EVALUATION OF A NEW SHEAR COEFFICIENT FOR DYNAMIC ANALYSIS OF HOMOGENEOUS ISOTROPIC PLATES AND BEAMS+

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Abctract:

A new developed formula for the shear deformation coefficient is successfully displayed in this work. The new expression is the more accurate one among all previous types in the dynamic analysis field of homogeneous isotropic beams and plates. The formalization of the new expression is based on the partial equating of the recent exact 3-dimensional frequency equation of rectangular plate with the correspondent approximate equation of the first-order shear plate theory of *Mindlin*. The modified shear coefficient is verified numerically with some typical examples of uniform free plates and bars oscillating in flexural shear modes. The data comparison of the theoretical resonant frequencies, using variety of familiar shear coefficients in literature, with the exact and experiment frequency values show, with out doubts, the excellent-standing of the new shear coefficient comparing with all other formulae till the appearance time of this paper.

المستخلص

"معامل للتشوه القصي أكثر دقة لغرض التحليل الديناميكي للصفائح والقضبان متجانسة التركيب-متناظرة الخواص"

نجح البحث بجدارة في تقديم معادلة رياضية جديدة وأكثر دقة لتمثيل معامل التشوه القصي لغرض تحليل التصرف الحركي لتراكيب القضبان والصفائح ذات المواصفات الميكانيكية المتجانسة والمتناظرة. أستند الاشتقاق الرياضي للمعامل على المساواة الجزئية لمعادلة التردد الطبيعي ثلاثية الابعاد الحديثة والدقيقة لـ (ليفنسن) مع نظيرتها التقريبية ثنائية الابعاد للصفائح القصية لـ (مندلن). تم إجراء اختبار عددي لصحة النتائج النظرية لهذه المعادلة من خلال أمثلة نوعية لصفائح وقضبان منتظمة ومهتزة بالنمط الانحنائي الحر. تم مقارنة النتائج النظرية للترددات الطبيعية، ولمختلف الاشكال الوضعية لمعامل القص من البحوث السابقة، مع النتائج الدقيقة (معادلات النظرية الحديثة للاهتزاز) ومختبرياً (بيانات من تجربة الاهتزاز)، وأثبتت المقارنة المقدرة العالية للتمثيل الرياضي الجديد لمعامل القص في تقريب النتائج الى مستوى عالٍ من الدقة.

NOTATIONS:

a, b	Rectangular plate length & breadth.	w	Flexural displacement.

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c	**constant.**	**x,y**	**Plate co-ordinate system.**
G	**Shear modulus.**	A_j	**Serial coefficients.**
h	**Plate thickness.**	α, β	**Constants.**
j	**Suffix integer (2,3,…etc.)**	ρ	**Plate mass density.**
k	**Shear coefficient.**	Ω_{mn}	**Natural frequency.**
m, n	**Vibration mode indices.**	ϕ_{mn}	**Phase angle.**
t	**Time.**	W_{mn}	**Oscillation amplitude.**

Introdaction;

The first published shear beam theory by *Timoshenko*[1] and the shear plate theory by *Mindlin*[2] had been succeeded powerfully in the field of static and dynamic analysis of mechanical structures of isotropic materials. The recent advanced third-order theories by *Reddy*[3], *Reddy and Liu*[4] as well as the exact 3-dimensional theory by *Levinson* [5] and *Reddy and Khdier*[6] confirmed very well the acceptance of the approximate results employing the mentioned first-order theories. However, a relative error from the exact value, in the range (3%-5%) may be often achieved when applying the former tools (see for example Ref.[7]). Normally, there was no doubt at all when expecting a range of error come out from these theories, since they are classified as approximate rather than exact theories. The major dialogistical agent, in evaluating the results of such theory, seems referred to the value of the "*shear coefficient*" inserted asymptotically in the content of *Timoshenko* and *Mindlin* approximate theories. The concepts of approximation lie in tow aspects: (a) Proposing the point-wise shear strain to be varied only with the position in the plane of bending. (b) Adjusting the shear stress value by a multiplier called "*shear coefficient*", whose magnitude is assigned by extraneous procedure independently. The period (1920-1975) involved so many theoretical attempts to quantify an acceptable value of the imposed shear coefficient. The majority of these attempts were found within the static bending field of homogeneous isotropic prism and plates. Beginning from the first suggested value by *Timoshenko*[1] himself, and forwards by *Goodman and Sutherland*[8], Riessner[9], *Cowper*[10] and lastly by *Stephen and Levinson*[11], the general expression of the coefficient differed between constant parameter and variable with the *Poisson*'s ratio as well as the cross-section configuration for some particular cases. *Mindlin*[2] displayed two famous formulae of the shear coefficient based on the crested-wave propagation phenomena in an infinite plate and the twist-shear mode concept. All of the suggested values of the shear coefficients lies in the neighbor of *Riessner* coefficient (5/6) which became, and still, the most acceptable simple figure.

