpha}-\frac{iv^\alpha}{(uv)^\beta(u^{2\alpha}-1)}+\frac{iv^\alpha}{(uv)^\beta(v^{2\alpha}-1)}+\frac{2iu^\alpha}{(uv)^\beta v^{2\alpha}}$$

$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^c(u,v)=\frac{2iu^\alpha}{(uv)^\beta v^{2\alpha}}-\frac{2i}{(uv)^\beta u^\alpha}+\frac{iv^\alpha}{(uv)^\beta(v^{2\alpha}-1)}-\frac{iv^\alpha}{(uv)^\beta(u^{2\alpha}-1)}$$

$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^c(u,v)=\frac{2i}{(uv)^\beta}\left[\frac{u^\alpha}{v^{2\alpha}}-\frac{1}{u^\alpha}\right]+\frac{iv^\alpha}{(uv)^\beta}\left[\frac{1}{v^{2\alpha}-1}-\frac{1}{u^{2\alpha}-1}\right]$$

$$(u^{2\alpha}-v^{2\alpha})\mathbf{F}^c(u,v)=\frac{2i(u^{2\alpha}-v^{2\alpha})}{(uv)^\beta v^{2\alpha}u^\alpha}+\frac{iv^\alpha(u^{2\alpha}-v^{2\alpha})}{(uv)^\beta(u^{2\alpha}-1)(v^{2\alpha}-1)}$$

Division by $u^{2\alpha}-v^{2\alpha}$, we get

$$\mathbf{F}^c(u,v)=\frac{2i}{(uv)^\beta v^{2\alpha}u^\alpha}+\frac{iv^\alpha}{(uv)^\beta(u^{2\alpha}-1)(v^{2\alpha}-1)}$$

Take inverse of both side, we obtain

$$u(x,t)=2t+\sin x\cos t$$

Conclusion

In this work, a new Complex double integral transformation is introduced for finding the solution of general partial differential equations.

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