

FORWARD , INVERSE POSITION KINEMATICS AND WORKSPACE ANALYSIS OF 3 – $RRR^{\perp}S$ PARALLEL MANIPULATORS⁺

Mazin Ismail^{*}

Abstract

In this paper, the equations for the inverse and direct kinematics problems of a 3 – $RRR^{\perp}S$ spatial parallel mechanism (revolute - revolute - revolute -spherical) of six degrees of freedom with two actuated joints and four passive joints in each of three parallel branches have been studied .

The number of solutions of the inverse kinematics problem is shown to be not more than 64 ,and the solution of the direct kinematics problem has been shown to reducible to a 16th order polynomial .This implies that for a given set of actuated angles , this 3 – $RRR^{\perp}S$ parallel mechanism can be assembled in at most 16 different configurations .Further numerical computations were performed to check the algebra and a numerical examples were solved to demonstrate the procedure .Finally a workspace analysis and visualization scheme is also presented .

المستخلص :

في هذا البحث تمت دراسة معادلات الحركة المباشرة و المعكوسة المجردة للآلية الفضائية $3 - RRR^{\perp}S$ المتوازية ذات درجات الحرية الستة والمتصفة بفعالية مفصلين و خمول المفاصل الاربعة المتبقية لكل ساق من السيقان الثلاث .

لقد تم التوصل الى ان عدد الحلول هي ٦٤ للحالة الحركية المعاكسة بينما أتضح بان حل معادلة الحركة المباشرة هي معادلة متعددة الحدود من الدرجة السادسة عشر ، أي يمكن الحصول على ستة عشر هيئة مختلفة لمجموعة زوايا مفعلة. تم اجراء مختلف الحسابات الرياضية الاخرى للتأكد من دقة المعادلات التي استنتجت ، كذلك تم حل العديد من الامثلة الرقمية لإثبات صحة ودقة الحلول .وأخيرا تم عمل تحليل رياضي لحيز العمل وكذلك تمثيل نموذج مرئي له .

Introduction

A parallel robot is a closed loop mechanism in which the mobile platform is connected to the base by at least two serial kinematic chains (legs).For the past few years ,since parallel robotic systems can offer high stiffness since the load is usually carried in compression-traction mode only ,high load capability since the payload is carried by several links in parallel ,a quick dynamic response and a good position accuracy due to non-cumulative joint error , they have received considerable research attention in the robotic field .Among the many aspects of parallel robotic systems, forward position analysis has been studied extensively but remains a challenging problem for researchers [1]. Forward position analysis is to determine the position and orientation of the moving platform provided that a set of

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^{*} Assist. Lecturer M.Sc.- Institute of Technical Trainers Preparing/ Foundation of Technical Education ,Baghdad ,Iraq
mazinismail712@yahoo.com

actuated joint variables is specified .Because a parallel robot usually has a limited workspace , trajectory planning and application development become difficult.

A six degrees of freedom parallel manipulator was introduced by Stewart in 1965 and since then has been commonly known as the "Stewart-Gough platform "[2].An atlas of parallel robots was composed by Merlet and can be found in the web site [http://www.inria.fr/prisme/personal/merlet/merlet_eng.html].Several researchers have analyzed the direct kinematics of Stewart-Gough platform .Kim and Tsai [3] studied the kinematics of 3-RPS parallel manipulator using the method of prescribed positions .Song and Kwon [4] solved the kinematics of parallel mechanism using the tetrahedron configuration. Hopkins and Williams [5] designed a 6-PSU parallel mechanism .Yang and his colleague have studied the design and kinematic analysis of modular reconfigurable parallel robots like 3-RRRS using the iteration method and used analytical algorithm to determine the workspace boundary [6] .The limitation of these methods is due to their dependent on the estimation of the initial configuration .Hence the solutions of the forward kinematics are generally calculated and preferred by numerical methods . For example ,Tahmasebi [7] studied a Novel Tip Tilt Piston parallel manipulator ,Ben-Horin, Shoham and Djerassi [8] analyzed a planarly actuated parallel robot .All these methods without initial estimation

In this paper the equations for the direct and inverse position kinematics for $3-RRR^{\perp}S$ mechanism -see Fig.(1)-have been developed and solutions have been obtained It is shown that ,at most ,sixty-four solutions exist for the inverse position kinematics problem, while the direct position kinematics solution has been shown to be reducible to a sixteenth order polynomial equation .

Notation

- θ_{2i}, θ_{3i} : the actively controlled joint angles are the two perpendicular revolutes at O_i .
- θ_{1i} : the passively controlled joint angle.
- m_{12}, m_{23}, m_{13} : the length of the sides of the moving triangle P_1P_2, P_2P_3 and P_1P_3 .
- $X_gY_gZ_g$: the fixed coordinate frame has its origin O_g at the centroid of the base triangle $O_1O_2O_3$., X_g axis points outward from the plane paper .
- Z_{o1}, Z_{o2}, Z_{o3} : the unit vectors along the passive revolute O_1, O_2 and O_3 respectively .
- α_i : the angle made by the normal of the Z_{oi} axis with the Z_g axis (positive in the clockwise sense) .
- λ : degree of freedom of the task space.
- n : total number of links .
- j :number of joints .
- f_i : degrees of relative motion permitted by joint i .
- q_i : the position vector of P_i relative to O_i .

Description of the Mechanism

The mechanism consists of two platforms connected to each other by three serial chains as shown in Fig. (1,2).One of the platforms is fixed to the ground while the other one is free .Each of the three serial chains has a total a six degrees of freedom, two of which are actively controlled . All the three serial chains are connected to the top platform by passive spherical joints at P_1, P_2 , and P_3 , respectively . Each chain is connected to the bottom platform at O_1, O_2 , and O_3 respectively, by a passive revolute joint .Each of these three chains has two actively controlled revolutes at A_1, B_1, A_2, B_2 , and A_3, B_3 , respectively ,whose axes are perpendicular. Revolute axes at A_1 and O_1 are parallel and so are the revolute axes at A_2, O_2 and A_3, O_3 . The angle of rotation about the X_g axis which is needed to align the Z_g axis with Z_{oi} is $(270-\alpha_i)$.This fixes the position and orientation of $X_{oi}Y_{oi}Z_{oi}$ relative to the base frame

$$D.F.O = \lambda(n-j-1) + \sum_{i=1}^{j=1} f_i = 6(8-9-1) + 3(2+1+3) = -12 + 18 = 6$$