Most of the applied finite element packages (like *NASTRAN*[12] and *ANSYS*[13]), utilizing recently as a fruitful tool for beam and plate analysis, make use of *Riessner* value in the content of the reduced stiffness matrix built up for solution. In his third-order rectangular beam theory, *Levinson*[14,15] had noticed that a value slightly larger that of *Riessner* (0.833) gives more accurate results of the resonant frequencies of free bending beams. Namely he tried artificially with a value close to (0.877) , just for testing, and found that the corresponding results were improved satisfactorily. In that test the uniform beam was of rectangular cross-section and of steel alloy with *Poisson*'s ratio equal to (0.3). The whole theoretical work, in that report, did not contain any mathematical deriving of the proposed coefficient and said nothing whether the coefficient value is valid for other selected *Poisson*'s ratio or not!. The present author tried, from his side, to investigate similar cases in this trend, and found it clear enough that there exists a shear coefficient other than the classical ones which is able to bring the approximate results closer to the exact theoretical and experimental values, or there must be some way to prove mathematically the excellent representation of the coefficient to be equal to (0.877) . Recently, much interest in composite structure analysis, utilizing the classical *Riessner* coefficient (5/6) and/or *Whitney*[16] coefficient is brought in literature. Ref.[17] represents a very good sample in this manner, while *Sadiq*[18] and *Saify etal*[19] had applied successfully the familiar (5/6) value in their approaches. The main thrust of present paper was devoted to prove mathematically the well-standing feature of the shear coefficient (0.877) for a material plate of *Poisson*'s ratio (0.3), as had been quantitatively verified by *Levinson*[14]. The procedure involves the complete derivation of a new expression for the coefficient as a function of *Poisson*'s ratio of arbitrary value. Two basic theories, in the field of free vibration, were employed for this achievement: the approximate shear plate theory of *Mindlin*[2], which implies the shear coefficient in its content, and the exact 3-dimensional plate theory of *Levinson*[5] (more information can be found also in Ref.[4]). The vibration case-study was simulated as a plate of homogeneous isotropic material of uniform rectangular cross-section and simply-supported at the four edges. In accordance to each theory, a frequency characteristic equation (FCE) governs and describes the relationship between the resonant frequency of free flexural mode of vibration, with the modal parameters and material properties of the oscillated plate. The main key to achieve the present goal, is represented in the algebraic factorization of the FCE (in form of frequency parameter being a polynomial function of other modal parameter). Equating of the first few terms appeared in the *Mindlin*'s and *Levinson*'s FCEs, will yield the required expression of the new shear coefficient. The developed formula of this coefficient would be then verified to check its validity through the comparison of *Mindlin* and *Timoshenko* theories (for typical selected examples of beam and plate and for variety of shear coefficient expressions) with the correspond exact results both experimentally or analytically obtained by the 3-dimensional theory.

THEORETICAL ANALYSIS:

Consider a uniform rectangular plate of homogeneous isotropic material with the specification G, υ and ρ denoting for the shear modulus, *Poisson*'s ratio and mass density respectively. Let the plate length, width and thickness be a, b and h respectively. The plate co-ordinate system (xyz) is distinguished by x-axis along a, y-axis along b and z-axis along h, with the origin being on one corner line and at the mid-thickness position. The structure is assumed to be simply-supported at the four edges and executing a normal flexural mode of displacement w along z-direction represented, as usual, by[4,5]:

$$\mathbf{w = W_{mn}.\sin(m\pi x/a).\sin(n\pi y/b).\sin(\Omega_{mn}t + \phi_{mn})} \qquad \text{......(1)}$$

where W_{mn} is referred to the mode shape amplitude, m and n the vibration modal indices (each takes the integer value of 1,2,…,etc.), Ω_{mn} the resonant bending frequency of the plate, t the elapsed time and ϕ_{mn} the associated phase angle of the mode. *Reddy and Liu*[4] and *Levinson*[5] had excellently displayed, in the same time, the exact 3-dimensional expression of the natural frequency as related to the geometrical and mechanical properties and for fixed integers of the modal indices in the form of a FCE as:

$$4\lambda_1\lambda_2.\tanh(\lambda_1 h/2) = P^4.\tanh(\lambda_2 h/2) \quad , \quad \text{with}$$

$$\lambda_1{}^2 = \delta_{mn}{}^2 - \frac{\rho\Omega_{mn}{}^2}{G} \quad , \quad \lambda_1{}^2 = \delta_{mn}{}^2 - \frac{\rho\Omega_{mn}{}^2}{(\lambda + 2G)}$$

$$\delta_{mn} = \sqrt{(m\pi/a)^2 + (n\pi/b)^2} \quad , \quad P = \sqrt{\lambda_1{}^2 + \delta_{mn}{}^2}$$

$$\lambda = \frac{2\nu G}{(1-2\nu)} \qquad \text{......(2)}$$

Evidently, the above formula serve to compute the exact value of Ω_{mn} using a suitable iteration procedure when all remaining parameters are pre-known. For present objectives, this formula would be altered to three independent variable functions as follows:

$$\frac{(1-\xi)}{(1-\xi/2)}\left(\frac{\tanh(\psi\sqrt{1-\xi})}{(\psi\sqrt{1-\xi})}\right) = \left(\frac{\tanh(\psi\sqrt{1-c\xi})}{(\psi\sqrt{1-c\xi})}\right) \quad , \quad \text{with}$$

$$\xi = \frac{\rho\Omega_{mn}{}^2}{G\delta_{mn}{}^2} \quad , \quad \psi = \delta_{mn} h/2 \quad \text{and} \quad c = \frac{(1-2\nu)}{2(1-\nu)} \qquad \text{......(3)}$$

Writing each bracket term of above formula in the form of algebraic series of the following type:

$$\left(\frac{\tan\theta}{\theta}\right) = 1 - \frac{1}{3}\theta^2 + \frac{2}{15}\theta^4 - \frac{17}{315}\theta^6 + \ldots\ldots \qquad (4)$$

then by simplifying and re-arranging the individual parameters, of the same eq.(3), would be resulted finally in:

$$\left[(\frac{1}{4})\xi\right] - \left[\frac{1}{3}(1-c) + (c - \frac{3}{4})\xi - (\frac{c}{4})\xi^2\right]\psi^2$$

$$+\left[\frac{2}{15}(2-c) + (c^2 + 2c - \frac{11}{4})\xi + (1 - \frac{c}{2} - c^2)\xi^2 - (\frac{c^2}{4})\xi^3\right]\psi^4 - \ldots\ldots = 0 \qquad \text{......(5)}$$

Clearly, eq.(5) implies $\psi=0$ when $\xi=0$ and asserts that the expansion elements of ψ are totally of even orders. Mathematically speaking, this serves to conclude that the parameter ψ^2 can be inversely formulized in terms of serial expansion of ξ, with multipliers, say A_1, A_2, and so on (i.e. $\psi^2=A_1\xi+A_2\xi^2+$….etc.). Substituting of this expansion back into eq.(5), re-arranging and collecting similar ξ-order terms, will give:

$$\psi^2 = A_1\xi + A_2\xi^2 + \ldots\ldots \quad \text{with}$$

$$A_1 = \frac{3(1-\nu)}{2} \quad , \quad A_j = A_{j-1}\left(\frac{\alpha_j - \beta_j\nu}{(3j-5).10^{j-1}}\right) \quad \text{for} \quad j = 2,3,4,\ldots.\text{etc.}$$

$$\beta_j = 13 - 3j \quad \text{and} \quad \alpha_j = \beta_j + 10 \qquad \text{......(6)}$$

The above equation represents the present factorized form of the exact FCE just for adequate terms of expansion.