therefore the mechanism is 6 *D.O.F.*

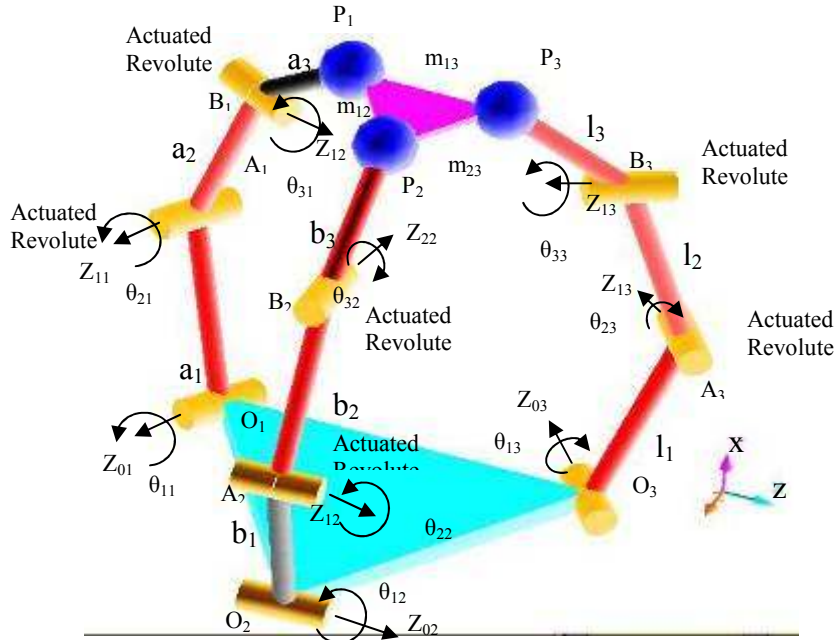


Fig. (1) schematic of $3-RRR^{\perp}S$ parallel mechanism.

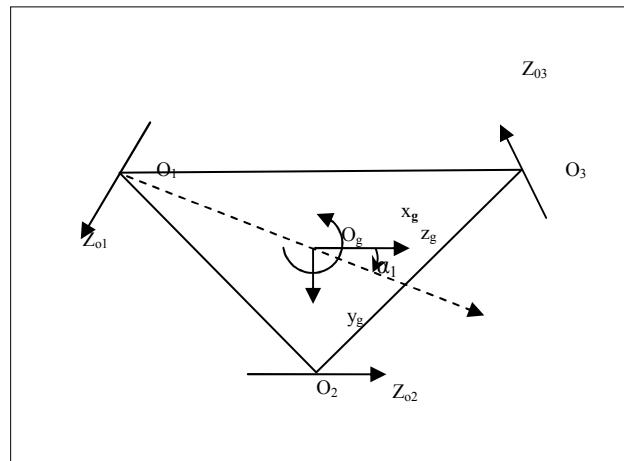


Fig. (2) Location of the base coordinate system.

Inverse Position kinematics

The inverse position kinematics problem for this structure can be stated as :Given the position and orientation of the moving platform $P_1P_2P_3$ with respect to the base platform $O_1O_2O_3$,find the intermediate actuator angles .

Since the position and orientation of a moving body in space can be uniquely determined by specifying the position of three noncollinear points embedded in the moving body, we can restate the inverse position kinematics problem as follows : Given : The position of the points P_1, P_2 and P_3 in the base coordinate frame $X_g Y_g Z_g$, Find : The intermediate actuator angles $\theta_{11}, \theta_{21}, \theta_{31}, \theta_{12}, \theta_{22}, \theta_{32}, \theta_{13}, \theta_{23}$ and θ_{33} .

Using standard , serial chain techniques ,the vector from O_g to P_i can be written in the form :

$$p_{pi} = [{}^gR_{oi}]q_i + \rho_{oi} \quad \text{for } i = 1,2,3 \quad (1)$$

where $[{}^gR_{oi}]$ is the coordinate transformation matrix of $X_{oi}Y_{oi}Z_{oi}$ frame with respect to $X_gY_gZ_g$ frame and is given by :

$${}^gR_{oi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin \alpha_i & \cos \alpha_i \\ 0 & -\cos \alpha_i & -\sin \alpha_i \end{bmatrix}$$

ρ_{oi} is the vector from O_g to O_i and is given by $\rho_{oi} = [0 \quad (\rho_{oi})_y \quad (\rho_{oi})_z]^T$

Detailed expressions for all of the three limbs are given below . The notation and conventions used here are those of Husain and Waldron [10] .they are consistent with Hartenberg and Denavit [11] notation ,but the inhomogeneous matrix from which has separate matrices for the notation and the position of the origin is preferred [12].The first subscript refers to the position of a revolute in the chain ,and the second subscript refers to the chain itself .

$$q_i = U_{1i}S_{1i} + U_{1i}U_{2i}S_{2i} + U_{1i}U_{2i}JU_{3i}U_{3i} \quad (2)$$

where

$$U_{ij} = \begin{bmatrix} \cos \theta_{ij} & -\sin \theta_{ij} & 0 \\ \sin \theta_{ij} & \cos \theta_{ij} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$S_{j1} = [a_i \ 0 \ 0]^T \quad S_{j2} = [b_i \ 0 \ 0]^T \quad S_{j3} = [l_i \ 0 \ 0]^T \quad j=1,2,3$$

combining Eq.(1),with Eq.(2) ,we get :

$$[U_{11}]^{-1}[{}^gR_{o1}]^{-1}(p_{p1} - \rho_{o1}) - S_{11} = U_{21}S_{21} + U_{21}JU_{31}JS_{31} \quad (3)$$

$$[U_{12}]^{-1}[{}^gR_{o2}]^{-1}(p_{p2} - \rho_{o2}) - S_{12} = U_{22}S_{22} + U_{22}JU_{32}JS_{32} \quad (4)$$

$$[U_{13}]^{-1}[{}^gR_{o3}]^{-1}(p_{p3} - \rho_{o3}) - S_{13} = U_{23}S_{23} + U_{23}JU_{33}JS_{33} \quad (5)$$

The numbers of solutions obtained are as follows:

Equation (3) gives:2 solutions for θ_{11} ,2 solutions for θ_{21} and 1 solution for θ_{31} ,

Equation (4) gives:2 solutions for θ_{12} ,2 solutions for θ_{22} ,and 1 solution for θ_{32} ,

Equation (5) gives:2 solutions for θ_{13} ,2 solutions for θ_{23} ,and 1 solution for θ_{33} .

Hence , the number of solutions to the inverse kinematics problem is not greater than 64 , although all of these solutions may not be physically realizable .