The time comes now to investigate *Mindline* FCE in the same manner outlined herein. The procedure is relatively easier because of the simple fashion of the approximate FCE which is often written as:

$$\frac{h^2}{12}\left(\delta_{mn}^{\ 2}-\frac{\rho\Omega_{mn}^{\ 2}}{kG}\right)\left(\frac{2}{1-\nu}\delta_{mn}^{\ 2}-\frac{\rho\Omega_{mn}^{\ 2}}{G}\right)-\frac{\rho\Omega_{mn}^{\ 2}}{G}=0 \tag{7}$$

in which k denotes the proposed shear coefficient to be determined. Employing the same notations of ξ and ψ, defined in eq.(3), into above expression, will give:

$$\psi^2=\left[\frac{3k}{\left(\frac{2k}{1-\nu}\right)-\left(\frac{2}{1-\nu}+k\right)\xi+\xi^2}\right]\xi \qquad \text{......(8)}$$

The bracket term, in above equation, is a fractional function of ξ. Applying the long division process on this term would be resulted in a serial expansion formula of ψ^2 in terms of ξ, very similar to that accomplished in eq.(6). The approximate *Mindlin* FCE may be then typed as:

$$\begin{aligned}&\psi^2=\overline{A}_1\xi+\overline{A}_2\xi^2+\ldots\ldots\ \text{with}\\&\overline{A}_1=\frac{3(1-\nu)}{2}\ ,\ \overline{A}_j=\overline{A}_{j-1}\alpha-\overline{A}_{j-2}\beta\ \text{ for }\ j=2,3,4,\ldots\ldots\\&\beta=\frac{1-\nu}{2k}\ \text{ and }\ \alpha=\left(\frac{1}{k}+\frac{1-\nu}{2}\right)\end{aligned} \qquad \text{......(9)}$$

A quantitative comparison of the present approximate FCE (eq.(9)) with the exact FCE (eq.(6)) shows that the frequency parameter ψ^2, in both equations, can be made approximately equal if the general coefficient A_j coincides with its correspondent $\overline{A}_j$. The first ones (A_1 and $\overline{A}_1$) are in fact already equal. However the coefficients (A_2 and $\overline{A}_2$) could be precisely same in magnitude, if and only if:

$$\frac{3(1-\nu)}{2}\cdot\frac{(17-7\nu)}{10}=\frac{3(1-\nu)}{2}\cdot\left(\frac{1}{k}+\frac{1-\nu}{2}\right)\quad \text{from which :}$$

$$k=\frac{5}{6-\nu} \qquad \text{......(10)}$$

While making the coefficients (A_3 and $\overline{A}_3$) equal would yield:

$$\frac{3(1-\nu)}{2}\cdot\frac{(17-7\nu)}{10}\cdot\frac{(14-4\nu)}{400}=\frac{3(1-\nu)}{2}\cdot\left(\frac{1}{k}+\frac{1-\nu}{2}\right)^2-\frac{3(1-\nu)}{2}\cdot\frac{1-\nu}{2k}\quad \textbf{which implies :}$$

$$(1/k)^2+\left(\frac{1-\nu}{2}\right)(1/k)+\left[\left(\frac{1-\nu}{2}\right)^2-\frac{(17-7\nu)(14-4\nu)}{4000}\right]=0 \qquad \text{......(11)}$$

The last 2^{nd}-order equation in (1/k) gives its roots as imaginary quantities for the complete range of the *Poisson*'s ratio for homogeneous isotropic metallic alloys ($\mathbf{0 \le \nu \le 0.5}$). The same result would be obtained for ($A_4=\overline{\mathbf{A}}_4$, $A_5=\overline{\mathbf{A}}_5$,…. etc.). Thus, the present k-expression, in eq.(10), seems to be the unique form of the new shear coefficient. It is worthily to note, here, that setting of υ=0.3 (for most of steel alloys), in eq.(10), leads to k=0.877, the value that was suggested by *Levinson*[5] but with no evidence to prove it mathematically as has been done here successfully, besides of its ability to deal with any other value of the *Poisson*'s ratio.

NUMERICAL RESULTS AND DISCUSSION:

The developed k-expression, explores some tasks of interest in the present field of research. At first, it seems simple in form for manipulating process. Second, it relates, in the same phase, with the *Poisson*'s ratio only (indeed it retains the classical value of *Riessner* 5/6 as a special case only when ignoring the effect of υ). Third, it gives always the value of k, slightly larger than *Riessner* value (0.833) for most isotropic metal selection of the plate ($\mathbf{0 \le \nu \le 0.5}$), and lastly it interpretates mathematically that (k=0.877) is truly the more accurate expression among other ones in literature, when the material *Poisson*'s ratio is nearly 0.3, as had been discovered before (refer to Ref/[14]) where there was no apparent proof being exist.

In order to check the validity of present k-expression (eq.(10)), two numerical examples were adopted. The first example concerns with the free bending vibration of uniform rectangular simply-supported plate of homogeneous isotropic material. The exact and approximate resonant frequencies were theoretically estimated for different values of the aspect ratio (h/a) and for five selected values of k familiarly found in literature, including the present one. Tables(1.a,b) show the corresponding results, of this example, for typical mode shape of vibration as indicated by the integer identification of m and n. Fig.(1) illustrated graphically the effect of different k-representations (from corresponding approaches) on the relative frequency error for variety (h/a) values as adapted from Table(1.a). From these tables and figure, it is obvious that the present representation of k makes the relative errors of frequencies, as compared with the exact values, smaller than those estimated by other reported values of k. The present maximum relative error does not exceed (0.54%) for all test cases of (h/a) and (m,n) values.

The second example deals with the fundamental frequency of free-free flexural vibration of uniform parallelepiped beam of homogeneous isotropic material. Four distinct values of the beam length were chosen and the recorded resonant frequencies, from a vibration test, were listed in Table(2) as had been adapted from Ref.[20]. The present theoretical frequencies, of this example, were computed using *Timoshenko* beam theory and for variety of k models. The governing free-free *Timoshenko* frequency equation was extracted from Ref.[7]. Fig.(2) show graphically the variation of the former results with the length of the beam and for variety of k-expressions. As illustrated in these table and figure, the new representation of k gives the lowest relative errors in frequencies among other results of familiar coefficients. The relative error **decreases with the beam length (increasing of** h/a), nevertheless the present k-formula shows less than (0.57%) relative error in all cases of the beam under consideration.

CONCUSIONS:

Pertinent useful remarks can be concluded from previous discussions and may be listed as followings:

(a) A new shear coefficient expression had been formulized and found succeeded to improve the dynamic analysis results for plates and beams of homogeneous isotropic material, compared with other suggested coefficients in related field, with maximum relative error less than (0.6%) in average.

(b) The new coefficient is simple in form and depends in magnitude upon the material *Poisson*'s ratio only, and both in the same phase of increasing or decreasing. The classical *Riessner* value (5/6) is found to be just a special case from the new k-expression (when $\upsilon=0$).

(c) The present mathematical work succeeded in interpreting why k=0.877 is seemed more accurate proposition for vibration analysis of steel alloys, a matter which has been left missing for more than two decades ago.

Table(1.a) The relative percentage decrease in the theoretical Ω_{mn} of simply-supported rectangular plate, for different (thickness/length) ratios and variety of shear coefficient representations.

h/a	Ω_{mn} (rad/sec) Exact[5]	General plate dimensions: a=50cm, b=50cm; Material properties: ρ=7860kg/m^3, G=79.6Gpa, υ=0.3; Modal indices: m=1, n=1				
		Mindlin[2] (k=0.822)	*Mindlin*[2] (k=0.860)	*Riessner*[9] (k=0.833)	*Cowper*[10] (k=0.850)	Present (k=0.877)
0.05	3036.834	0.0442(*)	0.0133	0.0349	0.0211	0.0003
0.10	5924.949	0.1670	0.0510	0.1324	0.0806	0.0022
0.20	10879.05	0.5574	0.1804	0.4452	0.2767	0.0211
0.40	17314.57	1.4181	0.5133	1.1500	0.7457	0.1274
0.60	20732.98	2.0862	0.8097	1.7092	1.1387	0.2614
0.80	22650.67	2.5544	1.0301	2.1054	1.4240	0.3720
1.00	23800.38	2.8730	1.1811	2.3756	1.6192	0.4476

(*) Relative percentage decrease=(Ω_{exact}-$\Omega_{theoretical}$/Ω_{exact})x100

Table(1.b) The relative percentage decrease in the theoretical Ω_{mn} of simply-supported rectangular plate, for different (thickness/length) ratios and variety of shear coefficient representations.