Direct Position Kinematics

The direct position kinematics problem is much more involved .It can be stated as: Given the actuator angles for all of the actively controlled joints ,find the position and orientation of the moving platform $P_1P_2P_3$ with respect to the base platform $O_1O_2O_3$.

From the previous discussion of inverse position kinematics it is obvious that the direct position kinematics problem can be re-stated as :

GIVEN: The actuator angles $\theta_{21}, \theta_{22}, \theta_{23}, \theta_{31}, \theta_{32}$ and θ_{33} , *FIND:* The joint angle $\theta_{11}, \theta_{12}, \theta_{13}$.

Expansion of Eq.(1) with the help of Eq.(2) gives the vector from O_g to P_i as :

$$p_{p1} = \begin{bmatrix} P_{11}c\theta_{11} + Q_{11}s\theta_{11} \\ P_{12}c\theta_{11} + Q_{12}s\theta_{11} + R_{12} \\ P_{13}c\theta_{11} + Q_{13}s\theta_{11} + R_{13} \end{bmatrix} \text{-----} (6)$$

where

$$P_{11} = a_1 + a_2c\theta_{21} + a_3c\theta_{21}c\theta_{31}$$

$$Q_{11} = -a_2s\theta_{21} - a_3s\theta_{21}c\theta_{31}$$

$$P_{12} = -a_2s\alpha_1s\theta_{21} - a_3s\alpha_1s\theta_{21}c\theta_{31}$$

$$Q_{12} = -a_1s\alpha_1 - a_2s\alpha_1c\theta_{21} - a_3s\alpha_1c\theta_{21}c\theta_{31}$$

$$R_{12} = a_3c\alpha_1s\theta_{31} + \rho_{1y}$$

$$P_{13} = -a_2c\alpha_1s\theta_{21} - a_3c\alpha_1s\theta_{21}c\theta_{31}$$

$$Q_{13} = -\alpha_1c\alpha_1 - a_2c\alpha_1c\theta_{21} - a_3c\alpha_1c\theta_{21}c\theta_{31}$$

$$R_{13} = -a_3s\alpha_1s\theta_{31} + \rho_{1z}$$

Similarly , the vector from O_g to P_2 is

$$p_{p2} = \begin{bmatrix} P_{21}c\theta_{12} + Q_{21}s\theta_{12} \\ P_{22}c\theta_{12} + Q_{22}s\theta_{12} + R_{22} \\ P_{23}c\theta_{12} + Q_{23}s\theta_{12} + R_{23} \end{bmatrix} \text{-----} (7)$$

where

$$P_{21} = b_1 + b_2c\theta_{22} + b_3c\theta_{22}c\theta_{32}$$

$$Q_{21} = -b_2s\theta_{22} - b_3s\theta_{22}c\theta_{32}$$

$$P_{22} = -b_2s\alpha_2s\theta_{22} - b_3s\alpha_2s\theta_{22}c\theta_{32}$$

$$Q_{22} = -b_1s\alpha_2 - b_2s\alpha_2c\theta_{22} - b_3s\alpha_2c\theta_{22}c\theta_{32}$$

$$R_{22} = b_3c\alpha_2s\theta_{32} + \rho_{2y}$$

$$P_{23} = -b_2c\alpha_2s\theta_{22} - b_3c\alpha_2s\theta_{22}c\theta_{32}$$

$$Q_{23} = -b_1c\alpha_2 - b_2c\alpha_2c\theta_{22} - b_3c\alpha_2c\theta_{22}c\theta_{32}$$

$$R_{23} = -b_3s\alpha_2s\theta_{32} + \rho_{2z}$$

and the vector from O_g to P_3 is

$$P_{p3} = \begin{bmatrix} P_{31}C\theta_{13} + Q_{31}S\theta_{13} \\ P_{32}C\theta_{13} + Q_{32}S\theta_{13} + R_{32} \\ P_{33}C\theta_{13} + Q_{33}S\theta_{13} + R_{33} \end{bmatrix} \text{-----} (8)$$

where

$$P_{31} = l_1 + l_2C\theta_{23} + l_3C\theta_{23}C\theta_{33}$$

$$Q_{31} = -l_2s\theta_{23} - l_3s\theta_{23}c\theta_{33}$$

$$P_{32} = -l_2s\alpha_3s\theta_{23} - l_3s\alpha_3s\theta_{23}c\theta_{33}$$

$$Q_{32} = -l_1s\alpha_3 - l_2s\alpha_3c\theta_{23} - l_3s\alpha_3c\theta_{23}c\theta_{33}$$

$$R_{32} = l_3c\alpha_3s\theta_{33} + \rho_{3y}$$

$$P_{33} = -l_2c\alpha_3s\theta_{23} - l_3c\alpha_3s\theta_{23}c\theta_{33}$$

$$Q_{33} = -l_1c\alpha_3 - l_2c\alpha_3c\theta_{23} - l_3c\alpha_3c\theta_{23}c\theta_{33}$$

$$R_{33} = -l_3s\alpha_3s\theta_{33} + \rho_{3z}$$

All of these P 's , Q 's and R 's can be considered to be known for the direct kinematics problem.

For the formulation of the direct kinematics equations ,we use geometric constraint that triangle $P_1P_2P_3$ embedded in the moving platform is invariant ,i.e.,

$$|p_{p1} - p_{p2}|^2 = |P_1 P_2|^2 = m_{12}^2 \quad (9)$$

$$|p_{p2} - p_{p3}|^2 = |P_2 P_3|^2 = m_{23}^2 \quad (10)$$

$$|p_{p1} - p_{p3}|^2 = |P_1 P_3|^2 = m_{13}^2 \quad (11)$$

Using Eqs.(6),(7),and (8) we can simplify Eqs.(9),(10) and (11) giving Eqs.(12), (13), and (14),respectively, which are stated below :

$$A_1 c_{11} c_{12} + A_2 c_{11} s_{12} + A_3 s_{11} c_{12} + A_4 s_{11} s_{12} + A_5 c_{11} + A_6 s_{11} + A_7 c_{12} + A_8 s_{12} + A_9 = 0 \quad (12)$$

$$B_1 c_{12} c_{13} + B_2 c_{12} s_{13} + B_3 s_{12} c_{13} + B_4 s_{12} s_{13} + B_5 c_{12} + B_6 s_{12} + B_7 c_{13} + B_8 s_{13} + B_9 = 0 \quad (13) \text{ where}$$