h/a	Ω_{mn} (rad/sec) Exact[5]	General plate dimensions: a=50cm, b=100cm; Material properties: ρ=7860kg/m^3, G=79.6Gpa, υ=0.3; Modal indices: m=2, n=3				
		Mindlin[2] (k=0.822)	*Mindlin*[2] (k=0.860)	*Riessner*[9] (k=0.833)	*Cowper*[10] (k=0.850)	Present (k=0.877)
0.05	9323.321	0.1325(*)	0.0422	0.1049	0.0637	0.0014
0.10	17394.43	0.4569	0.1457	0.3642	0.2251	0.0144
0.20	28576.15	1.2311	0.4358	0.9952	0.6398	0.0972
0.40	38691.40	2.3586	0.9372	1.9395	1.3041	0.3248
0.60	42526.30	2.9470	1.2153	2.4381	1.6639	0.4638
0.80	44269.96	3.2292	1.3330	2.6732	1.8254	0.5065
1.00	45164.95	3.3409	1.3499	2.7579	1.8677	0.5397

(*) Relative percentage decrease=(Ω_{exact}-$\Omega_{theoretical}$/Ω_{exact})x100

Table(2) The relative percentage decrease in the theoretical fundamental frequency of free-free rectangular beam, for different beam lengths and variety of shear coefficient representations.

Source	General beam dimensions: b=6.28cm, h=3.14cm Material properties: ρ=7850kg/m^3, G=80.11Gpa, υ=0.3 Mode shape: fundamental			
	a=7.88cm	a=7.55cm	a=7.25cm	a=7.01cm
Experiment[20] (rad/sec)	19203	20495	21789	22934
Mindlin[2] (k=0.822)	1.1518(*)	0.9827	1.0108	1.0910
Mindlin[2] (k=0.860)	0.7402	0.5569	0.5730	0.6437
Riessner[9] (k=0.833)	1.0293	0.8561	0.8805	0.9579
Cowper[10] (k=0.85)	0.8507	0.6713	0.6906	0.7638
Present (k=0.877)	0.5641	0.3749	0.3859	0.4526

(*) Relative percentage decrease=(Ω_{exact}-$\Omega_{theoretical}$/Ω_{exact})x100

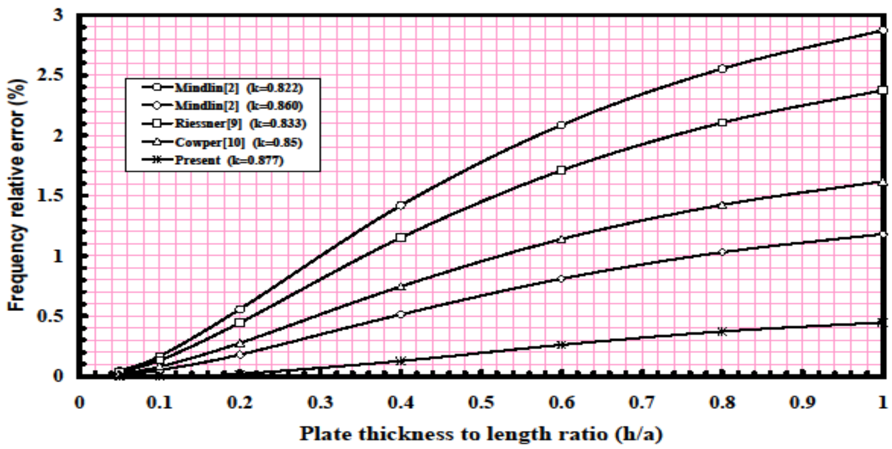

Fig.(1) Comparison of different shear coefficients effects upon the frequency relative errors(in %) of resonant frequency for variety of size factor (h/a) of rectangular simply-supported plates (refer to Table(1.a)).

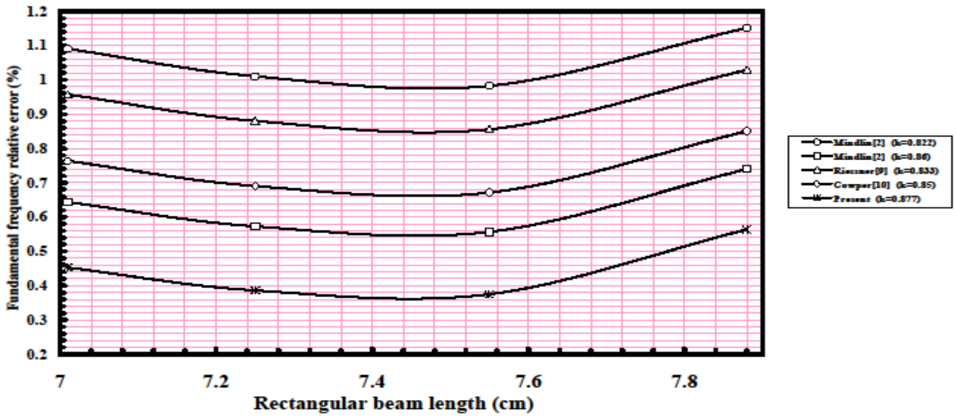

Fig.(2) Comparison of different shear coefficients effects upon the fundamental frequency relative errors(in %) for variety of lengths of free-free rectangular beams (refer to Table(2)).

References :

1. *Timoshenko S.P.*, "On The Correction For Shear Of The Differential Equations For

Transvers Vibrations Of Prismatic Bars", Phil. Magz., , Vol. 124, pp. 774-776. 1921
2. *Mindlin R.D.,* "Influence Of The Rotary Inertia And Shear On Flexural Motions Of Isotropic Plates", J. Appl. Mech., , Vol. 18, pp. 31-38. 1951
3. *Reddy J. N.,* "Exact Solution Of Moderately Thick-Laminated Shell", J. Engg. Mech., ASCE, , Vol. 110, pp. 794-809. 1984
4. *Reddy J.N. and Liu C.,* "A higher Order Shear Deformation Theory Of Laminated Shell", J. Engg. Soc., , Vol. 23, pp. 319-330. 1985
5. *Levinson M.,* "Free Vibration Of Simply-Supported Rectangular Plates: An Exact Elasticity Solution", J. Sound & Vib., , Vol. 98, pp. 289-298. 1985
6. *Reddy N.J. and Khdier A.,* " Dynamic Response Of Cross-ply Laminated Shallow Shells According To A refined Shear Deformation Theory", J. Accous. Soc. Amr., , Vol. 85, No. 6, pp. 2423-2431. 1989
7. *Khdier A. and Reddy N.J.,* "Free Vibration Of Beams With Arbitrary End Conditions", Int. J. Engg. Sci., , Vol. 32, pp. 1971-1986. 1994
8. *Goodman L.E. and Sutherland J.G.,* Discussion of the paper by Ayre R.S. and Jacobsen L.S., J. Appl. Mech., , Vol. 18, pp. 217-218. 1951
9. *Riessner E.,* "On Bending Of Elastic Plates", Quar. Appl. Math., , Vol. 5, pp. 55-56. 1947
10. *Cowper G.R.,* "The Shear Coefficient In Timoshenko's Beam Theory", J. Appl. Mech., , Vol. 33, pp. 335-340. 1966
11. *Stephen N.G. and Levinson M.,* "A second Order Beam Theory", , J. Sound & Vib., Vol. 67, pp. 293-305. 1979
12. *Jhones R.,* Msc/NASTRAN Application Manual, , ch. 2. 1982
13. *Pester C. and Kohnke J.,* ANSYS Theoretical Manual, , 8^{th} Ed., SAS Inc., Houston USA. 1999
14. *Levinson M.,* "A new Rectangular Beam Theory", 1981, J. Sound & Vib., Vol. 74, pp. 81-87.
15. *Levinson M.,* "Further Results Of A new Beam Theory", , J. Sound & Vib., Vol. 74, pp. 440-446. 1981
16. *Whitney J.M.,* "The Effect Of Transverse Shear Deformation On The Bending Of Laminated Plates", , J. Comp. Mat., Vol. 3, pp. 534-547. 1969
17. *Saify K.M.,* "Elasto-static And Dynamic Investigation Into Composite Plates And Shells With New Approaches To Estimate The Elastic Moduli", Ph. D Thesis, University of Baghdad, Iraq. 2000.
18. *Sadiq G.S.*, "Static And Dynamic Analysis Of Composite Thin Cylindrical Shells", Ph. D Thesis, University of Baghdad, Iraq. , 2004
19. *Saify K., Jweeg M. and Al-Timimi A.,* "Timoshenko Three Beams Technique (T3BT) To Estimate The Main Elastic Moduli Of Orthotropic Homogeneous Materials", accepted to be published at Al-Khawarizmy J. Engg, University of Baghdad, Iraq. ., 2005
20. *Hutchinson J.R. and Zillmer S.D.,* "Vibration Of A free Rectangular Parallelepiped", J. Appl. Mech., Vol. 50., pp. 123-130. , 1983