$$T_1 c_{11} c_{13} + T_2 c_{11} s_{13} + T_3 s_{11} c_{13} + T_4 s_{11} s_{13} + T_5 c_{11} + T_6 s_{11} + T_7 c_{13} + T_8 s_{13} + T_9 = 0 \quad (14) \quad A_1 = -2P_{11}P_{21} - 2P_{12}P_{21}$$

$$A_2 = -2P_{11}Q_{21} - 2P_{12}Q_{22} - 2P_{13}Q_{23}$$

$$A_3 = -2P_{21}Q_{11} - 2P_{22}Q_{12} - 2P_{23}Q_{13}$$

$$A_4 = -2Q_{11}Q_{21} - 2Q_{12}Q_{22} - 2Q_{13}Q_{23}$$

$$A_5 = 2P_{12}(R_{12} - R_{22}) + 2P_{13}(R_{13} - R_{23})$$

$$A_6 = 2Q_{12}(R_{12} - R_{22}) + 2Q_{13}(R_{13} - R_{23})$$

$$A_7 = -2P_{22}(R_{12} - R_{22}) - 2P_{23}(R_{13} - R_{23})$$

$$A_8 = -2Q_{22}(R_{12} - R_{22}) - 2Q_{23}(R_{13} - R_{23})$$

$$A_9 = P_{11}^2 + P_{12}^2 + P_{13}^2 + P_{21}^2 + P_{22}^2 + P_{23}^2 + (R_{12} - R_{22})^2 + (R_{13} - R_{23})^2 - m_{12}^2$$

and

$$B_1 = -2P_{21}P_{31} - 2P_{22}P_{32} - 2P_{23}P_{33}$$

$$B_2 = -2P_{21}Q_{31} - 2P_{22}Q_{32} - 2P_{23}Q_{33}$$

$$B_3 = -2P_{31}Q_{21} - 2P_{32}Q_{22} - 2P_{33}Q_{23}$$

$$B_4 = -2Q_{21}Q_{31} - 2Q_{22}Q_{32} - 2Q_{23}Q_{33}$$

$$B_5 = 2P_{22}(R_{22} - R_{32}) + 2P_{23}(R_{23} - R_{33})$$

$$B_6 = 2Q_{22}(R_{22} - R_{32}) + 2Q_{23}(R_{23} - R_{33})$$

$$B_7 = -2P_{32}(R_{22} - R_{32}) - 2P_{33}(R_{23} - R_{33})$$

$$B_8 = -2Q_{32}(R_{22} - R_{32}) - 2Q_{33}(R_{23} - R_{33})$$

$$B_9 = P_{21}^2 + P_{22}^2 + P_{23}^2 + P_{31}^2 + P_{32}^2 + P_{33}^2 + (R_{22} - R_{32})^2 + (R_{23} - R_{33})^2 - m_{23}^2$$

and

$$T_1 = -2P_{11}P_{31} - 2P_{12}P_{32} - 2P_{13}P_{33}$$

$$T_2 = -2P_{11}Q_{31} - 2P_{12}Q_{32} - 2P_{13}Q_{33}$$

$$T_3 = -2P_{31}Q_{11} - 2P_{32}Q_{12} - 2P_{33}Q_{13}$$

$$T_4 = -2Q_{11}Q_{31} - 2Q_{12}Q_{32} - 2Q_{13}Q_{33}$$

$$T_5 = 2P_{12}(R_{12} - R_{32}) + 2P_{13}(R_{13} - R_{33})$$

$$T_6 = 2Q_{12}(R_{12} - R_{32}) + 2Q_{13}(R_{13} - R_{33})$$

$$T_7 = -2P_{32}(R_{12} - R_{32}) - 2P_{33}(R_{13} - R_{33})$$

$$T_8 = -2Q_{32}(R_{12} - R_{32}) - 2Q_{33}(R_{13} - R_{33})$$

$$T_9 = P_{11}^2 + P_{12}^2 + P_{13}^2 + P_{31}^2 + P_{32}^2 + P_{33}^2 + (R_{12} - R_{32})^2 + (R_{13} - R_{33})^2 - m_{13}^2$$

For these equations we can substitute the following expression for the cosines and sines of the angle :

$$\cos \theta_{li} = \frac{1 - x_i^2}{1 + x_i^2} \quad \sin \theta_{li} = \frac{2x_i}{1 + x_i^2} \quad \text{where} \quad x_i = \tan \frac{\theta_{li}}{2} \quad \text{for } i = 1, 2, 3$$

Eqs.(12),(13) and (14) can be rewritten as:

$$V_1x_2^2 + V_2x_2 + V_3 = 0 \quad (15)$$

$$E_1x_3^2 + E_2x_3 + E_3 = 0 \quad (16)$$

$$F_1x_3^2 + F_2x_3 + F_3 = 0 \quad (17)$$

where

$$V_1 = M_1x_1^2 + M_2x_1 + M_3 \quad V_2 = M_4x_1^2 + M_5x_1 + M_6 \quad V_3 = M_7x_1^2 + M_8x_1 + M_9$$

$$E_1 = H_1x_2^2 + H_2x_2 + H_3 \quad E_2 = H_4x_2^2 + H_5x_2 + H_6 \quad E_3 = H_7x_2^2 + H_8x_2 + H_9$$

$$F_1 = N_1x_1^2 + N_2x_1 + N_3 \quad F_2 = N_4x_1^2 + N_5x_1 + N_6 \quad F_3 = N_7x_1^2 + N_8x_1 + N_9$$

and

$$M_1 = A_1 - A_5 - A_7 + A_9 \quad H_1 = B_1 - B_5 - B_7 + B_9 \quad N_1 = T_1 - T_5 - T_7 + T_9$$

$$M_2 = 2(-A_3 + A_6) \quad H_2 = 2(-B_3 + B_6) \quad N_2 = 2(-T_3 + T_6)$$

$$M_3 = -A_1 + A_5 - A_7 + A_9 \quad H_3 = -B_1 + B_5 - B_7 + B_9 \quad N_3 = -T_1 + T_5 - T_7 + T_9$$

$$M_4 = 2(A_8 - A_2) \quad H_4 = 2(B_8 - B_2) \quad N_4 = 2(T_8 - T_2)$$

$$M_5 = 4A_4 \quad H_5 = 4B_4 \quad N_5 = 4T_4$$

$$M_6 = 2(A_2 + A_8) \quad H_6 = 2(B_2 + B_8) \quad N_6 = 2(T_2 + T_8)$$

$$M_7 = -A_1 - A_5 + A_7 + A_9 \quad H_7 = -B_1 - B_5 + B_7 + B_9 \quad N_7 = -T_1 - T_5 + T_7 + T_9$$

$$M_8 = 2(A_3 + A_6) \quad H_8 = 2(B_3 + B_6) \quad N_8 = 2(T_3 + T_6)$$

$$M_9 = A_1 + A_5 + A_7 + A_9 \quad H_9 = B_1 + B_5 + B_7 + B_9 \quad N_9 = T_1 + T_5 + T_7 + T_9$$

We can eliminate x_3 from Eqs. (15) and (16) using Bezout's method [9,10] and the resulting equation will contain only x_1 and x_2 ;as follows :

* Step 1- Elimination of x_3 :

Multiplying eq.(15) by F_1 and eq.(16) by E_1 and subtracting we obtain :

$$(F_1E_2 - F_2E_1)x_3 + (F_1E_3 - F_3E_1) = 0 \quad (18)$$

Multiplying eq.(15) by F_3 and eq.(16) by E_3 and subtracting then dividing by x_3 we obtain :

$$(F_3E_1 - F_1E_3)x_3 + (F_3E_2 - F_2E_3) = 0 \quad (19)$$

therefore :

$$\begin{vmatrix} F_1E_2 - F_2E_1 & F_1E_3 - F_3E_1 \\ F_3E_1 - F_1E_3 & F_3E_2 - F_2E_3 \end{vmatrix} = 0 \quad (20)$$

Or in expanded form:

$$(F_1E_2 - F_2E_1)(F_3E_2 - F_2E_3) - [(F_1E_3 - F_3E_1)(F_3E_1 - F_1E_3)] = 0 \quad (21)$$

Expanding equation (21) and substituting the expressions for E_1, E_2, E_3, F_1, F_2 ,and F_3 results in the following equation :

$$O_1x_2^4 + O_2x_2^3 + O_3x_2^2 + O_4x_2 + O_5 = 0 \quad (22)$$

Where :

$$O_1 = G_1x_1^4 + G_2x_1^3 + G_3x_1^2 + G_4x_1 + G_5$$

$$O_2 = G_6x_1^4 + G_7x_1^3 + G_8x_1^2 + G_9x_1 + G_{10}$$

$$O_3 = G_{11}x_1^4 + G_{12}x_1^3 + G_{13}x_1^2 + G_{14}x_1 + G_{15}$$

$$O_4 = G_{16}x_1^4 + G_{17}x_1^3 + G_{18}x_1^2 + G_{19}x_1 + G_{20}$$

$$O_5 = G_{21}x_1^4 + G_{22}x_1^3 + G_{23}x_1^2 + G_{24}x_1 + G_{25}$$

* Step 2- Elimination of x_2 :

Multiplying equation (22) by V_1 and eq.(15) by $O_1x_2^2$,and subtracting ,we obtain :

$$(O_2V_1 - O_1V_2)x_2^3 + (O_3V_1 - O_1V_3)x_2^2 + O_4V_1x_2 + O_5V_1 = 0 \quad (23)$$

Multiplying equation (20) by $V_1x_2 + V_2$ and equation (15) by $O_1x_2^3 + O_2x_2^2$, and subtracting ,we get :

$$(O_3V_1 - O_1V_3)x_2^3 + (O_4V_1 + O_3V_2 - O_2V_3)x_2^2 + (O_5V_1 + O_4V_2)x_2 + O_5V_2 = 0 \quad (24)$$

Multiplying equation (15) by x_2 ,we obtain :

$$V_1x_2^3 + V_2x_2^2 + V_3x_2 = 0 \quad (25)$$

We can think of equations (23) ,(24),(25) ,and (15) as four linear equations in three unknowns x_2^3 , x_2^2 , and x_2 .Vanishing of their eliminant yields [7]:

$$\begin{vmatrix} O_2V_1 - O_1V_2 & O_3V_1 - O_1V_3 & O_4V_1 & O_5V_1 \\ O_3V_1 - O_1V_3 & O_4V_1 + O_3V_2 - O_2V_3 & O_5V_1 + O_4V_2 & O_5V_2 \\ V_1 & V_2 & V_3 & 0 \\ 0 & V_1 & V_2 & V_3 \end{vmatrix} = 0 \quad (26)$$

Expansion of equation (26) results in :

$$\begin{aligned} & -O_5V_1[(O_3V_1 - O_1V_3)(V_2^2 - V_1V_3) - V_1V_2(-O_2V_3 + O_3V_2 + O_4V_1) \\ & + V_1^2(O_4V_2 + O_5V_1)] + (O_2V_1 - O_1V_2)[O_5V_2(V_2^2 - V_1V_3) \\ & + V_3^2(-O_2V_3 + O_3V_2 + O_4V_1) - V_2(O_4V_2 + O_5V_1)V_3] + O_4V_1 \\ & [-V_1V_3(-O_2V_3 + O_3V_2 + O_4V_1) + V_2V_3(O_3V_1 - O_1V_3) + O_5V_1^2V_2] \\ & - (O_3V_1 - O_1V_3)[V_3^2(O_3V_1 - O_1V_3) - V_1(O_4V_2 + O_5V_1)V_3 + O_5V_1V_2^2] = 0 \end{aligned} \quad (27)$$

If we substitute the expressions for $V_1, V_2, V_3, O_1, O_2, O_3, O_4$, and O_5 into eq.(27) ,and expand ,we obtain :

$$D_1x_1^{16} + D_2x_1^{15} + D_3x_1^{14} + D_4x_1^{13} + + D_{15}x_1^2 + D_{16}x_1 + D_{17} = 0 \quad (26)$$

Detailed expressions for the O_i 's and D_i 's are not given here due to space limitation and for further information please contact the author .

After solving for x_1, x_2 and x_3 can be computed from Eqs.(16) and (17) ,respectively .Each of these equations is quadratic , and yields two solutions for x_2 and x_3 into Eq.(15) reveals that only one of the four possible combinations of solutions satisfies the equation.Hence we have a total of sixteen valid solutions for x_1, x_2 and x_3 .

The positions of points P_1, P_2 and P_3 relative to the fixed frame are now given by Eqs. (6),(7) and (8) ,respectively .This is sufficient for formulating the transformation from the moving to the fixed reference frame .

General Algorithm for Inverse and Direct Position Kinematics

The algorithm begins in both cases by taking as input the dimensions of a mechanism and the topology of the limbs :

Inverse case:-**Input** the geometry of the mechanism ;

Finding a two solutions for θ_{3i} using eqs.3,4,5

While θ_{3i} 1,2

For $-90 < \theta_{2i} < +90$

Calculate θ_{1i} invers

end

end

Direct case: **Input** the geometry of the mechanism ;

Calculate eqs. 12 ,13 ,14;15,16,17;

Finding the variables of G_i ; D_i ;

Finding X_{1i} using eq.(26)

Finding θ_{1i} direct

While θ_{1i} = real solutions

If $\theta_{1i \text{ direct}} = \theta_{1i \text{ invers}}$ **then** :

Finding θ_{3i} , θ_{2i} from eqs.(15,17) to check the results;

End

Else

End

End.

When following the program it can be observed that it is not only simple but also hasn't got a complex subroutine as well as it doesn't need initial trial and error values this program gives accurate results compared with Yang,G [6] who studied the 3-RRRS and used the iteration method, which leads to non accurate results because of the initial guess besides the long period where the desired solutions can be obtained in 3 to 4 iterations .

Numerical Example

Inverse Position Kinematics Problem

The dimensions of the mechanism were chosen to be :

$$\alpha_1 = 30^0, \alpha_2 = 270^0, \alpha_3 = 150^0$$

$$a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = l_1 = l_2 = l_3 = 1.0$$

Let the coordinates of the members of the mechanism in the base coordinate frame be
 $O_1=(0,-0.5,-0.866)$, $O_2=(0,1.0,0)$, $O_3=(0,-0.5,0.866)$

$P_1=(2.0824802, -1.6662539, 0.0379879)$, $P_2=(2.3824802, 1.4240243, 0.8566386)$,
 $P_3=(2.0824802, -0.4620121, 0.8002285)$

These coordinate of P_1, P_2 , and P_3 do satisfy the constraint that
 $m_{12} = m_{23} = m_{13} = 2.8860364$.

Solving Eq.(3) ;we get the following set of solutions for θ_{11} , θ_{21} and θ_{31} stated below :

$$\theta_{31} = (+60^0, +150^0)$$

$$\theta_{11} = (+50^0, -28.25^0) \text{ when } \theta_{31} = +60^0$$

No real solution for θ_{11} was obtained when $\theta_{31} = +150^0$

$$\theta_{21} = -63.2^0 \text{ when } \theta_{11} = +50^0 \text{ and } \theta_{31} = +60^0$$

$$\theta_{21} = +65.0^0 \text{ when } \theta_{11} = -28.25^0 \text{ and } \theta_{31} = +60^0$$

This is a total of 2 real solutions for θ_{11} , θ_{21} and θ_{31} .

Solving Eq.(4) ;we get the following set of solutions for θ_{12} , θ_{22} and θ_{32} stated below :

$$\theta_{32} = (+58.94^0, +148.94^0)$$

$$\theta_{12} = (+30^0, -10.2^0) \text{ when } \theta_{32} = +58.94^0$$

No real solution for θ_{12} was obtained when $\theta_{32} = +148.94^0$

$$\theta_{22} = -33.37^0 \text{ when } \theta_{12} = +30^0 \text{ and } \theta_{32} = +58.94^0$$

$$\theta_{22} = +32.92^0 \text{ when } \theta_{12} = -10.2^0 \text{ and } \theta_{32} = +58.94^0$$

This is a total of 2 real solutions for θ_{12} , θ_{22} and θ_{32} .

Solving Eq.(5) ; we get the following set of solutions for θ_{13} , θ_{23} and θ_{33} stated below :

$$\theta_{33} = (+60^0, +150^0)$$

$$\theta_{13} = (+50^0, -28.25^0) \text{ when } \theta_{33} = +60^0$$

No real solution for θ_{13} was obtained when $\theta_{33} = +150^0$

$$\theta_{23} = -63.2^0 \text{ when } \theta_{13} = +50^0 \text{ and } \theta_{33} = +60^0$$

$$\theta_{23} = +65.0^0 \text{ when } \theta_{13} = -28.25^0 \text{ and } \theta_{33} = +60^0$$

This is a total of 2 real solutions for θ_{13} , θ_{23} and θ_{33} .

Hence- for the inverse kinematics problem of this particular example- we have a total of 12 real solutions from the maximum possible 64 solutions.

Direct Position Kinematics Problem

As an example of solution of a direct kinematics problem , we shall take the same dimensions as were taken in the inverse kinematics problem ,and also for the angles of the

actuated joints we shall take one of the solutions obtained from the inverse kinematics problem so as to verify our result .

$$\begin{aligned} \text{Hence we select : } \theta_{21} &= -63.2^0, \quad \theta_{31} = +60.0^0 \\ \theta_{22} &= -33.37^0, \quad \theta_{32} = +58.94^0 \\ \theta_{23} &= -63.2^0, \quad \theta_{33} = +60.0^0 \end{aligned}$$

The polynomial equation obtained for x_I is :

$$\begin{aligned} &1.0e+16*(-0.0862X_1^{16} - 0.7272X_1^{15} - 2.1309X_1^{14} - 3.5645X_1^{13} - 6.2507X_1^{12} \\ &- 9.0553X_1^{11} - 9.5646X_1^{10} - 8.5934X_1^9 - 6.3117X_1^8 - 2.4643X_1^7 + 0.6082X_1^6 \\ &+ 1.204X_1^5 + 0.5128X_1^4 + 0.0322X_1^3 - 0.0439X_1^2 - 0.0145X_1 - 0.0015) = 0 \end{aligned}$$

The solutions of this equation are presented in Table 1. The angles in this table are in degrees and the lengths in meter .The final mechanism configuration obtained from this simulation is presented in Figure (3) .

Hence for this particular problem , we get **8** real solutions. All the solutions were checked by back-substitution into Eqs. (15) , (16) , and (17) . They are all valid .This shows that there are no spurious solutions and the minimum order of the polynomial Eq.(28) is sixteen .Also we can clearly verify that one of the real solutions obtained for direct kinematics problem (No. 7) does match with the solution obtained to the inverse kinematics problem ,which was expected .

Table -1- output results from eq-26

	X_I	θ_{I1}	θ_{I2}	θ_{I3}
1	- 4.1337	-152.80	-169.59	-152.75
2	- 2.7893	-140.55	-91.82	-32.43
3	0.5409 + 1.2536 i	-----	-----	-----
4	0.5409 - 1.2536 i	-----	-----	-----
5	0.1730 + 1.1164 i	-----	-----	-----
6	0.1730 - 1.1164 i	-----	-----	-----
7	0.4666	50.02	30.00	49.989
8	0.3422	37.78	-47.76	-70.32
9	- 0.6382	-65.09	-53.80	-131.96
10	- 0.6313	-64.52	30.78	-15.84
11	- 0.4473 + 0.2923 i	-----	-----	-----
12	- 0.4473 - 0.2923 i	-----	-----	-----
13	- 0.4485 + 0.2872 i	-----	-----	-----
14	- 0.4485 - 0.2872 i	-----	-----	-----
15	- 0.3467	-19.12	-170.37	-86.92
16	- 0.3412	-37.67	-85.79	29.20

Validation Example:

For the validity of this work ,if the example of Husain and Waldron [10] is modified to be 3 – $RRR^{\perp}S$ spatial mechanism with a dimensions according to the first limb as well as the other tow limbs are similar to it in the dimensions and angles as follows :

$$a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = l_1 = l_2 = l_3 = 1.0$$

$$\alpha_1 = 30^0, \quad \alpha_2 = 270^0, \quad \alpha_3 = 150^0$$

The coordinates of the members of the mechanism in the base coordinate frame be $O_1=(0,-0.866,-1.5)$, $O_2=(0,1.732,0)$, $O_3=(0,-0.866,+1.5)$

$P_1=(1.4763806, 0.8419763, -0.073787)$, $P_2=(1.4763806, -0.3571248, 0.7660444)$,
 $P_3=(1.4763806, -0.4848515, -0.6922573)$

According to this changes in the coordinate of P_1, P_2 , and P_3 , the side lengths of moving plate must be : $m_{12} = m_{23} = m_{13} = 1.463934986$.

Inverse Position Kinematics

Solving Eqs.(3,4,5) ;we get the following set of solutions for θ_{1i}, θ_{2i} and θ_{3i} stated below :

$$\theta_{3i} = (+50^\circ, +130^\circ)$$

$$\theta_{1i} = (+36.06^\circ, -73.48^\circ) \text{ when } \theta_{3i} = +50^\circ$$

No real solution for θ_{1i} was obtained when $\theta_{3i} = +130^\circ$

$$\theta_{2i} = +30^\circ \text{ when } \theta_{1i} = -73.48^\circ \text{ and } \theta_{3i} = +50^\circ$$

$$\theta_{2i} = -30^\circ \text{ when } \theta_{1i} = 36.06^\circ \text{ and } \theta_{3i} = +50^\circ$$

This is a total of 6 real solutions for θ_{1i}, θ_{2i} and θ_{3i} .

Hence- for the inverse kinematics problem of this particular example- we have a total of 12 real solutions from the maximum possible 64 solutions.

Direct Position Kinematics

we shall take the same dimensions as were taken in the inverse kinematics problem ,and also for the angles of the actuated joints we shall take one of the solutions obtained from the inverse kinematics problem so as to verify our result .

Hence we select : $\theta_{2i} = -63.2^\circ$, $\theta_{3i} = +60.0^\circ$

Therefore one of the solutions is $\theta_{11} = -73.9047^\circ$, $\theta_{12} = -73.9047^\circ$, $\theta_{13} = -73.9046^\circ$

Or in other word it is exactly the same results of Husain and Wldron [10].

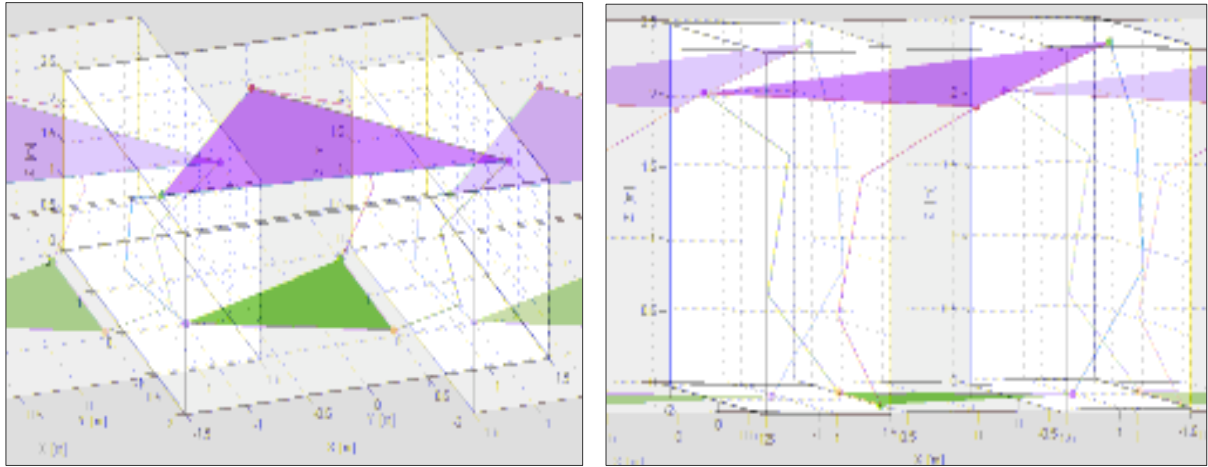


Fig.(3) The resulting 3 – RRR^1S mechanism using MATLAB

Workspace Visualization

Geometrical description of workspace

The complete workspace of a 6-DOF parallel robot is embedded in a 6-Dim. manifold which cannot be represented in a 3-Dim. space .Hence ,the workspace determined here is the reachable workspace ,i.e., the region of the 3-D cartesian space that can be reached by the mobile platform with a given orientation of the platform [6,13] .

Since the pose of each leg is:

$$p_{pi} = [{}^gR_{oi}]q_i + \rho_{oi} \quad \text{for } i = 1,2,3 \quad (1)$$

Where each leg is considered as an independent serial chain .For a given mobile platform orientation, the platform position can be derived by translating point through a fixed vector (q_i).In other word ,the reachable workspace of the mobile platform determined by leg i is a

fixed translation of point Pi's workspace .However ,the motion of 3-leg parallel robot is constrained by its three legs such that the position of the mobile platform must satisfy the three inverse kinematic equations ,i.e.,eq.(2),simultaneously. The reachable workspace of the mobile platform is ,therefore ,the intersection of the individual ones determined by each of the three legs

Workspace visualization scheme

For each of the three legs ,substituting eq.(2) into eq.(1),the individual workspace can be theoretically generated through infinitely varying the three joint displacements within their joint limitations .For simplicity ,the 3-Dim parametric surface plotting function provided by the MATLAB software is directly employed to generate each of the individual workspaces. However ,this 3-Dim plotting function only takes two parametric variables (θ_{1i} , θ_{3i}) at a time and the resulting plot is 3-Dim surface .We ,therefore ,evenly discretize the working range of one joint variable into many intermediate values ;then for each of them ,with the other two joint variable to generate a 3-D surface .Plotting all of these surfaces together ,a 3-D workspace can be visualized .Finally ,plotting the three individual workspaces in the same graph, the intersection of the three workspaces can be obtained ,which is the actual reachable workspace of the mobile platform .In addition ,the 2-Dim view provided by MATLAB can be used to investigate the internal structure of the workspace .

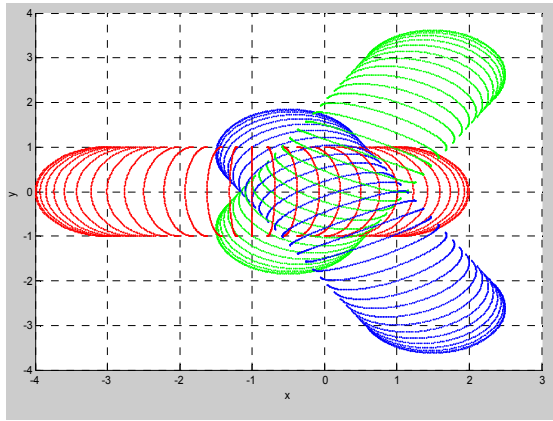


Fig.(4-a) y-axis verses x- axis

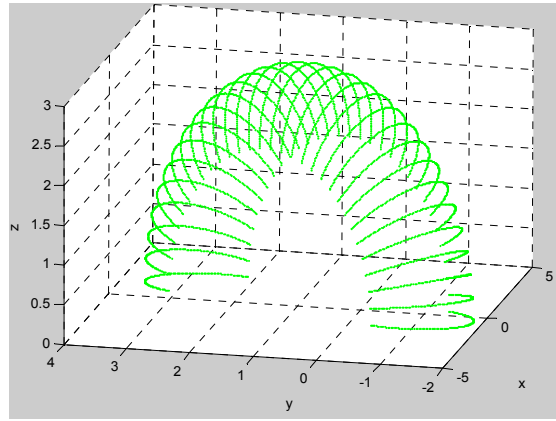


Fig.(4-b) workspace of first leg alone

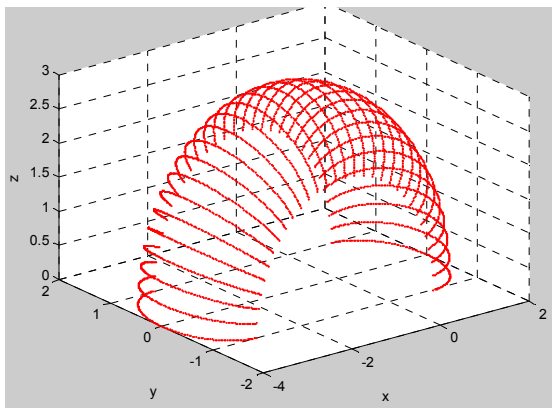


Fig.(4-c) workspace of second leg alone

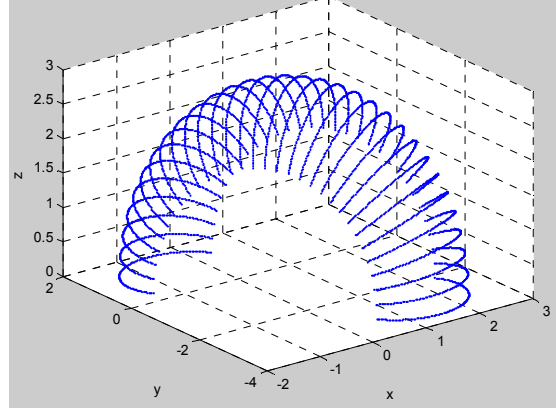


Fig.(4-d) workspace of third leg alone

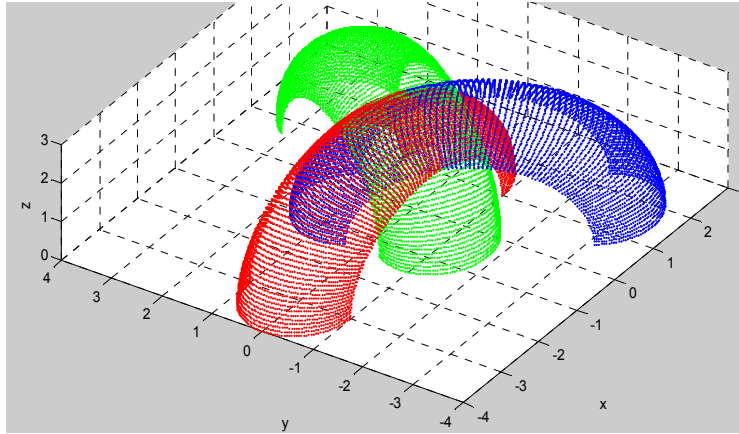


Fig.(4-e) 3-Dim workspace of three legs together

Visualization example :

In this example ,we employ the workspace visualization scheme developed above to graphically represent the workspace of the parallel robot given in the previous example. The workspace visualization results are shown in Fig .(4-a,b,c,d,e).

Fig.(a ,e) show the intersection part of the platform generated by the three legs .This 3-D workspace can be graphically studied by plotting it in different MATLAB function ,and observing it from different view points and angles .In order to understand it better ,the individual workspace generated by leg 1,2,3 is shown in Fig.(b,c,d) respectively .

In the planes parallel to the xy plane are shown in Fig. (a).With the help of the graphs , further study to the internal structure of the workspace , can be easily identified the working regions (intersection areas)in the composite workspaces based on the plotting density of the graphs.

Conclusion

The number of solutions of the inverse kinematics problem is shown to be not more than **64** ,and the solution of the direct kinematics problem has been shown to reducible to a **16th** order polynomial .This implies that for a given set of actuated angles , this $3-RRR^{\perp}S$ parallel mechanism can be assembled in at most **16** different configurations .

Further numerical computations were performed to check the algebra and a numerical examples were solved to demonstrate the procedure . It have been shown that the mechanism have a translational and rotational motion and this verify that the mechanism is a **6-D.O.F**. In spite of, the solution method is independent on the iteration technique, the solution results has been achieved simply where the initial configuration is not important, but all the equations of motion are trigonometric functions, thus the results are very sensitive for the input values .

It can be seen from these figures that the workspaces generated by each leg is a part of a half solid torus. There is a void in the lower part of the intersection of these three half toruses indicating that the platform has poor motion capability in the lower z coordinates .

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